A BERNOULLI TORAL LINKED TWIST MAP WITHOUT POSITIVE LYAPUNOV EXPONENTS

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Abstract. The presence of positive Lyapunov exponents in a dynamical system is often taken to be equivalent to the chaotic behavior of that system. We construct a Bernoulli toral linked twist map which has positive Lyapunov exponents and local stable and unstable manifolds defined only on a set of measure zero. This is a deterministic dynamical system with the strongest stochastic property yet it has positive Lyapunov exponents only on a set of measure zero. In fact we show that for any map g in a certain class of piecewise linear Bernoulli toral linked twist maps given any $\epsilon > 0$ there is a Bernoulli toral linked twist map g' with positive Lyapunov exponents defined only on a set of measure zero such that g' is within ϵ of g in the $\bar{d}$ metric.

1. Introduction

For the purposes of this paper a toral linked twist map (tltm) g is a map of the two dimensional torus onto itself of the form $g = \tau_2 \circ \tau_1$ where $\tau_1(x, y) = (x + h_1(y), y)$ and $\tau_2(x, y) = (x, y + h_2(x))$ are twist maps. The maps $h_i : [0, 1] \to \mathcal{R}$, $i = 1, 2$ are measurable and $h_i'(x) > 0$ where the derivative is defined. Such maps preserve Lebesgue measure.

If the defining maps h_i are piecewise linear with finitely many discontinuities then the tltm g possesses all the features of hyperbolicity (positive Lyapunov exponents, local stable and unstable manifolds) in a very explicit form. Furthermore an application of the work of Katok et. al. [3] shows that the torus decomposes into countably many ergodic components, on each of which the restriction of g is either a Bernoulli map or the cross product of a Bernoulli map with a rotation. Przytycki has shown that if in addition for almost all points $p, q \in T^2$ and every pair of integers m, n large enough $g^m(l^u(p)) \cap g^{-n}(l^s(q)) \neq \emptyset$ (where $l^u(p)$ is the local unstable manifold of p and $l^s(q)$ is the local unstable manifold of q) then g is a Bernoulli map [8]. The ergodicity of such a tltm may also be obtained from the more recent techniques described by Liverani and Wojtowski [4].

In much of the literature on chaotic dynamical systems the presence of positive Lyapunov exponents in a dynamical system is taken to be equivalent to the chaotic behavior of that system. We construct a Bernoulli tltm f for which positive Lyapunov exponents exist only on a set of measure zero. Furthermore we show that such maps are dense in the $\bar{d}$ metric in a certain class of piecewise linear tltm's. The class of tltm's we consider satisfy the following requirements:
‡
1) The maps h_i are piecewise linear functions with finitely many discontinuities.

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2) $h_1'(x) > 2$ a.e. and if $\{x_i\}$ are the discontinuities of h_1 (where $x_{i+1} > x_i$) then for all $x \in (x_i, x_{i+1})$, $h_1'(x)(x_{i+1} - x_i) > 1$. Analogous conditions hold for h_2.

Remark 1. The second stipulation is technical and allows us to check the Przytycki condition easily and hence show that the tltm's we consider are Bernoulli.

Theorem 1. *Given a Bernoulli tltm satisfying condition ‡ and $\epsilon > 0$ there exists a Bernoulli tltm g' which has positive Lyapunov exponents defined only on a set of measure zero and which is within ϵ of g in the $\bar{d}$ metric.*

The proof of this result uses several lines of argument from [5]. The density of the maps g' in Theorem 1 follows from the method of construction of one such map f. The Bernoulli property is usually established by arguments involving hyperbolic properties of a dynamical system. This hyperbolic structure is missing in our construction. For other reasons it is difficult to establish by the usual hyperbolic techniques that f is Bernoulli. To prove the Bernoulli property we use certain ideas from the Bernoulli theory of Ornstein and Weiss [6], in particular the notion of a finitely determined process and the fact that B processes are closed in the $\bar{d}$ metric (Theorem 2).

The idea of the construction is simple. The map f is constructed as a limit in the $\bar{d}$ metric of a sequence $\{f_i\}$ of hyperbolic toral linked twist maps. f_{i+1} is produced by altering f_i on a small set so as introduce a discontinuity. The difficult part of the construction lies in showing that it is possible to do this and yet still keep the two maps sufficiently close in the $\bar{d}$ metric so that the sequence converges to a Bernoulli tltm. By exploiting the hyperbolicity of each map in the sequence rather than the hyperbolicity of f itself we are able to do this. This construction allows us to show that f is Bernoulli even though f has no hyperbolic structure.

The following section is found in a slightly modified form in [5] but we include it for completeness.

1.1. **Some Background.** First we establish some notation. If $C \subset T^2$ is a set of positive Lebesgue measure then μ_C denotes the conditional measure on C, that is $\mu_C(A) = \frac{\mu(A \cap C)}{\mu(C)}$ for every set $A \subset T^2$. If P is a partition of T^2 and $E \subset T^2$ then $P|E$ denotes the induced partition of the measure space E with measure μ_E. We say that a property holds for ϵ a.e. atom of P if the property holds for all atoms of P except for a union of atoms which has measure less than ϵ. We denote by $H_S(P|Q)$ the relative entropy conditioned on the set S i.e. the set S is normalised to have measure one and the entropy of the partition $S \cap P$ relative to $S \cap Q$ is calculated. All finite partitions of the torus that we consider will be assumed to have atoms with piecewise smooth boundaries. $\mathcal{B}$ denotes the Borel sigma algebra.

Some Aspects Of The Bernoulli Theory

Let (f, X, μ) be a dynamical system (f is a transformation of the metric space X which preserves the measure μ) and let P be a finite partition of X . We say that the points x, y have the same $P - N$ name if $f^k(x)$ and $f^k(y)$ lie in the same set of P for all $0 \leq k \leq N$. The partition of X into sets with the same $P - N$ name we denote $\vee_0^N f^{-j} P$. If (f, X, μ) is a dynamical system then we say the partition P of X is a *generator* (or *generates*) if $\vee_{-\infty}^{\infty} f^{-j} P$ is the entire sigma algebra of the system (which in the present context is $\mathcal{B}$). A dynamical system is *Bernoulli* if it has an independent generator i.e. a generating partition P such that $f^n P$ is independent of $\vee_{-n+1}^{n-1} f^{-j} P$ for all $n \geq 1$ (for equivalent definitions see [6]).

We may model a stationary stochastic process consisting of random variables taking values in a set M as a dynamical system in a standard way [6, page 48]. If the resulting dynamical system is Bernoulli then the process is called a B-process (note that this is not necessarily an independent process).

Conversely if X is given the finite partition $P = \{P_1, ..., P_k\}$ then the dynamical system (f, X, μ) produces a stationary stochastic process by inducing a measure on the space $\Omega = \{1, ..., k\}^Z$. The shift map $S\omega_n = \omega_{n-1}$ preserves this measure. When X and μ are clear from context we will denote this process as (f, P). If (f, X, μ) is a Bernoulli dynamical system then for every finite partition P of X, (f, P) is a B-process.

The $\bar{d}$ metric

The $\bar{d}$ metric is a way of measuring how close the output of two stochastic processes are over a long time period. The awkwardness of the definition of the $\bar{d}$-metric arises from the fact that the two stochastic processes need not be defined on the same measure space. By constructing a joining of the two underlying measure spaces it becomes possible to make statements about the likelihood that two stochastic processes defined on the measure spaces (taking values in the same metric space) will have a similar output. Suppose (X, μ) and (Y, ν) are two measure spaces. A joining of (X, μ) and (Y, ν) is a measure Λ on $X \times Y$ such that $\Lambda(A \times Y) = \mu(A)$ and $\Lambda(X \times B) = \nu(B)$ for every measurable set $A \subset X$ and $B \subset Y$. Given a partition $P = \{P_1, ..., P_k\}$ of X we may think of the partition as a map from the space X to the set S of k symbols $S = \{1, ..., k\}$ such that $P(x) = j$ if $x \in P_j$. Similarly a partition $Q = \{Q_1, ..., Q_k\}$ of Y induces a map $Q : Y \to S$. We define the $\bar{d}$ distance between the partitions P of X and Q of Y by

$$\bar{d}(P, Q) = \inf_{\Lambda} \Lambda(\{x, y\} : P(x) \neq Q(y))$$

where the infinum is taken over joinings. Suppose $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ are a sequence of partitions of X and Y respectively. We define the the $\bar{d}$-distance between these sequences of partitions by

$$\bar{d}(\{A_i\}_{i=1}^n, \{B_i\}_{i=1}^n) = \inf_{\Lambda} \frac{1}{n} \sum_{i=1}^{n} \Lambda(\{x, y\} : A_i(x) \neq B_i(y))$$

Thus if $\bar{d}(\{A_i\}_{i=1}^n, \{B_i\}_{i=1}^n) < \epsilon$ there is a pairing of points in X with those in Y ($x \in X$ paired with $y_x \in Y$ say), such that the pairs (x, y_x) satisfy $A_i(x) = B_i(y_x)$ except for at most a number ϵN of the integers from 1 to N. Furthermore most of the measure of Λ is concentrated on these pairs in the sense that $\Lambda(\{x, y_x\}) > 1 - \epsilon$.

Hence if g_1 and g_2 are measure-preserving transformations of a metric space X and P is a partition of X then

$$\bar{d}(\{g_1^{-i} P\}_{i=1}^{\infty}, \{g_2^{-i} P\}_{i=1}^{\infty}) < \alpha$$

implies that there is a pairing of orbits such that corresponding orbits lie in the same set of the partition except for a set of times of density at most α. We also note that if Q is a partition that refines P then

$$\bar{d}((f_1, P), (f_2, P)) \leq \bar{d}((f_1, Q), (f_2, Q))$$

This fact is clear if one thinks of the characterization of the $\bar{d}$ distance in terms of the time that paired orbits spend in the same set of the partition.

It can be shown that

$$\bar{d}(\{A_i\}_{i=1}^{\infty}, \{B_i\}_{i=1}^{\infty})$$

$$= \lim_{n\to\infty} \bar{d}(\{A_i\}_{i=1}^n, \{B_i\}_{i=1}^n)$$
$$= \inf_n \bar{d}(\{A_i\}_{i=1}^n, \{B_i\}_{i=1}^n)$$

We denote
$$\bar{d}(\{g_1^{-i}P\}_{i=1}^\infty, \{g_2^{-i}P\}_{i=1}^\infty)$$
by
$$\bar{d}((g_1, P), (g_2, P)).$$
For a nice discussion of the $\bar{d}$ metric see [1].

Very Weak Bernoulli Partitions

Ornstein used the $\bar{d}$-metric to formulate the notion of a Very Weak Bernoulli partition. Let f be an invertible mpt of a metric space X and let P be a partition of X. The partition P is a Very Weak Bernoulli (VWB) partition if for every $\epsilon > 0$ there exists N_0 such that for all $N' > N > N_0$ and for all $n \geq 0$ ϵ a.e. atom E of $\vee_N^{N'} f^i P$ satisfies
$$\bar{d}(\{f^{-i}P \mid E\}_0^n, \{f^{-i}P\}_0^n) < \epsilon$$

The process (f, P) is a B-process if and only if the partition P is VWB. The next theorem motivates the construction of the $\bar{d}$ cauchy sequence.

Theorem 2. [6]

1 *If (f_n, P_n) are B-processes that converge in the $\bar{d}$-metric to (f, P) then (f, P) is a B-process.*
2 *If f is a mpt of X and $\{P_i\}$ is a sequence of partitions such that $P_i < P_{i+1}$, $\vee_i P_i = \mathcal{B}$ and each P_i is VWB then f is a Bernoulli transformation.*

The VWB property can be checked using hyperbolic structure and this allowed Ornstein and others to show that many physical systems are Bernoulli [2], [7].

Finitely Determined Processes

The property of being finitely determined is another characterization of B-processes.

Definition 2. A process (f, P) is finitely determined (FD) if given $s > 0$ there exists $\delta, N, \gamma > 0$ such that when $(\bar{f}, \bar{P})$ is an ergodic process which satisfies

1) (close in finite distribution)
$$\bar{d}(\{f^{-i}P\}_0^N, \{\bar{f}^{-i}\bar{P}\}_0^N) < \delta$$
2) (close in entropy)
$$h(\bar{f}, \bar{P}) \geq h(f, P) - \gamma$$
then
$$\bar{d}((f, P), (\bar{f}, \bar{P})) < s.$$

Hyperbolic Dynamical Systems

The hyperbolic structure of an invertible mpt of a space X can be used to show that partitions of X are VWB. The two features of hyperbolic dynamical systems that need mention are local (un)stable manifolds and hyperbolic blocks.

Local Stable and Unstable Manifolds.

Let g be a Bernoulli tltm such that it's defining maps h_i are piecewise linear and possess only finitely many discontinuities. Osceledec's theorem implies that there exist $\lambda > 0$ and a measurable splitting $T_x(T^2) = E_x^s \oplus E_x^u$ such that for $v \in E_x^u$ (resp. E_x^s) ,

$$\lim_{n\to\infty} \frac{1}{n} \log |D_x g^n(v)| = \lambda \ (resp. -\lambda).$$

for almost every $x \in T^2$.

A *local stable manifold* (lsm) of g is a C^1 curve such that there exists $0 < \rho < 1$ and a constant C such that $d(g^n(p), g^n(q)) < C\rho^n d(p, q)$ $n \geq 0$ for all points p, q contained in the curve and the curve is tangent to E^s_p (where defined) for every point in the curve. In the statement of Theorem 1 we refer to local stable and unstable manifolds corresponding to this definition.

In the *proofs* we give we impose stronger conditions on lsm's and lum's and we take into account the partition of T^2 that we are considering. This makes many of the proofs simpler but is nonstandard. Let (g, T^2, μ) be a tltm with P a finite partition of T^2. In our proofs we will also require that any two points on the same lsm have the same $P - N$ name for all N and that any two points on the same lum have the same $P - N$ name under the map g^{-1} for all N. It is instructive to think of points on the same lsm as possessing the same 'future' and points on the same lum as having had the same 'past'. When we speak of lsm's and lum's for a process (g, P) we are assuming this stronger condition. We will not introduce another notation since the meaning should be clear from context.

Hyperbolic Blocks

A *hyperbolic block* $\mathcal{H}$ of (f, P) is a measurable set with a distinguished point p (and distinguished lsm $l^s(p)$ and lum $l^u(p)$) constructed in the following way. Let $\tilde{\mathcal{H}}$ be an open, connected set in the shape of a four-sided polygon with p as center whose sides are parallel to $l^u(p)$ and $l^s(p)$ and of the same length. The hyperbolic block $\mathcal{H}$ is the subset of $\tilde{\mathcal{H}}$ such that if $q_1, q_2 \in \mathcal{H}$ then the local stable manifold $l^s(q_1)$ intersects the local unstable manifold $l^u(q_2)$ at precisely one point which lies in $\tilde{\mathcal{H}}$. Note that $p \in \mathcal{H}$ but that there may be points in $\tilde{\mathcal{H}}$ whose lsm and lum do not have this intersection property. A hyperbolic block is called *ϵ-good* if $\mu_{\tilde{\mathcal{H}}}\{\mathcal{H}\} > \infty - \epsilon$.

The grid structure of $\mathcal{H}$ defines a product measure (the product of the conditional invariant measures on $l^u(p)$ and $l^s(p)$).This product measure must be absolutely continuous with respect to the invariant measure for the concept of the block to be useful. For further discussion of these issues see [3] and [6].

Absolute continuity will be clear in our context as both the lsm's and the lum's may be taken to be straight line segments and the invariant measure is Lebesgue measure on T^2.

2. The Construction of f

The Bernoulli tltm with positive Lyapunov exponents defined only on a set of measure zero, which we denote f, is constructed as a $\bar{d}$ (and pointwise) limit of tltm's. This construction allows us to prove that f is Bernoulli as it is difficult to establish this directly from the standard hyperbolic theory. Instead, we establish these two properties by applying certain results of Ornstein and Weiss [6], in particular Theorem 2.

We construct a sequence of finite partitions (with piecewise smooth boundaries), $\{P_i\}$, such that $P_i < P_{i+1}$ and $\vee P_i = \mathcal{B}$. These are the only conditions on the partitions that we shall require. We also construct a sequence of maps $\{f_j\}$ of the form $f_j(x, y) = \tau_2^j \circ \tau_1(x, y)$. Note that $\tau_1(x, y) = (x + h_1(y), y)$ and $\tau_2^j(x, y) =$

$(x, y + h_2^j(x))$ where we take h_1 and h_2^j, $(j \geq 0)$ to be piecewise linear with finitely many discontinuities.
Let $f_0 \equiv \tau_2^0 \circ \tau_1$ satisfy condition ‡.

Now let A be a vertical strip of the form $(\alpha, \beta) \times [0, 1)$. Note here that we may take the area of A to be as small as we like.

Let $\{a_i\}$ be a countable, dense set of points in (α, β) and $\{\epsilon_i\}$ a sequence (to be specified later) such that $a_i \pm \epsilon_i \in (\alpha, \beta)$ for all i. We take ϵ_i sufficiently small so that $(a_i - \epsilon_i, a_i + \epsilon_i) \cap (a_j - \epsilon_j, a_j + \epsilon_j) = \emptyset$ if $i \neq j$. For each integer l let Q_l be a finite cover of (α, β) into intervals of length at most $\frac{1}{l}$.
We define $\tau_2^i(x, y) = \tau_2^{i-1}(x, y)$ for points in the complement of the strip $(a_i - \epsilon_i, a_i + \epsilon_i) \times [0, 1)$ and we define τ_2^i (i.e. define h_2^i) on $(a_i - \epsilon_i, a_i + \epsilon_i) \times [0, 1)$ so that h_2^i satisfies the conditions of ‡. Since $\epsilon_i \to 0$, this means that ${h_2^i}' \to \infty$ in the supremum norm. Now we will add discontinuities in such a way as to ensure that the limit map f is discontinuous on a set of positive measure. Fix an integer i'. Now suppose we have defined τ_2^i for $i < i'$. Fix an integer l' and consider the finite cover $Q_{l'}$ into intervals $\{I_{l'}^j\}$, $(j = 1, ..., N_{l'}$ say). We define $h_2^{i'}$ on $(a_i + \epsilon_i, a_i - \epsilon_i)$ in such a way that if there exist any points a_j $(j < i')$ such that $a_{i'}$ and a_j both lie in the same set $I_{l'}^k$ of the partition $Q_{l'}$ then for at least one such a_j and corresponding map h_2^j, $h_2^{i'}$ satisfies $|h_2^j(x') - h_2^{i'}(x)| > \frac{1}{2}$ for some points x, x' $(x \in (a_{i'} - \epsilon_{i'}, a_{i'} + \epsilon_{i'}) \cap I_{l'}^k, x' \in I_{l'}^k)$. We continue to define h_2^i for $i > i'$ in this way until $i = i_{l'}$ is sufficiently large that every interval $\{I_{l'}^j\} \in Q_{l'}$ has the property that there exist points $p_{l'}^j, q_{l'}^j$ in $I_{l'}^j$ such that $|h_2^{i_{l'}}(p_{l'}^j) - h_2^{i_{l'}}(q_{l'}^j)| > \frac{1}{2}$. Then in the construction of $h_2^{i_{l'}+1}$ we consider the intervals $\{I_{l'+1}^j\}$ of the finite cover $Q_{l'+1}$ and so on.

Note that this implies that the limiting map $f = \tau_2^f \circ \tau_1$ is discontinuous in ${\tau_1}^{-1}(A)$. To see this suppose (x, y) in ${\tau_1}^{-1}(A)$ and note that the condition $(a_i - \epsilon_i, a_i + \epsilon_i) \cap (a_j - \epsilon_j, a_j + \epsilon_j) = \emptyset$ implies that if $j \geq i$ then $h_2^j(x) = h_2^i(x)$ for all $x \in (\alpha_i - \epsilon_i, \alpha_i + \epsilon_i)$. Thus by construction in any interval about x there exists a point x' such that $|h_2^f(x) - h_2^f(x')| > \frac{1}{2}$ and hence f is discontinuous at all points in ${\tau_1}^{-1}(A)$.
It can be shown that such maps are Bernoulli ([5]). The idea of the proof is to show that condition ‡ implies the Przytycki condition and hence Bernoullicity.

We now show that it is possible to choose the sequence ϵ_i so that we have

$$\bar{d}((f_{i+1}, P_i), (f_i, P_i)) \leq 2^{-i}$$

We first stipulate that the sequence ϵ_i satisfies $\epsilon_i \to 0$ sufficiently fast to ensure that $\sum_{i \geq n} \epsilon_i \leq \frac{1}{2^n}$. This ensures that the sequence $\{f_i\}$ converges to some map f pointwise a.e. We then show that (f, P_i) is the $\bar{d}$ limit of (f_j, P_i) for each i and hence that (f, P_i) is a B process (Theorem 2). Since we have chosen the partitions P_i so that $P_i < P_{i+1}$ and $\vee P_i = \mathcal{B}$, Theorem 2 implies that f is Bernoulli. By construction f has a dense set of discontinuities on a set $\tau^{-1}(A)$ of positive measure. As a consequence of the ergodicity of f the orbit of a.e point will visit $\tau^{-1}(A)$ a fraction $|\alpha - \beta|$ of the time. If the orbit of a point enters $\tau^{-1}(A)$ then a Lyapunov exponent cannot be defined for that point since the map is differentiable at no point in $\tau^{-1}(A)$. Since the map is actually not continuous at any point in $\tau^{-1}(A)$ stable and unstable manifolds similarly cannot exist . Thus positive Lyapunov exponents and stable and unstable manifolds exist only for a set of Lebesgue measure zero.

We now show that we may construct such a sequence ϵ_i. The construction proceeds inductively. Assume that we have Bernoulli tltm's $\{f_j\}_{j\leq i}$ satisfying

a) $\bar{d}((f_j,P_j),(f_{j+1},P_j)) \leq \frac{1}{2^j}$ for $j<i$

Given f_i, P_i we now show that it is possible to choose a map f_{i+1} (i.e choose ϵ_{i+1}) such that

$$\bar{d}((f_{i+1},P_i),(f_i,P_i)) \leq \frac{1}{2^i}$$

To prove this we use the fact that B-processes are finitely determined. Since f_i is Bernoulli (f_i,P_k) is a B-process for any k.

Apply definition 2 to (f_i,P_i) with $s=2^{-i}$ to obtain $T>0, \gamma>0$ and $\delta>0$ such that if

1) $\bar{d}(\{f_{i+1}^{-j}P_i\}_0^T, \{f_i^{-j}P_i\}_0^T) < \delta$ and

2) $h(f_{i+1},P_i) > h(f_i,P_i) - \gamma$

then

$$\bar{d}((f_{i+1},P_i),(f_i,P_i)) < 2^{-i}.$$

Note that without loss of generality we may take $\gamma < \frac{1}{2^i}$.

The main technical difficulty is to ensure that condition 2) holds. To do this we use an argument involving hyperbolic blocks. For the remainder of this chapter $\gamma(\ \gamma < \frac{1}{2^i}), s, \delta,$ and T will refer to the corresponding parameters obtained by taking (f_i,P_i) for (f,P) in definition 2 .

We define a "successor" of f_i to be a map produced by some choice of ϵ_i subject to the conditions specified above. Thus a successor of the map f_i is a choice of the next map in the sequence f_{i+1}.

We now state the main lemma . It says that it is possible for both f_i and any succesor of f_i, to almost cover T^2 with the same number of hyperbolic blocks. These hyperbolic blocks are bounded below in measure, have a uniform bound on the Radon- Nikodym derivative of product measure with respect to Lebesgue measure and a uniform lower bound on the degree of ϵ- goodness (see Section 1). The proof of this lemma uses standard hyperbolic theory ,in particular the fact that the conditions imposed on f_i and any successor imply that there is a continuous (except at finitely many lines) family of cones in the tangent space [10]. Thus it is possible to construct local stable and unstable manifolds at a.e. point in T^2 . Since f_i and any successor is piecewise linear the lsm's and lum's are straight line segments. Finally the fact the the differential of f_i and any successor of f_i is the same except on a set of measure at most $2\epsilon_i$ and that we may choose ϵ_i sufficiently small that the two maps agree on a set of large measure ensures that the resulting hyperbolic block structure of f_i and any successor can be taken to be as similar as we wish. The proof of this lemma in a more general case has been given explicitly in [5] and and in essence in [10]. We do not repeat it here .

Lemma 3. *Let P_i be a finite partition of T^2. If Φ is a tltm of T^2 consisting of either f_i or a successor of f_i then given $\delta', \eta > 0$ there exists $\beta > 0$ such that we may cover T^2 disjointly up to measure $1-\delta'$ with hyperbolic blocks of Φ all of which have measure greater than β.*

The hyperbolic blocks have the following property:

- If $\mathcal{H}$ is such a hyperbolic block of Φ then

$$\frac{1}{k}|H_{\mathcal{H}}(\vee_0^k \Phi^{-j} P_i | \vee_1^{\infty} \Phi^j P_i) - H_{\mathcal{H}}(\vee_0^k \Phi^{-j} P_i)| \leq \eta$$

for all $k > 0$

This lemma is not saying that it is possible to use the same set of hyperbolic blocks for the two different maps f_i and a successor of f_i but rather that the properties of our cover by hyperbolic blocks (up to measure $1 - \delta'$) hold uniformly for all possible choices of successors. Here we summarise the ideas behind Lemma 3 and how it allows one to obtain the closeness in the $\bar{d}$ metric between f_i and a judicious choice of successor.

First note that any atom of the induced partition of $\vee_o^k \Phi^{-j} P_i$ on such a hyperbolic block $\mathcal{H}$ consists of whole lsm's (since two points on the same contracting fiber have the same future name for all time.) Similarly the induced partition of $\vee_1^{\infty} \Phi^j P_i$ on $\mathcal{H}$ has atoms which consist of whole lum's. Thus if the hyperbolic block $\mathcal{H}$ is composed of lsm's and lum's of sufficiently small length then lsm's in $\mathcal{H}$ approximate parallel straight line segments and lum's in $\mathcal{H}$ approximate parallel straight line segments which transverse the lsm's at almost the same angle. Thus uniformly over k the distribution of an atom in the induced partition of $\vee_0^k \Phi^{-j} P_i$ on $\mathcal{H}$ on each atom in the induced partition of $\vee_1^{\infty} \Phi^j P_i$ on $\mathcal{H}$ (with respect to product measure) is almost the same as its distribution on the whole of $\mathcal{H}$. Therefore the effect on the relative entropy by conditioning on the past is small and the difference between the conditioned and unconditioned entropy approaches 0 as the size of the lsm's and lum's approaches zero. This can be made precise [5, Lemma 1]. We alter f_i (to produce f_{i+1}) on a set of sufficiently small measure so that the finite distributions of f_i and f_{i+1} up to a time T are close on the hyperbolic blocks of f_{i+1} and the size of the atoms in the induced partition of $\vee_0^T f_i^{-j} P_i$ on these hyperbolic blocks reflects the entropy of (f_i, P_i) (a consequence of the Shannon-McMillan-Breiman Theorem).

When we condition on the past on a hyperbolic block of f_{i+1} the relative entropy of the P_i-names of f_{i+1} till time T does not decline much- hence the entropy of (f_{i+1}, P_i) is close to that of (f_i, P_i).

In definition 2 we take $\delta' < \delta$ such that $| \delta' \log \delta' | < \frac{\gamma}{10}$, $\delta' \log(\#P_i) < \frac{\gamma}{10}$ (this last requirement is because of the incompleteness of the cover by hyperbolic blocks and looks forward to lemma 6) and $\eta = \frac{\gamma}{10}$ to obtain a cover with the properties described in lemma 3 (we emphasize again the these properties hold uniformly for the hyperbolic blocks of admissible perturbations). Recall that the hyperbolic blocks in the cover have measure bounded below by $\beta > 0$ (a function of δ' and η) and this bound is uniform over admissible perturbations.

Lemma 4. *Given $0 < \beta < 1$ there exists $N' > N$ such that*

1) $|H_S(\vee_0^{N''} f_i^{-j} P_i) - N''(h(f_i, P_i))| \leq \frac{\gamma}{10}$ *for all $N'' > N'$ where S is any set of measure $\geq \beta$.*

This follows from an application of the Shannon-McMillan-Breiman Theorem and states that after a certain time N' the entropy of f_i can be estimated from the relative entropy of any set of measure greater than β.

Remark 5. Note that this lemma refers only to f_i.

We need one more lemma.

Lemma 6. *If f_{i+1} is an admissible perturbation of f_i then,*

$$H(\vee_0^{N'} f_{i+1}^{-j} P_i | \vee_1^{\infty} f_{i+1}^{j} P_i)$$
$$\geq \sum_j \mu(\mathcal{H}_|) H_{\mathcal{H}_|}(\vee_0^{N'} f_{i+1}^{-j} P_i | \vee_1^{\infty} f_{i+1}^{j} P_i)$$

where $\cup\mathcal{H}_|, j \neq 0$ is the cover of hyperbolic blocks of f_{i+1} and $\mathcal{H}_\prime$ ($\mu(\mathcal{H}_\prime) < \delta'$) is the complement of $\cup\mathcal{H}_|, j \neq 0$.

The proof of this lemma involves a routine calculation (see [5, Lemma 3] for details). It gives a lower estimate for the entropy of f_{i+1} in terms of its relative entropy on hyperbolic blocks.

We now combine lemmas 3, 4 and 6 together to show how to construct f_{i+1}. Recall that we have a $\gamma, (\gamma < \frac{1}{2^i}), \delta,$ and N from definition 2 applied to B_i with $s = 2^{-i}$. In lemma 3 we then take δ' and η so that

$\delta' < \delta, | \delta' \log \delta' | < \frac{\gamma}{10}$, $\delta' \log(\#P_i) < \frac{\gamma}{10}$ (this requirement is necessary because of the incompleteness of the cover in lemma 6) and $\eta = \frac{\gamma}{10}$. This choice of δ' and η defines a minimal measure β for the hyperbolic blocks in the cover as described in lemma 3. For this β we then have a N' as defined in lemma 4.

We need to choose ϵ_{i+1} and define τ_2^{i+1} on $(a_{i+1} - \epsilon_{i+1}, a_{i+1} + \epsilon_{i+1}) \times (0, 1]$ in order to define f_{i+1}. It is clear that once given ϵ_{i+1}, we may define τ_2^{i+1} on $(a_{i+1} - \epsilon_{i+1}, a_{i+1} + \epsilon_{i+1}) \times (0, 1]$ in the manner described earlier so we will just consider the choice of ϵ_{i+1}.

First choose ϵ_{i+1} small enough to ensure

a) $|H_S(\vee_0^{N'} f_{i+1}^{-j} P_i) - H_S(\vee_0^{N'} f_i^{-j} P_i)| < \frac{\gamma}{10}$ for any set S of measure $> \beta$

b) $\bar{d}(\{f_{i+1}^{-j} P_i\}_0^{N'}), (\{f_i^{-j} P_i\}_0^{N'})) < \delta$.

Note that condition b) is condition 2 of definition 2.

Then we have,

$$\begin{aligned}
N'h(f_{i+1}, P_i) &= H(\vee_0^{N'} f_{i+1}^{-j} P_i | \vee_1^{\infty} f_{i+1}^{j} P_i) \text{ (by stationarity and [9, Theorem 4.3 (i), page 81])} \\
&\geq (1 - \delta') \inf_{j>0} H_{\mathcal{H}_|}(\vee_0^{N'} f_{i+1}^{-j} P_i | \vee_1^{\infty} B_{i+1}^{-j} P_i) \text{ (by lemma 6 and a)} \\
&\geq \inf_{j>0} H_{\mathcal{H}_|}(\vee_0^{N'} f_i^{-j} P_i) - \frac{2N'\gamma}{10} \text{ (by lemma 3))} \\
&\geq N'(h(f_i, P_i) - \frac{2\gamma}{10}) - \frac{2N'\gamma}{10} \text{ (by lemma 4).}
\end{aligned}$$

Therefore $h(f_{i+1}, P_i) \geq h(B_i, P_i) - \gamma$ and condition 2 of definition 2 holds.

We now show that (f, P_i) is the $\bar{d}$-limit of (f_j, P_i) for all i and as $\vee P_j = \mathcal{B}$, by Theorem 2, f is Bernoulli.

Lemma 7. $\bar{d}((f, P_j), (f_k, P_j)) \leq \frac{1}{2^k}$ *for all* $k \geq j$.

Proof

It suffices to show

$$\bar{d}((f, P_k), (f_k, P_k)) \leq \frac{1}{2^k} \text{ for all } k$$

as then

$$\bar{d}((f, P_j), (f_k, P_j)) \leq \frac{1}{2^k} \text{ if } j \leq k \text{ (since } P_k \text{ refines } P_j)$$

Now fix P_k. Given s there exists n_s greater than k such that if $r \geq n_s$ we have

$$\bar{d}(\vee_0^s f^{-i} P_k, \vee_0^s f_r^{-i} P_k) \leq \frac{1}{2^s}$$

This is because the measure of the set upon which f and f_r differ tends to zero as $r \to \infty$ and thus if r is sufficiently large then, except for a set of points x of measure at most $\frac{1}{2^s}$, $P_k(F^i(x)) = P_k(f_r^i(x))$ (for $i = 0, ..., s$).
Since $\bar{d}$ is a metric, we have

$$\bar{d}(\vee_0^s f^{-i} P_k, \vee_0^s f_k^{-i} P_k) \leq \bar{d}(\vee_0^s f^{-i} P_k, \vee_0^s f_{n_s}^{-i} P_k) + \bar{d}(\vee_0^s f_k^{-i} P_k, \vee_0^s f_{n_s}^{-i} P_k)$$

By construction of the sequence,

$$\bar{d}(\vee_0^s f_k^{-i} P_k, \vee_0^s f_{n_s}^{-i} P_k) \leq \frac{1}{2^k}.$$

By definition of n_s,

$$\bar{d}(\vee_0^s f^{-i} P_k, \vee_0^s f_{n_s}^{-i} P_k) \leq \frac{1}{2^s}$$

Thus,

$$\bar{d}(\vee_0^s f^{-i} P_k, \vee_0^s f_k^{-i} P_k) \leq \frac{1}{2^s} + \frac{1}{2^k}$$

Since $\lim_{s\to\infty} \bar{d}(\vee_0^s f^{-i} P_k, \vee_0^s f_k^{-i} P_k) = \bar{d}((f, P_k), (f_k, P_k)$ we have the lemma. Once we have the lemma the result follows.

$\square$ Thus we have constructed a Bernoulli toral linked twist map with positive Lyapunov exponents defined only on a set of measure zero. We now prove the density of such maps in a certain class of piecewise linear Bernoulli toral linked twist maps.

Proof Of Theorem 1

To prove Theorem 1 we need little modification to the proof of the existence of a Bernoulli tltm possessing positive Lyapunov exponents only on a set of measure zero.

Let g be a Bernoulli piecewise linear tltm satisfying the conditions of Theorem 1. This map is finitely determined. Let $\epsilon > 0$. Using the same method given above we construct a sequence of maps f_i , with $f_0 = g$, converging to a map f_g. The maps in the sequence may be taken to satisfy (by our previous argument) $\bar{d}((f_i, P_i), (f_{i+1}, P_i)) < \frac{\epsilon}{2^{i+1}}$ where we choose the sequence of partitions P_i and maps f_i so that $P_i < P_{i+1}$ and $\vee P_i = \mathcal{B}$. This sequence converges to a Bernoulli toral linked twist map f_g with positive Lyapunov exponents defined only on a set of measure zero and such that

$\bar{d}((f_g, P_i), (g, P_i)) < \epsilon$ (for all i)

and hence

$\bar{d}(f_g, g) < \epsilon$

This proves theorem 1.

$\square$

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