Symmetries of the asymptotic dynamics of random compositions of equivariant maps.

Matthew Nicol
Department of Mathematics
University of Warwick
Coventry, CV4 7AL, UK *

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Abstract

Let Γ be a compact Lie group and let Γ^0 denote the connected component the identity of Γ. Suppose f is a map equivariant with respect to Γ. We consider perturbations of f which are modelled by random compositions of equivariant maps which are close to f pointwise. We show that under mild assumptions on the distribution governing the choice of maps any invariant measure for the resulting Markov process is Γ^0 invariant and absolutely continuous with respect to Lebesgue measure. Thus observations on the asymptotic dynamics of the perturbed system will have Γ^0 symmetry.

*Current address: Department of Mathematics, New Mexico State University, Las Cruces, NM, 88003

1 Introduction

Let $\Gamma \subset \mathbf{R}^n$ be a compact Lie group acting on $\mathbf{R}^n$ and let Γ^0 denote the connected component of the identity in Γ. Suppose $f : \mathbf{R}^n \to \mathbf{R}^n$ is a Γ-equivariant map. If A is a subset of $\mathbf{R}^n$ then the *symmetry group of A* is the group

$$\Sigma(A) = \{\gamma \in \Gamma : \gamma A = A\}$$

An important question for applications is what are the possible symmetries of the asymptotic dynamics of f. In other words, what subgroups of Γ can be realised as the symmetry groups of attractors (or more generally, ω-limit sets with varying degrees of structure) for the dynamics induced by f. In the case where Γ is finite there is a fairly good understanding of this problem, see ([2], [7], [12]).

The case where Γ is not finite is not as well understood. Recent work on the symmetries of maps commuting with a continuous compact Lie group Γ has shown that if an ω limit set is "sufficiently chaotic", and in particular not a relative periodic orbit, then one "expects" the ω limit set to possess at least as much symmetry as Γ^0- see ([11], [13], [1], [5]). The situation in the case where the ω limit set is a relative periodic orbit is well-understood by the work of Field [6] and Krupa [10], who show that generically the ω limit set has the symmetry of a maximal subgroup of the form $T^k \times Z_q$. It has often been suggested that the presence of small random fluctuations in systems with Γ symmetry should induce a drift along the Γ^0 orbit and thus experimental observations on physical systems should exhibit this symmetry. In this note we address this issue and give precise information on the behavior of systems with Γ symmetry subject to small random perturbations. In systems with symmetry it is appropriate to suppose that perturbations preserve some symmetry. This simplifies the analysis and hopefully provides a good model of the physical system subject to noise. Hence we perturb an equivariant map f with equivariant noise, that is a sequence of independent, identically distributed (i.i.d) random variables from some measure space taking values in the space of equivariant maps which are close to f pointwise. If one considers experiments in the presence of such noise then the resulting dynamical system may be modelled as a composition of i.i.d equivariant maps. The asymptotic behavior of such compositions should reflect experimental observations.

We discuss the asymptotic dynamics of maps commuting with Γ under a class of such random perturbations. The perturbations of an equivariant map f which we consider are given by random compositions of equivariant maps which are close pointwise to f. The resulting dynamical system is a discrete time Markov process. In this context the pertinent question is what are the possible symmetries of invariant measures for this Markov process. Leaving technicalities aside, the symmetries of such measures not only determine the symmetries of the asymptotic dynamics but also the symmetries of the time averages of observations on the system. We show that under mild assumptions on the transition probability function $P(x, A)$ (where $P(x, A)$ is the probability that the next map in the sequence will move the point x into the set A) then invariant measures for the resulting stochastic process are Γ^0 invariant. In addition the invariant measures are absolutely continuous with respect to Lebesgue measure and one obtains an exponentially rapid drift along the Γ^0 orbit. As a consequence, time averages of observations on the system will be Γ^0 invariant.

To state our results in more detail we need to describe the "haar lift" of a measure on $\mathbf{R}^n/\Gamma^0$ [14]. Let $\Pi : \mathbf{R}^n \to \mathbf{R}^n/\Gamma^0$ denote the projection onto orbit space and suppose that for all $x \in \mathbf{R}^n/\{0\}$ we have $h(\Gamma_0^x) = 0$, where $\Gamma_0^x = \{\gamma \in \Gamma^0 : \gamma x = x\}$ and h is haar measure on Γ^0. This last condition is a purely technical condition and is satisfied by most systems of interest. Alternatively we may require that Γ act faithfully and orthogonally and exclude from consideration those points x fixed by Γ_0. Suppose that $\tilde{m}$ is a nonatomic measure on $\mathbf{R}^n/\Gamma^0$. By the *haar lift* of $\tilde{m}$ we mean the measure m on $\mathbf{R}^n$ defined by

$$m(A) = \int_{\Pi A} [\int_{\Pi^{-1}\tilde{z}} I_A(\eta z) dh(\eta)] d\tilde{m}(\tilde{z})$$

where I_A is the indicator function of the set $A \subset \mathbf{R}^n$. If μ is a probability measure on $\mathbf{R}^n$ then the induced probability measure on $\mathbf{R}^n/\Gamma^0$,which we denote $\Pi\mu$, is defined by

$$\Pi\mu(\tilde{B}) = \mu(\Pi^{-1}(\tilde{B}))$$

for all sets $\tilde{B} \subset \mathbf{R}^n/\Gamma^0$.

The assumptions that we make on $P(x, A)$ are

I) $P(x, A)$ is a continuous function of x for fixed A,

II) $P(x, .)$ is absolutely continuous with respect to Lebesgue measure for each x.

III) $P(x, .)$ is absolutely continuous with respect to the haar lift of $\Pi P(x, .)$ for each $x \in \mathbf{R}^n$.

Condition III states (intuitively) that $P(x, .)$ is absolutely continuous with respect to a "product" transition probability function which determines where $f(x)$ will lie under perturbation by first choosing a map $\tilde{g}$ on orbit space from an induced distribution $\Pi\nu$ on the space of maps on orbit space and then independently choosing where $\tilde{g}(x)$ will land on the Γ^0 fiber. Conditions (I), (II) and (III) seem mild requirements to place on $P(., .)$.

The distribution ν on the space of equivariant maps induces a transition probability function $P(., .)$ by $P(x, A) = \nu\{g : g(x) \in A\}$. A measure μ on $\mathbf{R}^n$ is called invariant for this process if

$$\mu(A) = \int P(x, A) d\mu(x).$$

for every Borel set A of $\mathbf{R}^n$.

The transition probability function with invariant measure μ induces a measure P_μ on the space of possible paths for the perturbed system, $(\mathbf{R}^n)^{Z+}$. We write a path as $\omega = (\omega_0, \omega_1, ..., \omega_n, ...)$. First P_μ is defined on cylinder sets by

$$P_\mu\{\omega : \omega_0 \in A_0, \omega_1 \in A_1, ..., \omega_n \in A_n\}$$

$$= \int_{A_0} \mu(d\omega_0) \int_{A_1} P(\omega_0, d\omega_1) \ldots \int_{A_{n-1}} P(\omega_{n-2}, d\omega_{n-1}) P(\omega_{n-1}, A_n).$$

This extends to a measure on the space of paths $(\mathbf{R}^n)^{Z+}$. The shift map $\sigma : (\mathbf{R}^n)^{Z+} \to (\mathbf{R}^n)^{Z+}$ given by $(\sigma\omega)_j = \omega_{j+1}$ preserves this measure. Thus σ is a measure-preserving transformation of the space of paths $(\mathbf{R}^n)^{Z+}$, defining an abstract dynamical system with invariant measure P_μ. We say that μ is ergodic if σ is ergodic with respect to P_μ (or equivalently if the only sets A for which $P(x, A) = 1$ if $x \in A$, $P(x, A) = 0$ if $x \in A^c$ have μ measure 1 or 0).

Under the assumptions on $P(.,.)$ given above we show that if μ is an ergodic invariant measure then μ is Γ^0 invariant. The ergodic decomposition theorem then implies that any invariant measure is Γ^0 invariant. This has the following consequences : Suppose we have an observable for the system, that is a map $\phi : \mathbf{R}^n \to \mathbf{R}$. Then the time average of ϕ along a path ω is

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \phi(\omega_j) \tag{1.1}$$

Because of the Γ^0 invariance of the ergodic components of μ given $\gamma \in \Gamma^0$ for μ a.e. $x \in \mathbf{R}^n$ the time average of the observable along P_μ a.e. path beginning at x will be the same as the time average of the observable along P_μ a.e. path beginning at γx. This means that time averages of observations will be symmetric in the sense that if $\gamma \in \Gamma^0$ then the time average of an observation at the point γx will be independent of γ.

These findings may be phrased as a result about the stochastic stability of systems with Γ^0 symmetry ([19], [10]). Roughly speaking, a dynamical system is stochastically stable if, when the system is randomly perturbed, then the asymptotic distribution of points is close to that of the original (unperturbed) system. We show that the only systems with Γ^0 symmetry which are stable under the perturbations described above are those preserving a measure which is Γ^0 invariant.

A partial response to similar questions to these has been given in [1] using the notion of chain-recurrence introduced by Ruelle [18]. As a consequence of Ruelle's definition of attractor it may be shown [18] that if a probability measure is to have a vague limit (which will be an invariant measure) under random perturbations then the support of the vague limit must contain its' group orbit under Γ^0. This result applies even when one considers non-equivariant perturbations- a more realistic physical model than the one considered in this paper. The problem of obtaining more precise information on the structure of the vague limit under non-equivariant noise is however much more difficult and the presence of symmetry in the form of equivariant noise makes the system tractable enough to allow more precise information to be given on the structure of any such vague limit. In fact, under the assumption of equivariant noise we show that we expect any invariant dynamics under our perturbations to possess full Γ^0 symmetry- not just that the limit will do so. The model obtained under the assumption of equivariant noise ought, moreover, be reasonably assumed to be a good physical model.

In Section 2 we consider random cocycle perturbations (which preserve the dynamics of f on the orbit space) and in Section 3 the more general question of random compositions of equivariant maps. Random cocycle perturbations are of course a special case of the latter yet the ideas involved are clearest in the case of random cocycle perturbations and our proofs build upon this special case. The arguments of Section 3 are a modification of those of Section 2. Our results are stated precisely in Theorem 2.1 and Theorem 3.2.

2 random cocycle perturbations

Let $\Gamma \subset \mathbf{O}(n)$ be a compact Lie group and let Γ^0 denote the connected component of the identity of Γ. Let h denote haar measure on Γ^0. We suppose that $h(\Gamma_0^x) = 0$ for all $x \in \mathbf{R}^n/\{0\}$. This is so we may define the haar lift of nonatomic measures.

Much of the work on the symmetries of the asymptotic dynamics of deterministic equivariant maps is based on an analysis of the properties of cocycle extensions of equivariant maps ([1], [3], [5], [8], [11], [13]). A Γ-cocycle is a continuous map $\eta : X \to \Gamma$ which satisfies the equivariance condition

$$\eta(\gamma x) = \gamma\eta(x)\gamma^{-1}$$

for all $\gamma \in \Gamma$.

One perturbs a given equivariant map $f : X \to X$ by a cocycle $\eta : \mathbf{R}^n \to \Gamma^0$ to form the perturbed equivariant map $\tilde{f}(x) = \eta(x)f(x)$. The map $\tilde{f}$ is called the extension of f by the cocycle η. Note that any extension of f induces the same dynamics as f on the orbit space $\mathbf{R}^n/\Gamma$. Perturbations are restricted to cocycles with values in Γ^0 since it is desired to consider the effects of small perturbations. The iterates of $\tilde{f}$ have the form

$$\tilde{f}^k = \eta(f^j(x))...\eta(x)f^j(x).$$

and thus a "drift' is obtained along the group orbit of the trajectory of f. It is sometimes possible to show (perhaps with certain mixing conditions imposed on f) that most choices (in a given topology) of cocycle ensure that ω-limit sets have Γ^0 symmetry. It is natural to ask what happens if one chooses group elements randomly and in this section we address this question.

Suppose that $f : \mathbf{R}^n \to \mathbf{R}^n$ is a continuous Γ-equivariant map which preserves an ergodic measure μ concentrated on a set S which is the support of μ. Let I_A denote the indicator function of a set A. We say that x_0 is generic with respect to μ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} I_A(f^j(x_0)) = \mu(A).$$

for every set A such that $\mu(A) > 0$. Thus the orbit of x_0 lies in A a fraction $\mu(A)$ of the time. It is well known that μ a.e. point is generic.

Choose a right and left invariant metric d on Γ. Let $\Pi : \mathbf{R}^n \to \mathbf{R}^n/\Gamma^0$ denote the projection onto orbit space and if $B \subset \Gamma^0, A \subset \mathbf{R}^n$ let BA denote the set $\cup_{x\in A, g\in B}\{gx\}$ (a subset of $\mathbf{R}^n$). We will sometimes denote the projection of a set ΠA by $\tilde{A}$.

We consider the effects of random cocycle perturbations of f coming from a sequence of random variables taking values in Γ^0. To be more precise, let (Ω, ν) be a measure space and let $\eta_i : \Omega \to \Gamma^0$ be a sequence of i.i.d random variables of supremum norm less than r. That is $sup_\omega|\eta_i| < r$ with respect to the metric d on Γ^0. We suppose that

(I) the range of the support of ν contains an open neighborhood of the identity element $e \in \Gamma$ and that ν is absolutely continuous with respect to Haar measure.

The distribution ν defines a transition probability function $P(.,.)$ where $P(g, B) = \nu\{\gamma \in \Gamma^0 : \gamma \in g^{-1}B\}$. Intuitively $P(g, B)$ is the probability that a random walk on Γ^0 defined by ν will lie in the set B at time $n+1$ given that the walk was at the element g at time n. Let ν^n denote the convolution product of ν with itself n times. We define $P_k(g, B) = \nu^k\{\gamma \in \Gamma^0 : \gamma \in g^{-1}B\}$. Thus $P_k(g, B)$ is the probability that a walk at time $n+k$ will lie in the set B given that at time n the walk was at the group element g. Note that $P_k(g, B) = P_k(e, g^{-1}B)$. The distribution ν induces a measure P_e on the space of paths in Γ^0 beginning at the identity element e. The space of such paths is identified with the set ${\Gamma^0}^{Z^+}$, the set of sequences $\eta = (\eta_0, \eta_1, .., \eta_n, ..)$ such that $\eta_j \in \Gamma^0$ and $\eta_0 = e$.

First P_e is defined on cylinder sets of the form

$$C = \{\eta : \eta_{i_1} \in B_{i_1}, \eta_{i_2} \in B_{i_2}, ..., \eta_{i_n} \in B_{i_n}\}\ i_1 > 0$$

by

$$P_e(C) = \int_{B_{i_1}} P_{i_1}(e, d\eta_{i_1}) \int_{B_{i_2}} P(\eta_{i_1}, d\eta_{i_2}) ... \int_{B_{i_{n-1}}} P_{i_{n-1}-i_{n-2}}(\eta_{i_{n-2}}, d\eta_{i_{n-1}}) P_{i_n - i_{n-1}}(\eta_{i_{n-1}}, B_{i_n}).$$

This measure extends to a measure on the space of all paths beginning at e. We denote the realisation of such a perturbation acting on the point x by the map f_η. Thus $f^j_\eta(x) = \eta_j f^j(x)$.

Suppose that we perturb f as described above. We have the following result:

Theorem 2.1 *Let Γ be a compact Lie group. Let f be a Γ-equivariant map preserving an ergodic measure μ with support S. Perturb f by the sequence of random cocycles governed by the distribution ν satisfying assumption (I). Then for μ a.e. x and for P_e a.e. random walk η*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} I_{B\Pi_0 A}(f^j_\eta(x)) = h(B)\mu(A).$$

for every set $A \subset S$ and $B \subset \Gamma^0$.

Remark 2.2 *(1) This means that the Γ^0 orbit of every set of positive μ measure is visited uniformly by points under the perturbed map .*

(2) This result is independent of r- the size of the perturbations.

(3) The drift along the Γ^0 orbit is at an exponential rate. By this we mean that there are constants C, α with $0 < \alpha < 1$ such that $|P_k(g, B) - h(B)| < C\alpha^k$ for all $g \in \Gamma^0$ and all $B \subset \Gamma$. The constants k and α are independent of g and B. Thus the probability that a random walk beginning at the group element g lies in the set B after k steps converges to the haar measure of B exponentially in k uniformly in g and B. This is because in light of assumption (I) the Doeblin condition [17] is satisfied. That is to say, the Markov process with state space Γ^0 defined by $P(.,.)$ with invariant measure μ is ergodic and there exists an integer $m > 0$ and an ϵ, $0 < \epsilon < 1$ such that $P_m(g, B) \leq 1 - \epsilon$ for all g if $h(B) < \epsilon$. This implies that there exist constants C, α with $0 < \alpha < 1$ such that $|P_k(g, B) - h(B)| < C\alpha^k$ for all $g \in \Gamma^0$ and all $B \subset \Gamma^0$.

Proof

We will not give the simplest proof of this result since we wish to give a proof which will generalise to the case of random compositions of equivariant maps.

Let $A \subset \mathbf{R}^n$ with $\mu(A) > 0$. Take $B \subset \Gamma^0$ with $h(B) > 0$. Let $\epsilon > 0$. We will show that to within an error of ϵ for μ a.e. x a fraction $h(B)$ of those times n_i that x visits A under f , we have also $\eta_{n_i} \in B$. Since $\mathbf{R}^n$ and Γ^0 have a countable basis and ϵ is arbitrary , this will establish the theorem. Under assumption I) on the distribution of the random cocycle perturbations it is known that

$$\lim_{n\to\infty} P_n(g, B) = h(B).$$

uniformly in $g \in \Gamma^0$ [17]. Choose δ small enough so that $0 < \delta < 10\epsilon^3$. Choose N large enough to ensure that

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$$|P_n(g, B) - h(B)| < \delta.$$

for all $g \in \Gamma^0$ and all $n \geq N$. This implies that

$$(h(B) - \delta)^2 < |P_e\{\eta : \eta_j \in B, \eta_{j+k} \in B\}| < (h(B) + \delta)^2$$

if $j, k > N$.

Now take an initial condition x_0 with respect to μ. Define the return time function $n_A(x)$ of a point to A by $f^{n_A(x)}(x) \in A$ and $n_A(x)$ is the first positive integer with this property. First we will assume that for μ a.e. x, $|n_A(x)| > N$. This will not be true in general but we will show how to get around this assumption later.

Let $\{n_i\}$ be a sequence of return times for a generic point x. Thus $n_1 < n_2 ... < n_j < ...$ with $f^{n_i}(x) \in A$ and no other integers n have the property that $f^n(x) \in A$. We have also assumed that $n_{j+1} - n_j > N$ for all $j > 1$.

Define the real valued random variables

$$Y_i = I_B(\eta_{n_i}).$$

where I_B is the indicator function of the set $B \subset \Gamma^0$. Note that each Y_i has a finite second moment. Furthermore, by ‡

$$(1 - \delta)h(B) < E(Y_i) < (1 + \delta)h(B).$$

and

$$(h(B) - \delta)^2 < E[Y_i Y_j] < (h(B) + \delta)^2.$$

Let $S_k = Y_1 + Y_2 + ... + Y_k$. The standard proof of the weak law of large numbers (see [17]) shows that

$$\lim_{k\to\infty} m\{\omega : |\frac{S_k}{k} - h(B)| > \epsilon\} < \frac{4\delta h(B) + \delta^2}{\epsilon^2} < \epsilon.$$

the last inequality by our assumption on δ. As ϵ was arbitrary this means that a fraction $h(B)$ of those times $\{n_i\}$ that x returns to A we have $\eta_{n_i} \in B$. Since ϵ was arbitrary this establishes the theorem under the assumption that return times are bounded below by N. Of course, this condition is not necessarily satisfied but we are able to get around this. The following lemma enables us to do this.

Lemma 2.3 *Given $\epsilon' > 0$ there exist disjoint subsets $\{A_i\}$ of A of equal measure so that the union of these sets covers A except for a set of measure less than ϵ'. On each of these sets A_i it is true that for μ a.e $x \in A_i$, $n_{A_i}(x) > N$.*

Remark 2.4 *To see that this lemma establishes the theorem note that for each of the sets A_i a fraction $(h(B) \pm \epsilon)$ of those times n_{i_j} that x returns to A_i we have $\eta_{n_{i_j}} \in B$. Therefore there exists a set $\tilde{A} \subset A$ such that $\mu(A/\tilde{A}) < \epsilon'$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} I_{B\Pi A'}(f_\eta^j(x)) = (h(B) \pm \epsilon)\mu(\tilde{A}).$$

As ϵ and ϵ' were arbitrary this establishes the theorem.

Proof

Define the map $f^{n_A} : A \to A$ where $f^{n_A}(x) = f^{n_A(x)}(x)$. This map is ergodic with respect to the induced measure $\mu_A(C) = \frac{\mu(A\cap C)}{\mu(A)}$ [16]. Take $\epsilon' > 0$. Since $f^{n_A} : A \to A$ is ergodic given $k > 0$ and $\epsilon' > 0$ there exists a set $A' \subset A$ such that the sets $f^{n_A}(A'), f^{n_A 2}(A), ..., f^{n_A k}(A')$ are disjoint and cover A except for a set of measure at most $\frac{\epsilon'}{2}$ [16]. Clearly we may choose k large enough to ensure $\mu_A(\cup_{j=k-n}^k (f^{n_A j}(A')) < \frac{\epsilon'}{2}$. For each set $A_i = f^{n_A i}(A')$, $0 < i < k-n$, $n_{A_i}(x) > N$ for μ_A a.e. $x \in A_i$. The union of the sets $\{A_i\}$ cover the set A except for a set of measure at most ϵ'.

This concludes the proof of the theorem.

Remark 2.5 *We may prove theorem 1 more simply but in a way which unfortunately does not readily generalise to the case of random compositions of equivariant maps. Here is a sketch: Assumption (I) (see [17]) ensures that Haar measure is the unique invariant measure of this Markov process, that is*

$$\int P(g, B) dh(\gamma) = h(B).$$

for every Borel set $B \subset \Gamma^0$.

We induce a measure on the space of possible paths in Γ^0, the set ${\Gamma^0}^Z$. Elements of ${\Gamma^0}^Z$ are denoted $\omega = (..., \omega_j, ...\omega_{j+i}, ...)$ where for each j, $\omega_j \in \Gamma^0$. Each ω represents *a path in Γ^0 such that the path is at the group element ω_j at time j. This measure on paths is induced by using the transition probability function and Haar measure to first define a measure on cylinder sets in the following way: Let $B_{i_1}, B_{i_2}, ..., B_{i_k}$ be sets in Γ^0 and define*

$$P_h\{\omega : \omega_{i_1} \in B_{i_1}, ..., \omega_{i_k} \in B_{i_k}\}$$

$$= \int_{B_{i_1}} h(d\omega_{i_1}) \int_{B_{i_2}} P_{i_2-i_1}(\omega_{i_1}, d\omega_{i_2}) ... \int_{B_{i_{k-1}}} P_{i_{k-1}-i_{k-2}}(\omega_{i_{k-2}}, d\omega_{i_{k-1}}) P(\omega_{i_{k-1}}, B_{i_k}).$$

This extends to a measure (which we still denote P_h) on ${\Gamma^0}^Z$, the space of possible realisations of the Markov process. The shift map $(\sigma\omega)_j = (\omega_{j+1})$ preserves P_h . Under the assumption that the support of ν generates Γ^0 the Markov process we have defined is mixing (indeed Bernoulli) [17], [15]. This is equivalent to the fact that $\sigma : {\Gamma^0}^Z \to {\Gamma^0}^Z$ is mixing with respect to P_h . We now take the direct product of $\mathbf{R}^n$ and ${\Gamma^0}^Z$ and consider the map $F(x, \omega) = (f(x), S\omega)$. This map preserves the product measure $\mu \times h$. Since $\sigma : {\Gamma^0}^Z \to {\Gamma^0}^Z$ is mixing and $f : S \to S$ is ergodic the resulting dynamical system is ergodic with respect to the product measure. We define projection functions $\Pi_\mu(x, \omega) = x$ and $\Pi_\omega(x, \omega) = \omega_0$ onto the first and second components respectively. The paths of the perturbed system are given by the trajectories of the points (x, ω) such that $\omega_0 = e$ i.e. the sequences $\{\Pi_\omega F^n(x, \omega) \Pi_\mu F^n(x, \omega)\}$ such that $\omega_0 = e$.

Since the dynamical system is ergodic $\mu \times h$ a.e. point (x, ω) is generic in the sense that, given a Borel set of the form $A \times B$ where $\mu(A) > 0$ and $h(B) > 0$,

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} I_{A\times B}(F^j(x, \omega)) = \mu(A)h(B).$$

Recall that we have defined a measure P_e on the space of paths beginning at the identity element e. We then show that in fact given a μ generic $x \in A$, (x, ω_e) is generic for P_e a.e. random walk ω_e (where the subscript denotes that the random walk begins at e) and this also proves the theorem.

3 random compositions of equivariant maps

In this section we examine the more general question of the asymptotic behavior of points under random compositions of equivariant maps which are close pointwise to a continuous equivariant map f. Let T be a compact set such that $f(T) \subset T$ and let ν be a distribution on the space of continuous equivariant maps such that $\sup_{x\in T} |f(x) - g(x)| < \epsilon$. We model the cumulative effect of small random errors in the following way: suppose we take a point $x \in T$ and instead of applying f we apply an equivariant map g (coming from the distribution ν) to obtain the point $g(x)$. We repeat this process to obtain an asymptotic distribution μ_ϵ. One of our results says that independently of ϵ this asymptotic distribution is invariant under Γ^0 and so under perturbations of this kind we expect a uniform drift along the Γ^0 orbit. This result is made precise in Theorem 3.2. These results give an indication of what should be observed over long time periods in systems with compact Lie group symmetry subject to noise. Roughly speaking, in this section we partition the space of sequences of equivariant maps into sets which induce the same sequences on the orbit space $\mathbf{R}^n/\Gamma^0$ and then show that the choice of a sequence (from a set inducing the same sequence on orbit space) is "equivalent " to a random choice of cocycles. This is where we use our third assumption on the transition probability function. Using results from section 2 we then show that most of these choices have the "right" time averages and then integrate over the set of sequences on orbit space to show that most paths in $\mathbf{R}^n$ have time averages which are Γ^0 invariant. We then show that this implies that invariant measures are Γ^0 invariant.

Let f , T be as above and define $\Omega_\epsilon = \{g \in C^0_\Gamma : \sup_{x\in T} |f(x)-g(x)| < \epsilon \text{ for all } x \in T\}$. Let ν (we suppress the ϵ) be a probability measure on Ω_ϵ. We denote the projection of a map g onto orbit space by $\tilde{g} = \Pi g$ and similarly for sets of maps.

Note that ν induces a measure $\Pi\nu$ on maps on orbit space $\mathbf{R}^n/\Gamma^0$ by $\Pi\nu(\tilde{H}) = \nu(\Pi^{-1}\tilde{H})$ for every Borel set $\tilde{H}$ of maps on orbit space.

We define a transition probability function by $P(x, A) = \nu\{g : gx \in A\}$. We also define a transition probability function on orbit space by $P(\tilde{x}, \tilde{A}) = \nu\{g : \Pi g(z) \in \tilde{A}\}$ where $z \in \Pi^{-1}x$. This definition makes sense since the equivariance of g means that it doesn't matter which $z \in \Pi^{-1}x$ we choose. This is where the assumption that the maps are equivariant is crucial. Similarly given a point $\tilde{x}$ in orbit space and a set $A \in \mathbf{R}^n$ we define $h\{\eta : \eta\tilde{x} \in A\}$ to be the Haar measure of the set of η such that $\eta z \in A$ where $z \in \Pi^{-1}(x)$. This is also clearly well defined. We use the notation $m_1 << m_2$ to indicate that the measure m_1 is absolutely continuous with respect to m_2. It is easy to see that if μ is invariant for $P(x, A)$ then $\Pi\mu$ is invariant for $P(\tilde{x}, \tilde{A})$. We assume that

I) $P(., A)$ is continuous in x for every Borel set A.

II) $P(x, .)$ is absolutely continuous with respect to Lebesgue measure and

III) $P(x, .)$ is absolutely continuous with respect to the haar lift of $P(x, .)$ for each x.

Note that condition III) is equivalent to

$$P(x, A) << \int_{\tilde{A}} h\{\eta : \eta d\tilde{x} \in A\} P(\tilde{x}, d\tilde{x})$$

for every point $x \in \mathbf{R}^n$.

Remark 3.1 *Recall that condition (III) assumes $P(.,.)$ is absolutely continuous with respect to a "product" measure transition probability function. Intuitively according to the "product" transition probability function to determine where $f(x)$ will lie under perturbation one first chooses a map $\tilde{g}$ on orbit space from the distribution $\Pi\nu$ and then independently chooses where $\pi^{-1}\tilde{g}(x)$ it will land on the Γ^0 fiber.*

With this set-up we are able to state our second theorem:

Theorem 3.2 *Suppose that we perturb a map f as above with the distribution function $P(.,.)$ satisfying conditions (I), (II) and (III). Then every invariant measure is Γ^0 invariant and absolutely continuous with respect to Lebesgue measure.*

Remark 3.3 *1) This implies (see the discussion following 1.1) that time averages of observations on the system will be Γ^0 symmetric. In fact almost every path has a time average on sets of the from $\Pi A \times \Gamma^0$ which converges to $\Pi\mu(A)h(B)$. Thus any ergodic invariant measure μ is of the form $\Pi\mu \times h$.*

2) As a result of Doeblins theorem the drift along the Γ^0 orbit is exponentially fast.

Proof

Assumption (I) implies that there exists an invariant measure for this process [19]. The ergodic decomposition theorem insures that there are ergodic invariant measures. By considering the definition of an invariant measure for a transition probability function it is clear that assumption (II) implies that all invariant measures are absolutely continuous with respect to Lebesgue measure . In light of the ergodic decomposition theorem, without loss of generality we may assume that we are dealing with an ergodic invariant measure absolutely continuous with respect to Lebesgue measure. Now suppose that μ is an ergodic invariant measure for this process. Then $\Pi\mu$ is an ergodic invariant measure for the induced Markov process on the orbit space $\mathbf{R}^n/\Gamma^0$ with transition probability function $P(\tilde{x}, \tilde{A})$. The transition probability function with invariant measure μ induces a measure P_p on the space of paths in $\mathbf{R}^n$ in the usual way. We write the space of paths as $(\mathbf{R}^n)^{Z^+}$, the elements of which are $\omega = (\omega_0, \omega_1, .., \omega_n, ..)$. The shift map $(\sigma\omega)_j = \omega_{j+1}$ preserves the measure P_p. Each path $\omega \in \mathbf{R}^{n Z^+}$ defines an induced path in orbit space, which we denote by $\tilde{\omega}$. There is an induced measure on the space of paths $\mathbf{R}^n/{\Gamma^0}^{Z^+}$ which we denote by ΠP_p which is equal to the measure induced by the transition probability function $P(\tilde{x}, \tilde{A})$ with invariant measure $\Pi\mu$.

For each

$$\tilde{\omega} = (\tilde{\omega_0}, \tilde{\omega_1}, ..., \tilde{\omega_n}, ...) \in \mathbf{R}^n/{\Gamma^0}^{Z^+}$$

we may define a measure $m_{\tilde{\omega}}$ on the space of paths in $(\mathbf{R}^n)^{Z^+}$. We first do this on cylinder sets ,

$$m_{\tilde{\omega}}(\omega_{i_1} \in A_{i_1}, ..., \omega_{i_n} \in A_{i_n}) = h\{\eta : \eta\tilde{\omega}_{i_1} \in A_{i_1}\}...h\{\eta : \eta\tilde{\omega}_{i_n} \in A_{i_n}\}$$

By the Kolmogorov extension theorem [17] this extends to a measure on $(\mathbf{R}^n)^{Z^+}$. Note that ΠP_p a.e. path in orbit space visits the set ΠA with frequency $P_p(A)$. We call such paths generic. Furthermore a slight modification of the proof given in section 2 shows that for ΠP_p a.e. path $\tilde{\omega}$, $m_{\tilde{\omega}}$ a.e. path ω has a time average on sets of the form $\Pi A \times B$ which converges to $\Pi\mu(A)h(B)$. To see this note that the essential ingredient in the proof of this result was the fact that ergodicity allowed us to reduce to the case where return times to a set were bounded below. In the same way the ergodicity of the induced transformation on orbit space allows us to reduce to the case where return times to a set are bounded below. The measure defined by a generic point path ω is equivalent to that defined by a random choice of cocycles coming from a distribution satisfying condition I) and the proof proceeds in exactly the same way.

We now wish to combine these observations to show that P_p a.e. path has a time average on sets of the form $\Pi A \times B$ which converges to $\Pi\mu(A)h(B)$. To see this we make the following observation: let $A_0 \times A_1 \times ... \times A_n$ be a cylinder set in ${\mathbf{R}^n}^{Z^+}$. Then,

$$P_p\{\omega : \omega_0 \in A_0, \omega_1 \in A_1, ..., \omega_n \in A_n\}$$

$$= \int_{A_0} d\mu(d\omega_0) \int_{A_1} P(\omega_0, d\omega_1) ... \int_{A_{n-1}} P(\omega_{n-2}, d\omega_{n-1}) P(\omega_{n-1}, A_n)$$

$$<< \int_{\tilde{A}_n} \int_{\tilde{A}_{n-2}} ... \int_{\tilde{A}_0} h\{\eta : \eta d\tilde{\omega_0} \in A_0\}...h\{\omega : \eta d\tilde{\omega_n} \in A_n\} P(\tilde{\omega_0}, d\tilde{\omega_1})...P(\tilde{\omega_{n-1}}, \tilde{A_n})$$

$$\equiv \kappa\{\omega : \omega_0 \in A_0, \omega_1 \in A_1, ..., \omega_n \in A_n\}$$

The last expression is defined on all cylinder sets and by the Kolmogorov extension theorem converges to a measure κ on ${\mathbf{R}^n}^Z$ which we may think of as

$$\kappa(C) = \int \tilde{w}(C) d\Pi P_p(\tilde{w})$$

This measure has the property that if for ΠP_p a.e. $\tilde{\omega}$ we have $\mu_{\tilde{\omega}}(C) = 0$ then $\kappa(C) = 0$. The calculation given above shows that P_p is absolutely continuous with respect to κ. Since for ΠP_p a.e. path $\tilde{\omega}$, $m_{\tilde{\omega}}$ a.e. path ω has a time average on sets of the form $\Pi A \times B$ which converges to $\Pi\mu(A)h(B)$ we see that this must be true also for P_p a.e. path. This completes the proof of the theorem .

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