MEASURABLE COCYCLE RIGIDITY FOR SOME NON-COMPACT GROUPS

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Abstract. In this paper we consider extensions of Livsic's well-known classical results on rigidity of measurable cocycles. We consider transformations which need not be hyperbolic and cocycles which take values in certain non-abelian non-compact groups.

0. Introduction

Let $T : X \to X$ be a continuous map on a compact metric space and let $g : X \to G$ be a continuous map taking values in a connected Lie group G. We shall assume that μ is a T-invariant probability measure on X which is ergodic (i.e. for any Borel set C with $T^{-1}C = C$ either $\mu(C) = 0$ or $\mu(X - C) = 0$). In this paper we are interested in the regularity of functions $h : X \to G$ (with $h \in L^\infty(X, G)$) satisfying the cocycle equation

$$h(Tx) = g(x)h(x) \quad \text{a.e.} \tag{0.1}$$

In this note we shall be interested in when a measurable solution $h(x)$ to (0.1) must necessarily have a continuous version.

There are well-known classical results in this direction due to Livsic [6], where $g(x)$ is close to the identity and G is compact. There are also more recent results due to Parry and Pollicott [11], Guivarc'h and Hardy [4] and Walters [13] in the case that G is a compact connected Lie group. However, these results all require very strong hyperbolicity assumptions on $T : X \to X$.

We begin by observing that the original Livsic argument is valid for a measurable cocycle $h(x)$ taking values in any subgroup of $GL(n, \mathbb{C})$ provided that $T : X \to X$ is Anosov, the coboundary $g(x)$ is Hölder continuous and $h(x)$ is essentially bounded (Section 1).

The main part of the paper deals with a very general class of continuous endomorphism $T : X \to X$ and some particular examples of non-compact groups G. We introduce a property for G (Property (*)) which is useful in our subsequent analysis. This property holds, for example, in the case of $\mathbb{R}^n$ and $E(2)$ (cf. Section 2). In Section 3, we give our main result. We show that under the hypothesis that

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$T : X \to X$ is either a) a continuous and locally eventually onto endomorphism or b) a minimal transformation and that G satisfies Property (*) then provided g is continuous any essentially bounded solution $h(x)$ to (0.1) must have a continuous version. Our results apply to $E(2)$ and commutative groups.

In Section 4, we discuss some applications of these ideas to local solutions of (0.1) in more general groups of non-positive sectional curvature.

1. Hyperbolic maps and Hölder cocycles

In this section we shall assume that the map $T : X \to X$ is an Anosov diffeomorphism of a compact manifold X and $g : X \to G$ is Hölder continuous.

Definition. We call $T : X \to X$ an *Anosov diffeomorphism* if there is a DT-invariant splitting of the tangent space $TX = E^s \oplus E^u$ and constants $C > 0$ and $0 < \lambda < 1$ such that

$$||DT^n v|| \leq C\lambda^n ||v||, \forall n \geq 0\ \forall v \in E^s, \text{ and}$$
$$||DT^{-n} v|| \leq C\lambda^n ||v||, \forall n \geq 0\ \forall v \in E^u.$$

We shall consider only transitive Anosov diffeomorphisms (i.e those containing a dense orbit) preserving a measure μ equivalent to Lebesgue. Such a dynamical system is automatically ergodic with respect to μ.

We define the local stable manifolds and local unstable manifolds by

$$W^s(x, \epsilon) = \{y \in X : d(f^n x, f^n y) \leq \epsilon, \forall n \geq 0\}$$

and

$$W^u(x, \epsilon) = \{y \in X : d(f^{-n} x, f^{-n} y) \leq \epsilon, \forall n \geq 0\}$$

for $\epsilon > 0$ sufficiently small.

In this setting the measure μ has the important property that the induced measures on local unstable manifolds are absolutely continuous with respect to the induced Riemannian measure and the local holonomy along local stable manifolds, and similarly for the induced measures on local unstable manifolds. We have the following rigidity results for any matrix group $G \subset GL(n, \mathbb{C})$.

Theorem 1. *Assume that*

1. $T : X \to X$ *is a* C^∞ *transitive Anosov diffeomorphism and* μ *is an invariant measure equivalent to Lebesgue.*
2. $h : X \to GL(n, \mathbb{C})$ *is a measurable function such that* $h(x)$ *and* $h(x)^{-1}$ *are both essentially bounded.*
3. $g : X \to GL(n, \mathbb{C})$ *is an* α*-Hölder function such that* $g(x) = h(Tx)h(x)^{-1}$ μ *a.e.*

If $h(x)$ *is an essentially bounded measurable solution to* $g(x) = h(Tx)h(x)^{-1}$ *then* $h(x)$ *is continuous* μ *a.e.*

Nitica and Torok [9] considered the case where g is C^k ($k \geq 1$) and assumed that $h(x)$ was a continuous solution to (0.1). In particular, they showed that h must necessarily be C^{k-1}. Our proof follows the same general lines as their argument.

Proof. Assume that h and h^{-1} is bounded on a set Y with $\mu(Y) = 1$. Let $X_0 = \cap_{n\in\mathbb{Z}} T^n Y$ then $\mu(X_0) = 1$ and we can assume there exists $C > 0$ such that $\forall x \in X_0$, $\forall k \in \mathbb{Z}$ we have that $|h(T^k x)^{\pm 1}| \leq C$.

By Lusin's theorem, there is a set $L \subset X_0$ such that $\mu(L) > \frac{1}{2}$, say, and $h, h^{-1} : L \to GL(n, \mathbb{C})$ is uniformly continuous.

Assume that $x, y \in X_0$ lie on the same unstable manifold. Define x_n and y_n by $x = T^n(x_n)$ and $y = T^n(y_n)$. Denote $g(T^{n-1}x_n)g(T^{n-2}x_n)\dots g(Tx_n)g(x_n)$ by $g^n(x_n)$. By iterating identity (3) we have:

$$h(x)h(y)^{-1} = h(y_n)g^n(y_n)g^n(x_n)^{-1}h(x_n)^{-1} \tag{1.1}$$

We know that since x and y are on the same unstable manifold there exists $0 < \lambda < 1$ such that $d(x_k, y_k) \leq C\lambda^k d(x, y)$.

We first consider the expression $g^n(y_n)g^n(x_n)^{-1}$. Since g is Hölder continuous and T is C^∞ we can estimate

$$\begin{aligned} &g^n(y_n)g^n(x_n)^{-1} \\ &= I + (\sum_{k=0}^{n-1} g(T^{n-1}y_n)\dots g(T^{k+1}y_n)\left(g(T^k y_n) - g(T^k x_n)\right) g(T^{k-1}x_n)\dots g(x_n)).g^n(x_n)^{-1} \end{aligned}$$

By essential boundedness we can use (3) to write:

$$\begin{cases} g(T^{n-1}x_n)\dots g(T^{k+1}x_n) = h(x)h(x_{k+1})^{-1} \\ g(T^{k-1}x_n)\dots g(x_n) = h(T^k x_n)h(x_n)^{-1} \text{ and} \\ g^n(x_n)^{-1} = h(x_n).h(x)^{-1} \end{cases}$$

and thus the expression $g^n(y_n)g^n(x_n)^{-1}$ is bounded by

$$d(x, y)^\alpha \left(\sum_{n=0}^{\infty} |h|^3 |h^{-1}|^3 K C^\alpha \lambda^{n\alpha} \right) < +\infty.$$

where K is the Hölder constant for g. Thus for x and y close, $g^n(x_n)$ and $g^n(y_n)$ are close, independently of n.

By equation (1.1), the Theorem will follow providing we can find a subsequence $n_i \to +\infty$ such that $h(y_{n_i})h(x_{n_i})^{-1} \to I$

Let $\mathcal{G}$ be the set of points in X such that the $\lim_{N\to+\infty} \frac{1}{N}\sum_{i=0}^{N-1} \chi_L(T^i x) = \mu(L)$. By ergodicity of μ, we know $\mu(\mathcal{G}) = 1$.

Let $\{\mu_x\}$ be the conditional measures induced on local unstable manifolds. By absolute continuity $\mu_x(\mathcal{G} \cap W^u_\epsilon(x)) = \mu_x(W^u_\epsilon(x))$ (i.e. for μ almost all $x \in X$ and for (μ_x) almost all $y \in W^s_\epsilon(x)$ we have $x, y \in \mathcal{G}$). Since $\mu(L) > \frac{1}{2}$ we may choose for such points $x, y \in X$ a common sequence of times $n_i \to \infty$ such that $x_{n_i} \in L$ and $y_{n_i} \in L$.

Since h is uniformly continuous on L and $d(x_{n_i}, y_{n_i}) \to 0$ as $n_i \to \infty$ we conclude that $h(y_{n_i})h(x_{n_i})^{-1} \to I$. This shows uniform continuity of h on almost all unstable manifolds.

A similar argument with T^{-1} replacing T shows that h is uniformly continuous on almost all stable manifolds. By the local product structure this implies that h is uniformly continuous on a dense set of full measure and hence has a uniformly continuous extension to the whole of X. Hence h has a continuous version.

Problem. It is not known to us whether the conclusion to the theorem still holds if h is only assumed to be L^1.

2. A useful property on G.

Let us assume that $T : X \to X$ is a continuous map on a compact metric space and $g : X \to G$ is a continuous map into a connected Lie group G. Assume that G has a left or right invariant metric d.We let γa denote the left action of the group element $\gamma : a \to \gamma a$ and similarly $a\gamma$ denotes the right action of the element γ. We begin by introducing the following definition.

Definition. Let S denote the *essential range* of h. i.e.

$$S = \{k \in G : \text{ for any neighbourhood } V \ni k, \mu(h^{-1}(V)) > 0\}$$

Thus to say that h is essentially bounded is to say that the essential range of h is bounded. We define the diameter D of a set U by

$$D = \sup \{d(\gamma, \gamma') : \gamma, \gamma' \in U\}.$$

If the diameter of a set U is finite we say that U is bounded. If U is a bounded set then given $\gamma, \gamma' \in G$ we define

$$d_U(\gamma, \gamma') = \sup\{d(\gamma a, \gamma' a) : a \in U\}$$

Note that if G is a compact group then the essential range of any measurable function $h : X \to G$ is bounded.

We now describe a property that will enable us to describe a class of Lie groups for which strong regularity results hold for measurable cocycles taking values in these groups.

Property (*). If U is a bounded set in G, then for any $\epsilon > 0$ we can choose a, γ and $\rho > 0$ so that

(1) $a, \gamma a \in U$;
(2) $\gamma' (B(a, \epsilon)^c) \cap U = \emptyset$ whenever $d(\gamma', \gamma) < \rho$

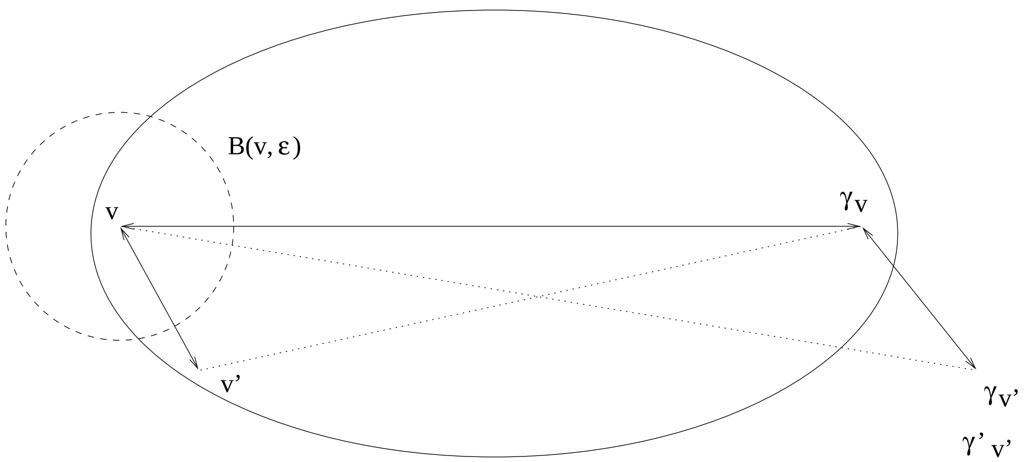

FIGURE 1: PROOF OF LEMMA 1

Lemma 1. *If G has a left (or right) invariant metric with non-positive sectional curvature then Property (*) holds.*

Proof. Because the proof is somewhat easier notationally in the case of left invariance we will assume left invariance.

Let $\rho_0 \equiv \frac{1}{10}\min\left\{1, \frac{1}{D}\right\}\epsilon^2$ and choose $\epsilon > 0$ be sufficiently small that $\epsilon - \rho_0 > 0$ and $D - \rho_0 > 0$. By the definition of diameter we may select $v, \gamma v \in U$ with $d(v, \gamma v) \geq D - \rho_0$. Choose $0 < \rho < \rho_0$ sufficiently small that $d(\gamma, \gamma') < \rho$ implies that $d_U(\gamma, \gamma') < \rho_0$ and henceforth assume $d(\gamma, \gamma') < \rho$.

Assume, to obtain a contradiction, that there exists $\gamma' v' \in U$ where $v' \in (B(v, \epsilon)^c) \cap U$ and $d(\gamma, \gamma') < \rho$. We claim that providing $\epsilon > 0$ is sufficiently small this implies that either $d(v, \gamma' v') > D$ or $d(\gamma v, v') > D$, either of which will be a contradiction to the assumption that the essential diameter of U is D.

Observe that

(1) $d(v, v') \geq \epsilon$ (by assumption),
(2) $d(v, \gamma v) \geq D - \rho_0$ (by assumption),
(3) $d(\gamma v, \gamma v') = d(v, v')$ (by left-invariance of the metric),

We claim that the angle $\theta = \langle v' v(\gamma v) \leq \pi/2$. Assume for a contradiction that $\theta \geq \frac{\pi}{2}$. For the purposes of comparison, it is convenient to consider the euclidean triangle with adjacent sides having lengths $d(v, v')$ and $d(v, \gamma v)$ and an angle $\theta \geq \pi/2$. By the cosine rule the opposite side in this euclidean triangle has a length whose square is at least

$$\begin{aligned}\epsilon^2 + (D - \rho_0)^2 &= D^2 + \epsilon^2 - 2D\rho_0 + \rho_0^2 \\ &\geq D^2 + \epsilon^2 - \frac{1}{5}\epsilon^2 + \frac{1}{100D^2}\epsilon^4\end{aligned}$$

In particular, in the last expression the term in ϵ^2 has coefficient $\frac{4}{5} > 0$. Thus there exists $\epsilon_0 = \epsilon_0(D)$ such that the opposite side of the eucliden triangle has length greater than D if $0 < \epsilon < \epsilon_0$. However, if we consider a triangle in a space with non-positive curvature then the euclidean estimate gives a lower bound on the length of the opposite side in the general case [5, Cor. 13.2 (i)] which is greater than D. This contradiction establishes that $\theta \leq \frac{\pi}{2}$.

Since the left action γ is an isometry, we see that the angle $\langle v(\gamma v)(\gamma v') = \pi - \theta \geq \pi/2$. We can now consider the euclidean triangle with sides of length $d(v, \gamma v)$ and $d(\gamma v, \gamma v')$ and the angle $\pi - \theta \geq \pi/2$. By the cosine formula, we again have that the opposite side in the euclidean triangle has side at least

$$\epsilon^2 + (D - \rho_0)^2 = D^2 + \frac{4}{5}\epsilon^2 + \frac{1}{100D^2}\epsilon^4.$$

Thus, by comparison, we have a lower bound on the hyperbolic distance by

$$d(v, \gamma v') \geq \sqrt{D^2 + \frac{4}{5}\epsilon^2 + \frac{1}{100D^2}\epsilon^4} = D^2 + \frac{4}{10}\epsilon^2 + O(\epsilon^4)$$

[5, Cor. 13.2 (i)] and, in particular, $d(v, \gamma' v') \geq d(v, \gamma v') - d(\gamma v', \gamma' v') \geq D^2 + (\frac{4}{10} - \frac{1}{10})\epsilon^2 + O(\epsilon^4)$. Thus there exists $\epsilon_1 = \epsilon_1(D)$ such that if $0 < \epsilon < \epsilon_1$ then $d(v, \gamma' v') > D$.

Examples. As a particular case, Lemma 1 applies to Lie groups for which the left invariant metric is flat (i.e. all curvatures equal to zero). All commutative groups are flat (cf. [8, Theorem 1.5]). The simplest non-commutative example is probably $E(2)$, the group of affine transformations on the real line [8,p. 298].

More generally, if a connected Lie group has a left invariant metric with all sectional curvatures being non-positive then it must be solvable [8, Theorem 1.6]. There exist examples of soluble Lie groups with left invariant metrics with strictly negative sectional curvatures (cf. [8, Example 1.7] and [8, Theorem 4.11]).

3. Continuity of the cocycle

We begin with a few relevant definitions.

Definition. We say that a continuous map $T : X \to X$ is *locally eventually onto* if for every open set $U \subset X$ we can find $n \in \mathbb{N}$ such that $T^n(U) = X$

Simple examples of locally eventually onto systems include mixing one-sided subshifts (not necessarily of finite type) and expanding maps on manifolds (not necessarily uniformly expanding).

Definition. We say that a continuous map $T : X \to X$ is minimal if every point $x \in X$ has a dense orbit under T.

Simple examples of minimal transformations include irrational rotations of the circle.

Our main result is the following.

Theorem 2. *Assume that T is locally eventually onto or minimal and that G has a left invariant metric with all sectional curvatures non-positive then if there exists a $L^\infty(X, G)$ solution h to (0.1) there must exist a continuous solution h' such that $h = h'$ μ a.e.*

The theorem applies when G is $\mathbb{R}^n$, $E(2)$ or is commutative.

Our result was motivated by a recent note of Quas [12]. When $G = \mathbb{R}$ we recover a result from [12].

The proof of Theorem 2 comes from the lemmas in the previous section combined with the results in this section.

Lemma 2. *Let $A, B \subset X$ be Borel sets with $\mu(A) > 0$ and $\mu(B) > 0$. Assume that μ is T-invariant and ergodic then there exists $n \geq 0$ with $\mu(T^{-n}A \cap B) > 0$.*

Proof. Let $C = \cup_{n=0}^{\infty} T^{-n}A$ then $T^{-1}C = C$. By ergodicity, we see that $\mu(C) = 1$ thus $\mu(C \cap B) > 0$ and $\mu(T^{-n}A \cap B) > 0$ for some $n \geq 0$.

Remark. The above Lemma is easily seen to hold if we assume that μ is ergodic but relax the hypothesis of T-invariance to the weaker hypothesis that μ is incompressible (i.e. if $T^{-1}C \subset C$ then $\mu(C) = \mu(T^{-1}(C))$).

Lemma 3. *Assume that $T : X \to X$ is locally eventually onto or minimal. If Property (*) holds then every essentially bounded measurable solution to $h(Tx) = g(x)h(x)$ has a continuous version.*

Proof.
First assume $T : X \to X$ is locally eventually onto. The function h is only defined, a priori, on a set of full measure $X' \subset X$. By virtue of the ergodicity of the measure μ, the set $X_0' := \cap_{n=0}^{\infty} T^{-n}X'$ has full measure, and the function $h(x)$ is well defined on this set. The set $X_0 \subset X_0'$ of points which have a dense orbit in X is also of full measure.

In the definition of Property (*) we take U to be the essential range of h restricted to X_0, thus U is closed and bounded. Let $\epsilon > 0$.

With this choice of ϵ there exist a, γ and ρ satisfying

$$a, \gamma a \in U$$

and

$$\gamma'(B(a, \epsilon)^c) \cap U = \emptyset$$

whenever $d(\gamma', \gamma) < \rho$.

With this choice of $\rho > 0$ there exists $0 < \epsilon' < \frac{\rho}{2}$ such that if $\gamma_1, \gamma_2 \in U$ and $d(\gamma_1, \gamma_2) < \epsilon'$ then $d(\gamma_3\gamma_1\gamma_2^{-1}, \gamma_3) < \frac{\rho}{2}$ for all $\gamma_1, \gamma_2, \gamma_3 \in U$.

Given $\epsilon' > 0$ we can consider the sets

$$A = \{x \in X_0 : d(h(x), a) < \epsilon'\} \text{ and } B = \{x \in X_0 : d(h(x), \gamma a) < \epsilon'\}.$$

Since a and γa are in U we know that $\mu(A) > 0$ and $\mu(B) > 0$. By ergodicity, we can choose $x_0 \in A$ and $n > 0$ such that $T^n(x_0) \in B$.

By the cocycle identity we have that $h(T^n x_0) = g^n(x_0)h(x_0)$ where $g^n(x_0) = g(T^{n-1}x_0) \dots g(Tx_0)g(x_0)$.

Now $d(g^n(x_0)h(x_0), \gamma a) < \epsilon'$ hence $d(g^n(x_0)h(x_0)a^{-1}, \gamma) < \epsilon'$ (by left invariance) and so $d(g^n(x_0), \gamma) < \frac{\rho}{2}$ (by choice of ϵ'). By continuity, there exists $\eta > 0$ such that $d(x_0, x) < \eta$ implies $d(g^n(x_0), g^n(x)) < \frac{\rho}{2}$. Thus $d(\gamma, g^n(x)) < \rho$. We claim that $d(h(x_0), h(x))$ is less than ϵ. Otherwise, if $d(h(x_0), h(x)) > \epsilon$ then we can apply Property (*) with $\gamma' = g^n(x)$ to conclude $g^n(x)h(x) = h(T^n x) \notin U$. This would give a contradiction and so the claim is proved.

Recall that all points in the full measure, dense set X_0 have dense orbits. Given $y \in X_0$ we now produce an open ball $B(y, r(y))$ such that $d(h(y'), h(y'')) < 2\epsilon$ if $y', y'' \in B(y, r(y)) \cap X_0$. Since y has a dense orbit there exists $m > 0$ such that $d(x_0, T^m y) < \eta$. By continuity of g^m and T^m there is a neighbourhood N of y such that $y', y'' \in N$ implies $d(g^m(y), g^m(y')) < \epsilon$, $d(x_0, T^m(y')) < \eta$ and $d(x_0, T^m(y'')) < \eta$.

Now fix $y', y'' \in N \cap X_0$, then $h(T^m y') = g^m(y')h(y')$ and $h(T^m y'') = g^m(y'')h(y'')$ hence $d(h(y'), h(y'')) < 2\epsilon$.

Given any $\epsilon > 0$ we can choose a point $y \in X_0$ and a neighbourhood $N(y, \epsilon)$ as above and choose $n \geq 0$ such that $T^n N(y, \epsilon) = X$. Given $x_1, x_2 \in X_0$ we choose $z_1, z_2 \in N(y, \epsilon)$ with $T^n z_1 = x_1$ and $T^n z_2 = x_2$. We know that $d(h(z_1), h(z_2)) < \epsilon$ and $h(x_1) = g^n(z_1)h(z_1)$ and $h(x_2) = g^n(z_2)h(z_2)$. Providing z_1 is sufficiently close to z_2 we can use the uniform continuity of $g^n(z)$ to ensure that $d(h(x_1), h(x_2)) < 2\epsilon$. Hence h is uniformly continuous on X_0 and thus has a continuous extension to X.

Now assume that $T : X \to X$ is minimal. Then we may cover X by the balls $B(y, r(y))$ (since every point y has a dense orbit) and choose a finite subcover $B(y_i, r(y_i))$ by compactness. Let ν be the Lebesgue number for this cover. Then any two points $x_1, x_2 \in X_0$ satisfy $d(h(x_1), h(x_2)) < 2\epsilon$ and hence h is uniformly continuous on X_0. Thus h has a continuous extension to X.

Theorem 2 now follows immediately from the preceding lemmas.

4. Local results for arbitrary groups

The restrictions on the type of groups we considered in the previous section were constrained by the weakness of the assumption that $h \in L^\infty(X, G)$. If we make more restrictive assumptions on the function h then we can deal with arbitrary groups. The following result encapsulates this idea.

Lemma 4. *Given any connected Lie group G there exists $\epsilon > 0$ such that if the diameter of a set U is less than ϵ then the conclusions of Property (*) hold for U.*

Proof. The proof follows the same lines as that of Lemma 3. Locally we can consider the domain of a Euclidean chart given by the exponential map from the Lie algebra to the Lie group. Locally, such charts can always be defined. The bound on the size of the domain is determined by the injectivity radius of the Lie group G (viewed as a manifold). The proof then follows as in Lemma 3.

Comparing Lemma 4 with Lemma 3 we have the following local version of Theorem 2.

Theorem 3. *Let G be any connected Lie group. There exists $\epsilon > 0$ such that for any $L^\infty(X, G)$ solution to (0.1) with diameter of the essential range $S < \epsilon$ there exists a continuous version of the solution.*

Remark. Assume that we are given a connected Lie group which does *not* satisfy Property (*) then clearly we are unable to apply Theorem 2. However, observe that if we consider the Universal Cover $\tilde{G}$ for G (viewed as a topological space) then it is easy to see that the argument for exponential groups still applies and an analogue of the theorem holds for cocycles taking values in $\tilde{G}$.

Consider the special case of $G = SL(2, \mathbb{R})$. We easily see that Property (*) fails by considering the subset $E = \{kak : k \in SO(2) \text{ and } a \in R\}$, where

$$R = \left\{ \begin{pmatrix} e^\alpha & 0 \\ 0 & e^{-\alpha} \end{pmatrix} : |\alpha| < \epsilon \right\}$$

However, the covering space $\tilde{S}L(2, \mathbb{R})$ is a non-matrix connected Lie group to which our techniques apply[3].

Remark.

We would hope that the results in this note might shed light on a number of open problems. For example, a problem related to the measurable Livsic theorem for non-hyperbolic systems occurs in the study of rational maps (i.e. the invariant line field problem [7]). The measurable Livsic theorem is also known to have important consequences for the study of skew products [10], [9], [11]. We refer to these papers for more details.

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