Livšic theorems and stable ergodicity for group extensions of hyperbolic systems with discontinuities

Matthew Nicol Andrew Scott

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Abstract

We prove measurable Livšic theorems for Lie group valued cocycles over certain classes of uniformly hyperbolic maps with discontinuities, in particular the β–transformation, C^2 Markov maps and a simple class of toral linked twist maps. As an application we establish stable ergodicity results for compact group extensions of certain classes of uniformly hyperbolic systems with discontinuities.

Introduction

Livšic theorems establish that solutions to dynamical cohomological equations which are assumed to have a certain degree of regularity have in fact a greater degree of regularity. For hyperbolic systems the techniques in this area are still mainly based on the seminal ideas of Livšic [15, 16]. By measurable Livšic theorems we mean theorems that apply to dynamical cohomological equations in which solutions are a priori only measurable. Measurable Livšic theorems have applications to the study of ergodicity of compact group extensions of hyperbolic systems, since an obstruction to ergodicity of such systems is given by the existence of group valued measurable solutions to a cohomological equation. Good references on the state of the art for Livšic theorems are [8, 17, 23].

Let G be a connected Lie group with Lie algebra $L(G)$ and a right invariant metric d_G. We are particularly interested in the case in which $G \subset U(d)$ is a compact connected Lie subgroup of the unitary group of $d \times d$ matrices with a bi–invariant metric d induced by the norm $\|A\|^2 = \text{trace}(A\bar{A}^t)$. In this section we consider the regularity of G–valued measurable solutions h to the cocycle equation $g(x) = h(fx)h(x)^{-1}$ in two simple settings. First where the base transformation $f\colon X \longrightarrow X$ is a piecewise C^2 expanding map of the interval and where g is a Lipschitz (or Hölder) G–valued cocycle. Second where f is from a class of hyperbolic mappings of the two torus (a special case of which is a toral linked twist map of the two–dimensional torus X which is smooth except on finitely many lines of discontinuity) and g is a C^1 G–valued cocycle.

In Section 2 we show, as applications of these results, that if G is a compact Lie group and $g\colon X \longrightarrow G$ is C^1 then in the C^1 topology a prevalent and residual set of group extensions of these base transformations are ergodic and if G is a semisimple Lie group then an open, dense and prevalent set of group extensions of these base transformations are ergodic. If the base transformation is a Lasota-Yorke map, then our stable ergodicity results hold in the Hölder topology for Hölder cocycles.

Pollicott and Yuri [24, Theorem 1] have related results for the β–transformation. Suppose that $X := [0,1)$ (mod 1), $f\colon X \longrightarrow X$ is a β-transformation, $g\colon X \longrightarrow \mathbb{R}$ is Hölder and μ is an equilibrium measure corresponding to the Hölder potential g. Pollicott and Yuri [24, Theorem 1] have shown that if $h\colon X \longrightarrow \mathbb{R}$ is an essentially bounded measurable solution to the cohomological equation $g(x) = h(fx) - h(x)$ then for any $\epsilon > 0$, there exists $h' = h$ μ a.e. such that h' is of bounded variation on $[0, 1-\epsilon]$. Their method of proof, based upon the transfer operator, does not generalize to the general compact Lie group case. Let $G = \mathbb{R}^n$, a compact Lie group or a connected Lie group satisfying a partial hyperbolicity and center–bunching condition (to be specified later). In Theorem 1 we show that if m is the unique invariant probability measure for the β-transformation which is equivalent to Lebesgue and $g\colon X \longrightarrow G$ is Hölder (Lipschitz) then for any measurable solution h to $g(x) = h(fx)h(x)^{-1}$ there exists $h' = h$ m a.e. and h' is Hölder (Lipschitz) on X.

1 Livšic Theorems

Let G be a connected Lie group with Lie algebra denoted by $L(G)$ which we identify with the tangent space at the identity, T_eG. We let r_g denote right multiplication by $g \in G$. Given a norm $\|\cdot\|$ on T_eG we define a norm on T_gG by $\|\mathbf{v}\|_g = \|r_{g^{-1}}\mathbf{v}\|_e$. This norm induces a right invariant metric d_G on G so that $d_G(gk, hk) = d_G(g, h)$ [23, Section 4]. Throughout this paper we will write $d(\cdot,\cdot)$ instead of $d_G(\cdot,\cdot)$ when it is clear from context that we mean the metric on G.

We define the adjoint map $\mathrm{Ad}\colon G \longrightarrow \mathrm{Aut}(L(G))$, for $g \in G$ and $X \in T_eG$ by

$$\mathrm{Ad}(g)\mathbf{v} = \frac{d}{dt}(g\exp(t\mathbf{v})g^{-1}).$$

A calculation [23, Section 4] shows that

$$d(gh, gk) \leqslant \|\,\mathrm{Ad}(g)\| d(h, k). \tag{1}$$

We make spectral assumptions on our dynamics and cocycle. In this paper we consider one–dimensional piecewise expanding maps and two–dimensional piecewise C^2 diffeomorphisms $f\colon X \longrightarrow X$, where X is either an interval or $\mathbb{T}^2$ respectively. In both settings f is ergodic with respect to an invariant measure m. We suppose that in the one-dimensional case $|f'(x)| > \lambda_u > 1$ for all but a finite number of points. We assume in the two-dimensional setting that for μ a.e. point $x \in X$ there is a splitting of the tangent space $T_x = E^s_x \oplus E^u_x$ and two numbers $0 < \lambda_s < 1 < \lambda_u$ such that $|Df_x v| \leqslant \lambda_s |v|$ for all $v \in E^s_x$ and $|Df_x^{-1} v| \leqslant \lambda_u |v|$ for all $v \in E^u_x$.

We will state our assumptions on partial hyperbolicity in the invertible case. In the one–dimensional case any assumptions involving the quantities μ_s, λ_s are vacuously satisfied.

Let $g\colon X \longrightarrow G$ be Hölder of exponent $\gamma > 0$. We define

$$\begin{aligned} \mu_s &:= \lim_{n\to\infty} \left(\sup_{x\in X} \| \operatorname{Ad}(g_n(x)^{-1}) \| \right)^{-\frac{1}{n}} \\ \mu_u &:= \lim_{n\to\infty} \left(\sup_{x\in X} \| \operatorname{Ad}(g_n(x)) \| \right)^{\frac{1}{n}}, \end{aligned}$$

where $g_n = g(f^{n-1}x)\dots g(fx)g(x)$.

We assume that the group extension is partially hyperbolic and satisfies the partial hyperbolicity condition (PH)

$$\lambda_s < \mu_s \leqslant 1 \leqslant \mu_u < \lambda_u$$

Define

$$\theta_{PH} := \max\left\{ \frac{\log \mu_s}{\log \lambda_s}, \frac{\log \mu_u}{\log \lambda_u} \right\}$$

The assumption (PH) implies that $\theta_{PH} < 1$.

Note that if G is compact, nilpotent or abelian then $\mu_s = \mu_u = 1$ and condition (PH) is satisfied.

1.1 Lasota–Yorke Maps

We consider maps $f\colon [0,1) \longrightarrow [0,1) \pmod 1$ of the following form. Let $S = \{0 = s_0, s_1, \dots, s_d = 1\}$ and suppose f restricted to each open interval (s_i, s_{i+1}) is piecewise C^2 (with C^2 extension to $[s_i, s_{i+1}]$ with left and right derivatives) and $f'(x) \geq \lambda_u > 1$ for all $x \in [0,1) \setminus S$. We call this class of maps Lasota-Yorke maps. It is known that such maps have a finite number of distinct ergodic absolutely continuous invariant measures and that if p is an f invariant density then the support of p is a finite union of open intervals Lebesgue a.e. [1, Theorem 8.2.1, Theorem 8.2.2]. Furthermore the f-invariant density p (with respect to Lebesgue measure l) is bounded below and above on its support so that there exists $M > 0$ such that

$$\frac{1}{M} \leqslant p(x) := \frac{dm}{dl} \leqslant M \text{ if } x \in \operatorname{supp}(p). \tag{2}$$

To simplify notation and to avoid trivialities we assume such a map is weak-mixing with unique absolutely continuous invariant measure m with support $X \subset [0,1)$. This assumption implies that the map is exact.

We further note that Lasota-Yorke maps satisfy a bounded distortion condition [1, Lemma 6.1.1] in that there exists some $R > 0$ such that for all n and all x, y such that if $f^j(x)$ and $f^j(y)$ both belong to the same connected component of $X \setminus S$ for $j = 0, \dots, n$, then

$$\frac{1}{R} \leqslant \frac{|(f^n)'(x)|}{|(f^n)'(y)|} \leqslant R. \tag{3}$$

The class of β-transformations are a subclass of Lasota-Yorke maps. Let $b < \beta < b+1$ where b is a positive integer. We let (x) denote $x \pmod 1$ though we write x instead of (x) when it is the argument of a function. Define the β-transformation $f\colon X \longrightarrow X$ by $f(x) = (\beta x)$. Hence $\lambda_u = \beta$. It is known that this system is exact with respect to a unique absolutely continuous invariant measure μ whose support is $[0, 1)$.

We will also distinguish the subclass of Lasota-Yorke maps known as C^2 Markov maps, which have the additional property that for each i, $f(s_i, s_{i+1})$ is a union of intervals of form (s_j, s_{j+1}) [1, Chapter 9].

We prove the following regularity results,

Theorem 1. *Let $f\colon X \to X$ be a β-transformation or a C^2 Markov map, G a connected Lie group and $g\colon X \longrightarrow G$ a cocycle which is Lipschitz or Hölder of exponent γ. Assume condition (PH) and if g is γ-Hölder assume $\gamma > \Theta_{PH}$. Then, if $g\colon X \to G$ is Lipschitz (Hölder) and $h\colon X \to G$ is a measurable solution to the equation*

$$g(x) = h(fx)h(x)^{-1}$$

then there exists h' which is Lipschitz (Hölder) on X and $h' = h$ m a.e.

Theorem 2. *Let $f\colon X \to X$ be a weak-mixing Lasota-Yorke map with respect to an absolutely continuous invariant measure m with support X. Assume the same (PH) condition on $g\colon X \longrightarrow G$ as in Theorem 1. Then if $g\colon X \to G$ is Lipschitz (Hölder) and $h\colon X \to G$ is a measurable solution to the equation*

$$g(x) = h(fx)h(x)^{-1}$$

then there exists h' such that $h' = h$ m a.e. and for m a.e. $x \in X$ there exists an interval $U(x)$ with $h'|U(x)$ uniformly Lipschitz (Hölder) over all such intervals $U(x)$.

Remark. *It is easy to construct nontrivial examples of Lipschitz $\mathbb{R}^d$ or group-valued coboundaries where the base transformation is the β-transformation and the cobounding function is also Lipschitz. For example, let $h(x)$ be an $\mathbb{R}^d$ valued Lipschitz function on X with the additional property that $h(\beta) = h(0) = h(1)$ and define $g(x) = h(fx) - h(x)$. Since h is Lipschitz on X and f has a discontinuity only at the point $0 = 1$ the composition $h \cdot f$ is Lipschitz except possibly at the point $0 = 1$. By checking left and right limits it follows that $h \cdot f$ is Lipschitz on X and hence $g(x)$ is Lipschitz on X. Hölder and Lie group valued examples may be constructed in the same manner.*

The proof of Theorem 2 proceeds in four steps. The proof for Lipschitz group extensions is almost the same as that for Hölder group extensions so we prove only the Hölder case.

Step 1: The natural extension

Suppose $f\colon X \longrightarrow X$ is a Lasota-Yorke map. We suppose $g\colon X \longrightarrow G$ is Hölder of exponent γ and satisfies the (PH) conditions with $\gamma > \Theta_{PH}$. Since Lasota-Yorke maps are noninvertible we form the natural extension. We recall its construction. Let

$$\Omega\colon = \{(x_0, x_1, x_2, \ldots) : x_n = f(x_{n+1}), x_n \in X, n = 0, 1, 2, \ldots\}$$

with $\sigma\colon \Omega \longrightarrow \Omega$ defined by $\sigma((x_0, x_1, x_2, \ldots)) = (f(x_0), x_0, x_1, \ldots)$. The map σ is invertible on Ω. Given the invariant measure m we define the invariant measure for the natural extension $\tilde{m}$ on Ω by defining it first on cylinder sets $C(A_0, A_1, \ldots, A_k)$ by

$$\tilde{m}(C(A_0, A_1, \ldots, A_k)) = m(f^{-k}(A_0) \cap f^{-k+1}(A_1) \ldots \cap f^{-k+i}(A_i) \ldots \cap A_k)$$

and then extending it to Ω using Kolmogorov's extension theorem. We think of $(x_0, x_1, x_2, \ldots)$ as being an inverse branch of $x_0 \in X$ under the mapping $f\colon X \longrightarrow X$. The map $\sigma\colon \Omega \longrightarrow \Omega$ is ergodic with respect to $\tilde{m}$ [30]. The natural extension is in fact Bernoulli because f is exact but we will only use the fact that σ^{-1} and σ are ergodic with respect to $\tilde{m}$.

Step 2: A Borel-Cantelli argument

Let B denote the image of S under f; strictly speaking we take the images of the points s_i under the appropriate C^2 extension of $f|_{(s_i, s_{i+1})}$. In other words

$$B := \left\{ \lim_{x \searrow s_i} f(x), \lim_{x \nearrow s_i} f(x) : i = 0, 1, \ldots, k \right\}$$

Note that B may consist of twice as many points as S. Let $x = (x_0, x_1, \ldots)$ be a point in Ω such that $x_i \notin B$ for all $i \geqslant 0$ and $x_0 \notin S$. A full measure set of points satisfies this condition so we lose nothing by dealing only with such points.

Take a point $x \in \Omega$ and an open interval I disjoint from S which contains x_0. Consider the preimage $f^{-1}I$, consisting of a finite number of intervals. Divide this partition further by deleting points in S. Thus we have:

$$f^{-1}(I) \setminus S = I_{(1)} \dot{\cup} I_{(2)} \dot{\cup} \ldots \dot{\cup} I_{(l_0)}$$

where all $I_{(i)}$ are open intervals which may or may not map onto I. Note that if $I \cap B = \emptyset$ then $f(I_i) = I$ for all i.

Now take one of these intervals I_{i_1} and carry out the same procedure as before giving us a partition into open intervals:

$$f^{-1}(I_{(i_1)}) \setminus S = I_{(i_1,1)} \dot{\cup} I_{(i_1,2)} \dot{\cup} \ldots \dot{\cup} I_{(i_1, l_{(i_1)})}$$

where each of the intervals will map onto $I_{(i)}$ if $I_{(i)} \cap B = \emptyset$.

We can repeat this process indefinitely giving us partitions, in general, of the form

$$I_{(i_1,i_2,\ldots,i_{n-1})} \setminus S = I_{(i_1,i_2,\ldots,i_{n-1},1)} \dot{\cup} I_{(i_1,i_2,\ldots,i_{n-1},2)} \dot{\cup} \ldots \dot{\cup} I_{(i_1,i_2,\ldots,i_{n-1}, l_{(i_1,i_2,\ldots,i_{n-1})})}.$$

Again if $I_{(i_1,i_2,\ldots,i_{n-1})} \cap B = \emptyset$ then all of the intervals $I_{(i_1,i_2,\ldots,i_{n-1},j)}$ will map onto I.

This process enables us to define modified inverse branches of the interval I. For ease of notation we write I_n for such a branch (that is we define $I_n := I_{(i_1,i_2,\ldots,i_n)}$ with $I_{i_0} = I$). In particular we choose our branch so that $x_n \in I_n$ for all $n \geqslant 0$.

It may happen that for each $i \geqslant 0$ we have that $f(I_{i+1})$ is strictly contained in I_i. The following Lemma shows that for $\tilde{m}$ almost every $x \in \Omega$ there exists some $N(x)$ such that for all $i > N(x)$, $f(I_{i+1}) = I_i$.

Lemma. *For $\tilde{m}$ a.e. $x \in \Omega$ there exists $N = N(x) \in \mathbb{N}$ s.t. for all $n \geqslant N$, $f(I_{n+1}) = I_n$.*

Proof. We define a sequence of cylinder sets E_n by:

$$E_n = C(X, \ldots, X, N(B, \lambda_u^{-n}), X, \ldots)$$

where $N(B, \lambda_u^{-n}) = \{x \in X : d(x, B) < \lambda_u^{-n}\}$. Note that $\tilde{m}(E_n) \leqslant M\lambda_u^{-n}$ for $M > 0$ so that

$$\sum_{n=0}^{\infty} \tilde{m}(E_n) < \infty.$$

Thus we can use the Borel-Cantelli Lemma to conclude that $\tilde{m}$ almost every $x = (x_0, x_1, \ldots)$ belongs to only a finite number of the E_n. Thus there exists a $N = N(x) \in \mathbb{N}$ such that for all $n \geqslant N$, $I_n \cap B = \emptyset$ which implies $I_n = f(I_{n+1})$ proving the lemma. $\square$

If we take the open interval neighbourhood I still smaller then we have an open interval J_0 and inverse interval branch satisfying $x_n \in J_n$ and $f^n(J_n) = J_0$ for all $n \geqslant 0$. Note also that by construction $J_n \cap S = \emptyset$ for all $n \geq 0$.

Step 3: h has a Hölder version on J_0

Take such an $x_0 \in J_0$. In what is to follow if we refer to a point $y_0 \in J_0$ we naturally associate with it the inverse branch $(y_0, y_1, \ldots)$ such that $y_n \in J_n$ for all $n \geqslant 0$ as constructed in Step 2.

Now we take points $y = (y_0, y_1, \ldots y_n \ldots)$, $z = (z_0, z_1, \ldots z_n \ldots) \in \Omega$ such that $y_k, z_k \in J_k$ for all $k = 0, 1, \ldots$. We remark that the set of points, $y = (y_0, y_1, \ldots y_n) \in \Omega$ such that for all $i \in \mathbb{N}$ $g(y_{i-1}) = h(y_i)h(y_{i-1})$ is of full measure and without loss of generality we will only consider such points.

On J_0 we define a function $\Phi^u \colon J_0 \longrightarrow G$ by

$$\Phi^u(y_0) = \lim_{n\to\infty} g_n(y_0) g_n(x_0)^{-1}$$

where $g_n(x_0) = g(x_1)\dots g(x_n)$. This is well defined. In fact

$$\begin{aligned}
& d(g_{n+1}(y_0)g_{n+1}(x_0)^{-1}, g_n(y_0)g_n(x_0)^{-1}) \\
= \;& d(g_n(y_0)g(y_{n+1})g(x_{n+1})^{-1}g_n(x_0)^{-1}, g_n(y_0)g_n(x_0)^{-1}) \\
= \;& d(g_n(y_0)g(y_{n+1})g(x_{n+1})^{-1}g_n(x_0)^{-1}, g_n(y_0)g(x_{n+1})g(x_{n+1})^{-1}g_n(x_0)^{-1}) \\
= \;& d(g_n(y_0)g(y_{n+1}), g_n(y_0)g(x_{n+1})) \qquad \text{(right invariance)} \\
\leqslant \;& \|\operatorname{Ad}(g_n(y_0))\| d(g(y_{n+1}, g(x_{n+1}) \qquad \text{(by (1))} \\
\leqslant \;& C\left((\mu_u)\lambda_u^{-\gamma}\right)^n \\
\leqslant \;& C\kappa^n,
\end{aligned}$$

where $|\kappa| < 1$ (since $\gamma > \Theta_{PH}$). Thus the sequence $g_n(y_0)g_n(x_0)^{-1}$ is Cauchy and so converges.

Next we show Φ_u is Hölder. Let $y_0, z_0 \in J_0$, then

$$\begin{aligned}
& d(g_n(y_0)g_n(x_0)^{-1}, g_n(z_0)g_n(x_0)^{-1}) = d(g_n(y_0), g_n(z_0)) \\
\leqslant \;& \sum_{i=0}^{n-1} d(g_i(y_0)g(y_i)g_{n-i-1}(z_{i+1}), g_i(y_0)g(z_i)g_{n-i-1}(z_{i+1})) \\
\leqslant \;& \sum_{i=0}^{n-1} \|\operatorname{Ad}(g_i(y_0))\| d(g(y_i), g(z_i)) \\
\leqslant \;& \sum_{i=0}^{n-1} C(\mu_u)^i \lambda_u^{-i\gamma} d(y_0, z_0)^\gamma.
\end{aligned}$$

Letting $n \to \infty$ gives $d(\Phi_u(y_0), \Phi_u(z_0)) \leqslant K d(y_0, z_0)^\gamma$, K constant. It is clear that if g is Lipschitz then Φ_u is also Lipschitz.

Since h is measurable, by Lusin's Theorem there is a positive measure set H such that h restricted to H is uniformly continuous. We claim the following:

Lemma. *Given $\eta > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$*

$$m\{y_0 \in J_0 : y_n \in J_n \cap H\} > (1-\eta)m(J_0)$$

Let l denote Lebesgue measure. We use the Lebesgue Density Theorem to find positive numbers $\rho_k \longrightarrow 0$ such that if $H_k \subset H$ is defined as the set

$$H_k := \left\{ p \in H : \frac{l((p-\rho, p+\rho) \cap H)}{l(p-\rho, p+\rho)} > 1 - \frac{1}{2^k} \quad \text{for all} \quad 0 < \rho \leqslant \rho_k \right\}$$

then $l(H_k) > (1 - \frac{1}{2^k})$. The sets H_k contain points of 'uniform density'. Note that by the ergodicity of $\sigma^{-1}\colon \Omega \longrightarrow \Omega$ a full measure set of $x = (x_0, x_1, \dots)$ have the property that for all k, $x_n \in H_k$ for infinitely many n.

Fix k large enough that $\frac{M^2R^2}{2^{k-2}} < \eta$ and choose n large enough so that $\lambda_u^{-n} < \rho_k$ and also $x_n \in H_k$. This guarantees that $J_n \subset (x_n - \rho_k, x_n + \rho_k)$ and consequently

$$l(J_n \cap H^c) \leqslant l(J_n \cap H_k^c) < \frac{1}{2^{k-1}} l(J_n).$$

Thus by bounded distortion

$$l(f^n(J_n \cap H^c)) \leqslant \frac{R^2}{2^{k-1}} l(J_0).$$

where l is Lebesgue measure. Since $\frac{1}{M} \leqslant \frac{dm}{dl} \leq M$

$$m(f^n(J_n \cap H^c)) \leq \frac{M^2 R^2}{2^{k-1}} m(J_0).$$

Since we have chosen k so that $\frac{M^2 R^2}{2^{k-2}} < \eta$ this proves the lemma.

Given $0 < \delta < 1$ we may choose a subsequence n_i such that

$$m(f^{n_i}(J_{n_i} \cap H^c)) < \frac{\delta}{2^i} m(J_0).$$

and hence

$$m\{y_0 \in J_0 : y_{n_i} \in J_{n_i} \cap H\} > (1 - \frac{\delta}{2^i}) m(J_0)$$

Since $\sum_{i=1}^{\infty} \frac{\delta}{2^i} < 1$ we may choose a point $t_0 \in J_0$ such that $t_{n_i} \in J_{n_i} \cap H$ for all n_i.

Define $\Phi^{u,n}(y_0) = g_n(y_0) g_n(t_0)^{-1}$. Then

$$h(y_0) = \Phi^{u,n}(y_0) \Phi^{u,n}(t_0)^{-1} h(t_0) h(t_n)^{-1} h(y_n)$$

It may seem at first a cumbersome way of expressing $h(y_0)$ but it simplifies the following calculations. Using the triangle inequality we estimate

$$\begin{aligned} d(h(y_0), h(z_0)) \ &\leq \ d(\Phi^{u,n}(y_0)\Phi^{u,n}(t_0)^{-1}h(t_0)h(t_n)^{-1}h(y_n), \Phi^{u,n}(z_0)\Phi^{u,n}(t_0)^{-1}h(t_0)) \\ &+ \ d(\Phi^{u,n}(y_0)\Phi^{u,n}(t_0)^{-1}h(t_0)h(t_n)^{-1}h(z_n), \Phi^{u,n}(y_0)\Phi^{u,n}(t_0)^{-1}h(t_0)) \\ &+ \ d(\Phi^{u,n}(y_0)\Phi^{u,n}(t_0)^{-1}h(t_0)h(t_n)^{-1}h(z_n), \Phi^{u,n}(z_0)\Phi^{u,n}(t_0)^{-1}h(t_0)h(t_n)^{-1}h(z_n)) \end{aligned}$$

By right-invariance of the metric,

$$d(\Phi^{u,n}(y_0)\Phi^{u,n}(t_0)^{-1}h(t_0)h(t_n)^{-1}h(z_n), \Phi^{u,n}(z_0)\Phi^{u,n}(t_0)^{-1}h(t_0)h(t_n)^{-1}h(z_n))$$

$$= d(\Phi^{u,n}(y_0)\Phi^{u,n}(t_0)^{-1}h(t_0), \Phi^{u,n}(z_0)\Phi^{u,n}(t_0)^{-1}h(t_0))$$

The group valued function of y_0, $\Phi^{u,n}(y_0)\Phi^{u,n}(t_0)^{-1}h(t_0)$, is uniformly bounded and converges to a Hölder function of y_0. The function $\Phi^{u,n}(y_0)\Phi^{u,n}(t_0)^{-1}h(t_0)$ is uniformly bounded. The group elements $h(t_n)^{-1}h(y_n)$ and $h(t_n)^{-1}h(z_n)$ converge to the identity element uniformly along the subsequence n_i. Thus

$$d(\Phi^{u,n}(y_0)\Phi^{u,n}(t_0)^{-1}h(t_0)h(t_n)^{-1}h(y_n), \Phi^{u,n}(y_0)\Phi^{u,n}(t_0)^{-1}h(t_0))$$

and

$$d(\Phi^{u,n}(y_0)\Phi^{u,n}(t_0)^{-1}h(t_0), \Phi^{u,n}(y_0)\Phi^{u,n}(t_0)^{-1}h(t_0)h(t_n)^{-1}h(z_n))$$

converge to zero uniformly along the subsequence n_i.

Thus given $\eta > 0$ there exists N such that if $n_i \geqslant N$ and $z_{n_i}, y_{n_i} \in H$ then

$$d(h(y_0), h(z_0)) \leq K' d(y_0, z_0)^\gamma + \eta$$

where K' is constant.

Hence, for any $\eta > 0$

$$m \times m\{(y_0, z_0) \in J_0 \times J_0 : d(h(y_0), h(z_0)) \leq K' d(y_0, z_0)^\gamma + \eta\} \geq (1 - 2\eta) m \times m(J_0 \times J_0)$$

Since η was arbitrary, this implies

$$m \times m\{(y_0, z_0) \in J_0 \times J_0 : d(h(y_0), h(z_0)) \leq K' d(y_0, z_0)^\gamma\} = m \times m(J_0 \times J_0)$$

and hence $h|J_0$ has a Hölder version. This proves Theorem 2.

$\square$

The β-transformation and Markov maps: Extending h as a Hölder function from J_0 to X

We now show how in the case that the base transformation is a Markov map or a β-transformation the version h may be extended to be Hölder on X. We consider the β-transformation first. Suppose there exists an open interval $J_0 = (\gamma_1, \gamma_2) \subset (0, 1)$ on which h has a Hölder version. The key idea is to extend h to a larger domain by using the equation $h(f^i p) = g(f^{i-1} p) \cdots g(fp) g(p) h(p)$. There exists a first $i = j$, say, such that $0 \in (\beta^j \gamma_1, \beta^j \gamma_2)$ in which case h has a uniformly Hölder version on $(0, \beta^j \gamma_2)$. We now extend h as a Hölder function from $(0, \beta^j \gamma_2)$ to $(0, 1)$. Since there exists a first integer r such that $(0, 1) \subset f^r(0, \beta^j \gamma_2)$, h has a Hölder version on $(0, 1)$. Since h is Hölder on a neighbourhood of the point $\frac{1}{\beta}$ using the equation $h(fx) = g(x)h(x)$ we may extend h as a Hölder function on X.

Now suppose (f, X, m) is a C^2 Markov map. First note that we may assume that $|f'(x)| > 2$ for all x (where we take right or left derivatives at endpoints of (s_i, s_{i+1})). If this is not the case then we may take a power f^n of f for which $|f'(x)| > 2$ and a refinement $\{(\tilde{s}_j, \tilde{s}_{j+1})\}$ of the partition $\{(s_i, s_{i+1})\}$ which is Markov for f^n. If $g(x) = h(fx)h(x)^{-1}$ then $h(f^n x)h(x)^{-1} = g(f^{n-1}) \ldots g(x)$ where $g(f^{n-1}) \ldots g(x)$ is Hölder on the interior of each partition element $(\tilde{s}_j, \tilde{s}_{j+1})$. Hence if we obtain a Hölder version of h on X solving $h(f^n x)h(x)^{-1} = g(f^{n-1}) \ldots g(x)$ (recall h also solves $g(x) = h(fx)h(x)^{-1}$) then we have the Hölder version we are seeking. Suppose $|f'(x)| \geq 2 + \alpha$ where $\alpha > 0$. Take an interval J_0 such that $f^n(J_n) = J_0$ and on which h is Hölder. We may extend h as a Hölder solution to $f(J_0)$ unless $f(J_0) \cap S \neq \emptyset$. If $f(J_0) \cap S = \emptyset$ then the length of J_0 has been increased by a factor of at least 2. However if $f(J_0) \cap S \neq \emptyset$ then either (i) $(s_j, s_{j+1}) \subset f(J_0)$ for some j or (ii) we may divide $f(J_0)$ into two intervals one of which, say $J(1)$, has length strictly greater than that of J_0 by a factor of $1 + \frac{\alpha}{2}$. Continuing this process we produce an interval $J(i)$ such that h is Hölder on $J(i)$ and $(s_j, s_{j+1}) \subset f(J(i))$. Thus we may extend h as a Hölder function to (s_j, s_{j+1}) and then the Markov property allows us to extend h as a Hölder version on X. The argument in the case of a Lipschitz extension is identical. This proves Theorem 1.

Remark. *The extension of h as a Lipschitz or Hölder version from an interval J_0 to X depends upon the form of the Lasota-Yorke map and must be investigated on a case by case basis. In the case of the β-transformation and Markov maps this is straightforward but unfortunately we were not able to prove the existence of a Hölder version on X for a general Lasota-Yorke map.*

1.2 Two-dimensional systems

In this section we consider two-dimensional examples of uniformly hyperbolic systems with discontinuities, concentrating on a simple class of toral link twist maps. First we describe a setting slightly more general than the toral linked twist maps we consider in which our results hold.

Uniformly hyperbolic toral maps with discontinuities.

Suppose f is a a piecewise C^2 Lebesgue measure preserving mapping of the two-torus $\mathbb{T}^2$.

We suppose that

(I) f is piecewise C^2 in the sense that there exists a finite set of C^2 one-dimensional submanifolds $\{S_1^+, \dots, S_r^+\}$ with $S_i^+ \cap S_j^+$ $i \neq j$, either empty or consisting of a finite set of points and f is C^2 on $\mathbb{T}^2/(\cup_{j=1}^r S_j^+)$ with C^2 extension to the boundary of open components of $\mathbb{T}^2/(\cup_{j=1}^r S_j^+)$. Similarly there exists a finite set of C^2 one-dimensional submanifolds $\{S_1^-, \dots, S_t^-\}$ with $S_i^- \cap S_j^-$ $i \neq j$, either empty or consisting of a finite set of points and f^{-1} is C^2 on $\mathbb{T}^2/(\cup_{j=1}^t S_j^-)$ with C^2 extension to the boundary of open components of $\mathbb{T}^2/(\cup_{j=1}^t S_j^-)$.

(II) there exist piecewise continuous Df invariant cone fields $C(x), D(x)$ in the tangent space to $\mathbb{T}^2$ such that $DfC(x) \subset C(fx)$ (strictly) and $Df^{-1}D(x) \subset D(f^{-1}x)$ (strictly) and that vectors inside the cone $C(x)$ resp. $(D(x))$ are strictly expanded in length by $Df($ resp. $Df^{-1})$. Furthermore suppose that the angle between $C(x)$ and $D(x)$ is strictly bounded away from zero.

(III) the submanifolds S_j^- have no tangents contained in the cones $D(x)$ and the submanifolds S_j^+ have no tangents contained in the cones $C(x)$.

By results of Pesin [22] and Katok and Strelcyn [10] through almost every point $p \in \mathbb{T}^2$ there are local stable and unstable manifolds of dimension one and the foliations are absolutely continuous. We will need stronger properties of the foliations and we note here that such maps satisfy conditions A-F of [14]. We call this class of maps *uniformly hyperbolic toral maps with discontinuities.*

To fix ideas we first analyse in detail a special class of such maps, discontinuous toral linked twist maps of the two-torus. The structure of the toral linked twist maps enables us, as in the case of the β-transformation to show that measurable solutions to cohomological equations are Lipschitz on the whole phase space. For hyperbolic toral maps with discontinuities we are only able to show that there is a version of h such that almost every point $x \in \mathbb{T}^2$ has a neighbourhood $U(x)$ and h is Lipschitz on each $U(x)$.

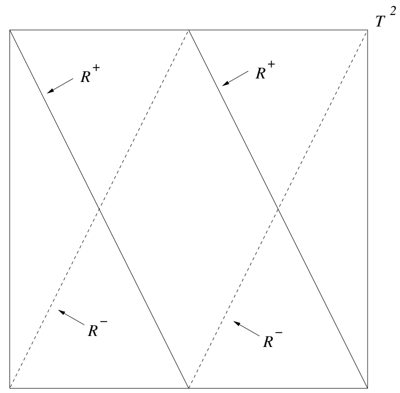

Figure 1: Discontinuity lines for the case $a = \frac{1}{2}$.

The toral linked twist mapping (tltm) we consider has been analysed in detail by Liverani and Wojtkowski [14] as a particularly simple setting in which to study the Hopf method.

Let $a > 0$ and let $f_i \colon \mathbb{T}^2 \to \mathbb{T}^2$ be defined by $f_1(x, y) = (x + ay, y)$ and $f_2(x, y) = (x, y+ax)$. Let $f \colon \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ be the toral map $f := f_2 \cdot f_1$. This map has linearization

$$\begin{pmatrix} 1 & a \\ a & 1 + a^2 \end{pmatrix}$$

Note that if a is a non-negative integer then f is a hyperbolic toral automorphism. Here, though, we are interested in the case in which a is not an integer. The map f has a finite set of discontinuity lines, preserves Lebesgue measure m and is Bernoulli [14]. Our analysis and results hold equally well for a tltm in which $f_2(x, y) = (x, y + bx)$ for $a \neq b > 0$ but in the tradition of Liverani and Wojtowski we will focus on the simplest case notationally.

Let R^+ and R^- denote the (finite) set of discontinuity lines for f and f^{-1} respectively (see Figure 1).

If $h \colon \mathbb{T}^2 \to G$ $(\mathbb{R}^d)$ is measurable we say h has a Lipschitz version if there exists $h' \colon \mathbb{T}^2 \longrightarrow G$ $(\mathbb{R}^d)$ such that $h' = h$ m a.e. and h' is Lipschitz.

Theorem 3. *Let $f \colon \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ be a toral linked twist map. Let G be a connected Lie group and $g \colon \mathbb{T}^2 \longrightarrow G$ satisfy (PH). If $g \colon \mathbb{T}^2 \longrightarrow G$ is Lipschitz then any measurable solution $h \colon \mathbb{T}^2 \longrightarrow G$ of*

$$g(x) = h(fx)h(x)^{-1} \tag{4}$$

has a Lipschitz version $h' = h$ m a.e.

Theorem 4. *Let $f \colon \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ be a uniformly hyperbolic toral map with discontinuities. Let G be a connected Lie group and $g \colon \mathbb{T}^2 \longrightarrow G$ satisfy (PH). If $g \colon \mathbb{T}^2 \longrightarrow G$ is Lipschitz then any measurable solution $h \colon \mathbb{T}^2 \longrightarrow G$ of*

$$g(x) = h(fx)h(x)^{-1} \tag{5}$$

has a version $h' = h$ m a.e. such that m a.e. $x \in \mathbb{T}^2$ has a neighbourhood $U(x)$ on which h' is Lipschitz with the same Lipschitz constant.

Proof of Theorem 3

Our proof of Theorem 3 follows with little modification the exposition of [14, Section 1]. Our first step is to construct, for m almost every $x \in \mathbb{T}^2$ stable and unstable leaves $W^s(x)$ and $W^u(x)$. We may construct unstable leaves for m a.e. x such that $x \notin \bigcup_{i=0}^{\infty} f^i R^-$ (we say x is smooth in the past). Similarly we may construct stable leaves for m a.e. x such that $x \notin \bigcup_{i=0}^{\infty} f^{-i} R^+$ (we say x is smooth in the future).

The stable and unstable leaves are line segments along which we have exponential contraction under f (respectively f^{-1}) for $W^s(x)$ ($W^u(x)$). In addition these leaves may be constructed so that all forward iterates of $W^s(x)$ and all backward iterates of $W^u(x)$ avoid the singular set.

The argument giving regularity on local unstable manifolds is the same as that for local stable manifolds, so we give only the argument for local stable manifolds. Fix $x \in X$ and on $W^s(x)$ we define a function $\Phi^s_x \colon W^s(x) \longrightarrow G$ by

$$\Phi^s_x(y) = \lim_{n\to\infty} g_n(y)^{-1} g_n(x).$$

where $g_n(x) = g(f^{n-1}x)\dots g(x)$. Note

$$\begin{aligned}
&d(g_{n+1}(y)^{-1}g_{n+1}(x), g_n(y)^{-1}g_n(x)) \\
&\quad = d(g_n(y)^{-1}g(f^ny)^{-1}g(f^nx)g_n(x), g_n(y)^{-1}g_n(x)) \\
&\quad = d(g_n(y)^{-1}g(f^ny)^{-1}g(f^nx)g_n(x), g_n(y)^{-1}g(f^nx)^{-1}g(f^nx)g_n(x)) \\
&\quad = d(g_n(y)^{-1}g(f^ny)^{-1}, g_n(y)^{-1}g(f^nx)^{-1}) \qquad \text{(right invariance)} \\
&\quad \le \|\operatorname{Ad}(g_n(y)^{-1})\| d(g(f^ny)^{-1}, g(f^nx)^{-1}) \qquad \text{(by (1))} \\
&\quad \le C\left((\mu_s)^{-1}\lambda_s^{\gamma}\right)^n \\
&\quad \le C\kappa^n,
\end{aligned}$$

where $|\kappa| < 1$.

Thus the sequence $g_n(y)^{-1}g_n(x)$ is Cauchy and so converges. Next we show Φ^s_x is Hölder. Let $y, z \in W^s(x)$, then for large n.

$$\begin{aligned}
&d(g_n(y)^{-1}g_n(x), g_n(z)^{-1}g_n(x)) \\
&\quad = d(g_n(y)^{-1}, g_n(z)^{-1}) \\
&\quad \leqslant \sum_{i=0}^{n-1} d(g_i(y)^{-1}g(f^iy)^{-1}g_{n-i-1}(f^{i+1}z)^{-1}, g_i(y)^{-1}g(f^iz)^{-1}g_{n-i-1}(f^{i+1}z)^{-1}) \\
&\quad \leqslant \sum_{i=0}^{n-1} \|\operatorname{Ad}(g_i)^{-1}\| d(g(f^iy)^{-1}, g(f^iz)^{-1}) \\
&\quad \leqslant \sum_{i=0}^{n-1} C\mu_s^i\lambda_s^i d(y,z).
\end{aligned}$$

Letting $n \to \infty$ gives $d(\Phi^s_x(y), \Phi^s_x(z)) \leqslant Kd(y,z)^{\alpha}$, K constant.

Let $Y' := \{x \in X \mid g(x) = h(fx)h(x)^{-1}\}$ and define $Y = \bigcap_{n\in\mathbb{Z}} f^{-n}Y'$. Then $m(Y) = 1$, the cocycle equation holds on Y and Y is f-invariant. For $x, y \in Y$ we may iterate the cocycle equation to give

$$g_n(y)^{-1}g_n(x) = h(y)h(f^n y)^{-1}h(f^n x)h(x)^{-1}.$$

By Lusin's theorem we can find a set H, $m(H) > \frac{1}{2}$ such that h and h^{-1} are uniformly continuous when restricted to H. For almost all stable leaves $W^s(x)$ and any full measure set A, the intersection of $W^s(x)$ with A has full m_x measure, where m_x is the conditional induced measure on $W^s(x)$ (i.e. scaled one-dimensional Lebesgue measure). In particular if we let $\mathcal{A}$ be the set of points $x \in \mathbb{T}^2$ such that

$$\begin{aligned} m(H) &= \lim_{N\to\infty} \frac{1}{N}\sum_{i=0}^{N-1} \chi_H(f^i x) \\ &= \lim_{N\to\infty} \frac{1}{N}\sum_{i=0}^{N-1} \chi_H(f^{-i} x) \end{aligned}$$

(by the ergodicity of f a set of full measure) then we can conclude that for m almost every $W^s(x)$, $m_x(\mathcal{A} \cap W^s(x)) = m_x(W^s(x))$. Since $m(H) > \frac{1}{2}$ for each $x, y \in W^s(x) \cap \mathcal{A}$ there exists a sequence $n_i \to \infty$ such that $f^{n_i}x, f^{n_i}y \in H$. Thus if we let $n \to \infty$ through the sequence $\{n_i\}$ we obtain

$$h(y)h(x)^{-1} = \Phi_x^s(y)$$

Hence for m-almost all stable leaves $W^s(x)$ and m_x-almost all $y \in W^s(x)$, $h(y) = \Phi_x^s(y)h(x)$. Thus there is version of h whose restriction to m a.e. stable leaf $W^s(x)$ is Hölder. A similar argument shows that there is a version of h whose restriction to m a.e. stable leaf $W^s(x)$ and m a.e. unstable leaf $W^u(x)$ is Hölder.

The Sinai Method

Our next aim is to prove that almost every x has a neighbourhood $U(x)$ on which h has a Hölder version. For this we recall the Sinai Method used in [14] to prove the local ergodicity of f.

In the proof of local ergodicity two sectors $\mathcal{K}_1$ and $\mathcal{K}_2$, containing the stable and unstable directions respectively are chosen. These sectors are chosen sufficiently narrow that for any square intersected by a line with direction in $\mathcal{K}_i$, $i = 1, 2$, the areas shown in Figure 1.2 are no more than $\frac{1}{8}$ the area of the square. The constants M and N are then chosen to be the smallest positive integers so that all line segments $\{f^{-i}R^+\}_{i=M}^{\infty}$ or $\{f^i R^-\}_{i=N}^{\infty}$ fall inside $\mathcal{K}_1$ or $\mathcal{K}_2$ respectively. We may choose such M, N as R^+ is not contained in the unstable direction and R^- is not contained in the stable direction.

The neighbourhood of x, $\mathcal{U}(x)$, is defined as the largest neighbourhood not intersecting the collection of lines $L := \bigcup_{i=0}^{N-1} f^{-i}R^+ \cup \bigcup_{i=0}^{M-1} f^i R^-$.

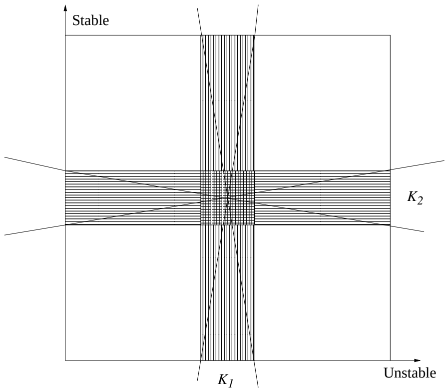

Figure 2: Areas shaded are each less than $\frac{1}{8}$ the area of the square.

Let $x \in \mathbb{T}^2/L$ and $\mathcal{U}(x)$ be such a neighbourhood of x. To simplify the exposition we assume $\mathcal{U}(x)$ has the shape of a square. We construct a sequence of open coverings of $\mathcal{U}(x)$ by dividing it into n^2 squares of equal size and then expanding the squares by a factor of two, keeping centers fixed to obtain an open cover Γ_n. We make the following definitions.

Definition. *Let H be a square in Γ_n, y a point in H which has a stable leaf and an unstable leaf. If $W^s(y)(W^u(y))$ intersects both horizontal (vertical) edges of H then we say that $W^s(y)(W^u(y))$ is long or connecting in H. Otherwise $W^s(y)(W^u(y)))$ is called short in H.*

A square $H \in \Gamma_n$ is called connecting if:

$$m(\{y \in H : W^s(y) \text{ and } W^u(y) \text{ is long in } H\}) \geq \frac{3}{4}m(H).$$

Otherwise H is called nonconnecting.

A stable (unstable) leaf is said to completely intersect a row (column) of squares if it is connecting in one of the squares in the row (column).

Note that if H_1, H_2 are two adjacent connecting squares then

$$m\{p \in H_1 \cap H_2 : W^u(p) \text{ and } W^s(p) \text{ are long in } H_1 \text{ and } H_2\} \geq 1/4m(H_1)$$

This implies that it is possible to construct a $W^{u,s}$ path between two adjacent connecting squares.

A standard argument, a nice account of which is given in [14] to demonstrate the Sinai-Chernov method shows :

Proposition 5. *[14, Sinai Theorem] Let $x \in \mathbb{T}^2$ be smooth in past and future. Then*

$$\lim_{n\to\infty} nm\left(\cup H \subset \Gamma_n \middle|\ H \text{ is non-connecting}\right) = 0$$

Let $\mathcal{U}(x)$ be a square neighbourhood of a point x constructed as above and on which the Sinai Theorem holds. Suppose y and z are points in $\mathcal{U}(x)$ which are smooth in both past and future and such that h is uniformly Hölder on their stable and unstable leaves (we will call such points *typical*). Also assume that $W^u(y)$ and $W^s(z)$ are such that m_y, m_z a.e. point on them is typical and contained in $\mathcal{A}$. The set of such points is of full measure in $\mathcal{U}(x)$.

As a consequence of Proposition 5 it is straightforward to show:

Lemma. *There exists $n \in \mathbb{N}$ so that at least one row of connecting squares in Γ_n is completely intersected by $W^s(z)$ and one column of connecting squares by $W^u(y)$.*

Proof. Suppose that for any n all rows which are completely intersected by $W^s(z)$ contain at least one bad square. Since the number of such rows grows linearly with n and the measure of one square is $\frac{4m(\mathcal{U}(x))}{n^2}$ this implies that $\lim_{n\to\infty} nm\,(\cup H \subset \Gamma_n|\text{ H is bad}) = O(1)$ contradicting the Sinai theorem. The statement regarding $W^u(y)$ is proven similarly. ☐

We may choose a row and column of connecting squares which are intersected completely by $W^s(z)$ and $W^u(y)$ respectively. We are thus able to construct a $W^{u,s}$ path of stable and unstable leaves which links $W^s(z)$ and $W^u(y)$. By absolute continuity of the foliation the intersections of the leaves of the path may be chosen to be typical points $\{x_i\}$, $i = 1, \ldots, N$ say, which also lie in $\mathcal{A}$. More explicitly $x_1 \in W^s(z)$, $x_N \in W^u(y)$ and $x_i \in (W^s(x_{i-1}) \cup W^u(x_{i-1})) \cap (W^s(x_{i+1}) \cup W^u(x_{i+1}))$, $i = 2, \ldots, N-1$. Note that h is uniformly Lipschitz (with same Lipschitz constant K) along each leaf $W^r(x_i)$ where $r \in \{s, u\}$.

Furthermore since $x_i \in \mathcal{A}$, $d(h(x_i), h(x_{i+1}) \leq Kd(x_i, x_{i+1})$ for each $i = 0, \ldots, N+1$ (where for convenience we take $x_0 = z$ and $x_{N+1} = y$). Since $\sum_{i=0}^{N+1} d(x_i, x_{i+1}) \leq \sqrt{2}d(z, y)$, it follows that h has a Lipschitz version on $\mathcal{U}(x)$ and hence h has a version which is Lipschitz on each connected open component of $\mathbb{T}^2/L$.

We now show that h is uniformly Lipschitz on $\mathbb{T}^2$. Let C_1 and C_2 be two such components separated by a line segment $l \in f^{M-1}R^-$. Take a point z on l, not lying on any other line in L and a small neighbourhood A of z.

The point fz is contained in the interior of one of the components of $\mathbb{T}^2/L$. For A sufficiently small, $f(A)$ will lie wholly inside one of the components. This is because the only invariant lines have direction vectors equal to the eigenvectors of the linearization of f. As g is C^1 on A we can thus, in view of the equation $g(x) = h(fx)h(x)^{-1}$ conclude that h is Lipschitz on A, allowing us to extend the Lipschitz version of h to the union of C_1 and C_2. We then argue inductively to take care of all lines $l \in \bigcup_{i=0}^{M-1} f^i R^-$. In a similar way we can also inductively extend h as a Lipschitz version across the boundaries of the components given by lines $l \in \bigcup_{i=0}^{N-1} f^{-i}R^+$ using the identity $h(x) = g(f^{-1}x)h(f^{-1}x)$. Thus h has a Lipschitz extension to $\mathbb{T}^2$. This concludes the proof of Theorem 3.

Proof of Theorem 4

In the set-up we have conditions (I)-(III) imply conditions A-F of [14]. The same argument as in the case of the toral linked twist map shows that there is a full measure set $A \subset \mathbb{T}^2$ and a version of h which has the same Lipschitz constant C when restricted to $W^u_\epsilon(x)$ and $W^s_\epsilon(x)$ for $x \in A$ (note that ϵ does not indicate the induced length of local stable or unstable manifolds, over which there is no uniformity, merely that they are not global). Take a neighbourbood $U(x)$ of x and two points $y, z \in A \cap U(x)$ for which there exists a $W^{u,s}$ path which links $W^s_\epsilon(y)$ and $W^u_\epsilon(z)$. The absolute contiunity of the foliations into local stable and unstable leaves allows us to assume that if x_i, $i = 1, .., N$ are the intersections of the leaves of the $W^{u,s}$ path then $x_i \in A$. Conditions (I) and (II) imply that $\sum_{i=1}^N d(x_i, x_{i+1}) \leq Kd(y, z)$ and hence

$$\begin{aligned} d(h(y), h(z)) &\leq \sum_{i=1}^N d(h(x_i), h(x_{i+1})) \\ &\leq \sum_{i=1}^N Cd(x_i, x_{i+1}) \\ &\leq KCd(y, z) \end{aligned}$$

Thus h has a version h' which is Lipschitz on each neighbourhood $U(x)$ with the same Lipschitz constant.

2 The ergodicity of compact Lie group extensions of hyperbolic systems with discontinuities

Suppose $f\colon X \to X$ is an ergodic dynamical system with respect to an invariant measure m. Let G be a compact Lie group and let dh denote Haar measure on G. Suppose $g\colon X \longrightarrow G$ is measurable.

Proposition 6. *[11, 18, Keynes-Newton, 1974]*

(a) The compact group extension $T(x, g) = (f(x), g(x)g)$ is ergodic with respect to $m \times dh$ if and only if the only solutions to

$$h(fx) = R(g(x))h(x) \tag{6}$$

where R is an irreducible (unitary) representation of degree d and $h\colon X \longrightarrow \mathbb{C}^d$ is measurable, are h constant and R the trivial representation.

(b) Suppose $f\colon X \to X$ is weak-mixing and $T\colon X \times G \to X \times G$ is ergodic. Then $T(x, g) = (f(x), g(x)g)$ is weak-mixing with respect to $m \times dh$ if and only if for any $e^{i\alpha} \neq 1$ and any non-trivial one-dimensional representation χ of G the equation

$$h(fx) = e^{i\alpha}\chi(g(x))h(x) \ m \ a.e. \tag{7}$$

has no nontrivial measurable solution $h\colon X \to \mathbb{C}$.

We now give two results which pertain to the class of systems we have considered. For a discussion of the notion of prevalence see [6]. We now consider C^1 rather than Lipschitz extensions because it is simpler to state our results in the C^1 topology, though in the case of Lasota-Yorke maps analogous statements may be made in the Hölder topology.

Theorem 7. *Suppose $f\colon X \to X$ is a Lasota-Yorke map which is weak mixing with respect to an absolutely continuous invariant measure m with support X. Let F denote the class of C^1 cocycles $g\colon X \longrightarrow G$.*

(a) If G is semisimple then there exists a C^1 open and dense subset F' of F such that $T(x,g) = (fx, g(x)g)$ is Bernoulli.

(b) If G is a compact connected Lie group then there exists a C^1 prevalent and residual subset F' of F such that if $g \in F'$ then $T(x,g) = (fx, g(x)g)$ is Bernoulli.

Theorem 8. *Suppose $f\colon \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ is a uniformly hyperbolic toral map with discontinuities. Let F denote the class of C^1 cocycles $g\colon X \to G$.*

(a) If G is semisimple then there exists a C^1 open and dense subset F' of F such that $T(x,g) = (fx, g(x)g)$ is Bernoulli.

(b) If G is a compact connected Lie group then in the C^1 topology there exists a prevalent and residual subset F' of F such that if $g \in F'$ then $T(x,g) = (fx, g(x)g)$ is Bernoulli.

In the next sections we establish Theorem 7 and Theorem 8. We consider first abelian and semisimple extensions and then, following Field and Parry [3, Section 5.1], use the fact that a compact connected Lie group is covered by the direct product of a compact group with a semisimple Lie group to obtain our results for general compact connected Lie groups.

2.1 Proof of Theorem 7 and Theorem 8

We may modify the proof of the Livšic regularity theorems above to conclude that if $f\colon X \to X$ is a map satisfying the conditions of Theorem 7 or Theorem 8 and there exists a measurable solution h to Equation (6) then there exists a solution which is Lipschitz on an open set. The modification is straightforward and we sketch the main points only in the case of Lasota-Yorke maps. For m a.e. $x_0 \in X$ there exists a sequence of intervals $\{J_n\}$ and an inverse branch $x_n \in J_n$, $f^n J_n = J_0$. Let $z_0, y_0 \in J_0$ and $f^n(z_n) = z_0$ and $f^n(y_n) = y_0$ define the inverse branches of z_0 and y_0 respectively with the property $z_n, y_n \in J_n$. We calculate

$$\begin{aligned}
\|h(z_0) - h(y_0)\| &= \|R(g(z_1)) \cdots R(g(z_{n-1}))h(z_n) - R(g(y_1)) \cdots R(g(y_{n-1}))h(y_n)\| \\
&\leqslant \sum_{j=1}^{n-2} \|R(g(y_1)) \cdots R(g(y_j))(R(g(z_{j+1})) - R(g(y_{j+1})))R(g(z_{j+2})) \cdots R(g(z_{n-1}))h(z_n)\| \\
&+ \|R(g(y_1)) \cdots R(g(y_{n-1}))(h(z_n) - h(y_n))\|
\end{aligned}$$

The sum

$$\sum_{j=1}^{n_k-2} \|R(g(y_1)) \cdots R(g(y_j))(R(g(z_{j+1})) - R(g(y_{j+1})))R(g(z_{j+2})) \ldots R(g(z_{n_{k-1}-1})h(z_{n_k})\|$$

is bounded above by $Kd(z_0, y_0)$ for some $K > 0$ since $R(g(z_{j+2}))\dots R(g(z_{n_{k-1}-1})h(z_{n_k})$ is uniformly bounded in k and $d(z_j, y_j) < \lambda_u^{-j} d(z, y)$. Since h is measurable there is a set H, $m(H) > \frac{1}{2}$, such that h restricted to H is uniformly continuous (hence bounded). The same argument as that of Step 3 in Section 1.1 using the Lebesgue density theorem and the fact that f has bounded distortion shows that given $\eta > 0$

$$m\times m\{(y_0, z_0) : \|R(g(y_1))\cdots R(g(y_{n-1}))(h(z_n)-h(y_n))\| \leqslant \eta\} \geq (1-2\eta)(m\times m)(J_0\times J_0)$$

and hence for m a.e. $x_0 \in X$ there is an interval $x_0 \in J_0$ such that h restricted to J_0 is Lipschitz.

Thus in Equation (6) we may assume that m a.e. point has a neighbourhood $U(x)$ on which h is Lipschitz. The same analysis shows that in Equation (7) we may also assume that m a.e. point x has a neighbourhood $U(x)$ on which h is Lipschitz.

2.2 Abelian and semisimple extensions of Lasota-Yorke maps

In the case that $G = \mathbb{T}^r$ is a toral group we may, following Field and Parry [3, Theorem 2.4.2, Remark 2.4.3] recast Equation 6 in multiplicative form as the statement that the group extension is ergodic iff there are no nontrivial Lipschitz solutions $h\colon X \longrightarrow \mathbb{T}^r$ to the equation $h(fx)h(x)^{-1} = g(x)^k$ for any integer $k = (k_1, \dots, k_r)$ and representation $g^k = (g_1^{k_1}, \dots, g_r^{k_r})$. If p is a periodic point of period n that avoids the singular set for f then $h(fx)h(x)^{-1} = g(x)^k$ implies that $Id = (g(f^{n-1}p)\cdots g(p))^k$. But $(g(f^{n-1}p)\cdots g(p))^k = Id$ only if $g(f^{n-1}p)\cdots g(p)$ is not a generator of $\mathbb{T}^r$.

Lasota-Yorke maps have a dense set of periodic points. We may choose a periodic orbit (i.e. a point x such that $f^{n(x)}x = x$) that avoids the singular set $\{0\}$. For a prevalent and residual set of extensions $g\colon X \longrightarrow G$, $g(f^{n-1}x)\cdots g(x)$ will generate the toral group G [3]. Thus for a prevalent and residual set of extensions g the toral extension is ergodic (for a discussion of the notion of prevalence see [6]).

To establish the prevalence and residualness of weak-mixing first note that we may as well assume that T is ergodic as a prevalent and residual set of extensions are ergodic. It is easy to see that we may choose two periodic points x, y with distinct orbits of periods m, n say. If T is not weak-mixing then neither $e^{im\alpha}g(f^{n-1}x)\cdots g(x)$ nor $e^{in\alpha}g(f^{n-1}y)\cdots g(y)$ generate G, which implies that $g(f^{m-1}x)\cdots g(x)$ is rationally related to $g(f^{n-1}y)\cdots g(y)$. For a prevalent and residual set of extensions $g(f^{m-1}x)\cdots g(x)$ will not be rationally related to $g(f^{n-1}y)\cdots g(y)$ and hence a prevalent and residual set of extensions are weak-mixing. A result of Rudolph [28, Theorem 1] states that a weak-mixing compact group extension of a Bernoulli dynamical system is Bernoulli, and hence we have shown a prevalent and residual set of extensions are Bernoulli.

Assume now that G is semisimple. Kuranishi [13] has shown that an open and dense set of pairs (g_1, g_2) of elements of G generate G (Field has shown the set of such pairs is Zariski open [2]). We may choose a periodic point for f, x_0, and a point $y = (y_0, y_1, \dots) \in \Omega$ homoclinic to an inverse branch of x_0, $x = (x_0, x_1, \dots, x_0, ldots)$. Presumably it is well-known that such homoclinic points exist for piecewise C^2 expanding maps but we were unable to find a reference in the literature so we have included a proof here.

To see that we may assume the existence of such homoclinic points and inverse branches note that by taking a suitable power of f we may assume $|f'(x)| > 2$.

As we saw before, assuming $|f'(x)| > 2$, for each i and every subinterval $I \subset (s_i, s_{i+1})$, there exists $J(0) \subset I$ such that $f^{k_1}(J(0)) \supset (s_j, s_{j+1})$ for some j, k_1 and $f^i(J(0)) \cap S = \emptyset$ for $i = 0, \ldots, k_1$. Hence we may extend h to be Hölder on (s_j, s_{j+1}). Furthermore, for any periodic point z_0 for any sufficiently small interval $I(z_0)$, $z_0 \in I$, there exists an inverse branch $\{I_n(z_0)\}$ of $f^{-n}(I(z_0))$ which contains an iterate of the point z_0 at each stage i.e. $f^j(z_0) \in I_n(z_0)$ for some j and which limits on z_0 in the sense that $\lim_{n\to\infty} l(I_n(z_0)) = 0$.

So we may take a subinterval $J(0) \subset I(z_0) \subset (s_{j_0}, s_{j_0+1})$, $z_0 \notin J(0)$ and an integer k_0 such that $f^{k_0}(J(0)) \supset (s_{j_1}, s_{j_1+1})$ for some integers k_0, j_0. If we may take $j_1 = j_0$ then there exists a point $y_0 \in J(0)$ distinct from z_0 for which $f^{k_0}(y_0) = z_0$ and which has an inverse branch limiting on z_0 so we have found a point homoclinic to z_0. It may be the case that $j_1 \neq j_0$. In this case we choose a periodic orbit $z_1 \in (s_{j_1}, s_{j_1+1})$ with corresponding interval $I(z_1)$ and subinterval $J(1) \subset I(z_1)$ such that $f^{k_1}J(1) \supset (s_{j_2}, s_{j_2+1})$. Continuing this process, since there are only finitely many intervals (s_i, s_{i+1}), we find a cycle of (s_j, s_{j+1}) interval indices $(j_0 j_1 \ldots \ldots j_t \ldots j_t)$. By refining the interval $I(t) \subset J(z_t) \subset (s_t, s_{t+1})$, we find $I(t)$, $z_t \notin I(t)$, such that $f^{k_t}I(t) \supset (s_t, s_{t+1})$ and $f^i(I(t)) \cap S = \emptyset$ for $i = 0, \ldots, k_t$.

Hence we may find a periodic point $x_0 \in (s_i, s_{i+1})$ for some i on which h has a Hölder version and an interval $I \subset (s_i, s_{i+1})$ such that $f^k(I) \supset (s_i, s_{i+1})$ and an inverse branch of I limits on x_0.

We assume without loss of generality that x_0 is a fixed point (otherwise take an appropriate power of the mapping) and that $\lim_{n\to\infty} f^n y_0 = x_0$ and $\lim_{n\to\infty} y_n = x_0$. We may also assume by construction that forward iterates of y_0 and x_0 avoid $\{0\}$ and $x_k, y_k \notin B$ for any $k > 0$.

Consider the cohomology equation

$$R(g(z))h(z) = h(fz)$$

where R is an irreducible unitary representation and h is Lipschitz (by Livšic theorem) and of constant modulus $|h| = 1$ (since R is unitary). To simplify notation we write $R(g)$ as g. Iterating the cohomology equation we obtain, $h(y_0) = g(f^{n-1}y_n)\ldots g(y_n)h(y_n)$ and $g(f^{n-1}y_0)\ldots g(fy_0)h(y_0) = h(f^n y_0)$ for each positive integer n. Since the representation is linear (hence smooth) the cocycles g,h are Lipschitz and the products $g(f^{n-1}y_n)\ldots g(y_n)$ and $g(f^{n-1}y_0)\ldots g(fy_0)$ converge. Let $\gamma^- = \lim_{n\to\infty} g(f^{n-1}y_n)\ldots g(y_n)$ and $\gamma^- = \lim_{n\to\infty} g(f^{n-1}y_0)\ldots g(y_0)$. Since

$$\lim_{n\to\infty} h(f^n y_0) = \lim_{n\to\infty} h(y_n) = h(x_0)$$

we have the equations $h(x_0) = \gamma^+ h(y_0)$, $h(y_0) = \gamma^- h(x_0)$. Furthermore since x_0 is fixed, $g(x_0)h(x_0) = h(x_0)$. Thus $h(x_0) = \gamma^+ h(y_0) = \gamma^+\gamma^- h(x_0)$ and $h(x_0) = g(x_0)h(x_0)$.

Since generating pairs are open and dense for semisimple Lie groups it is easy to show that for a C^1 open and dense set of extensions the pair $(g(x_0), \gamma^+\gamma^-)$ generate

G and hence the entire group fixes $h(x_0)$. This is a contradiction as $|h(x_0)| = 1$ and the representation was assumed irreducible.

Since the only characters of semisimple Lie groups are trivial ergodicity implies weak-mixing, so that an open and dense set of extensions are weak-mixing (hence Bernoulli [28]).

2.3 Abelian and semisimple extensions of uniformly hyperbolic toral maps with discontinuities

We may, using the same arguments as in the case Lasota-Yorke maps, assume that m a.e. $x \in T^2$ has a neighbourhood $U(x)$ on which solutions to equations (6) and (7) are Lipschitz.

The proofs to establish prevalence and residualness of Bernoullicity of compact connected Lie group extensions and the openness and density of Bernoullicity of semisimple group extensions proceed in much the same way as in the setting of the β-transformation, with the proviso that our arguments using periodic and homoclinic points rely on the work of Krüger and Troubetskoy [12] who have constructed Markov partitions for a large class of hyperbolic systems with discontinuities. We have included the details in an Appendix.

2.4 General Compact Lie Group Extensions

Our proof follows that of Field and Parry [3, Section 5.1], who use a structure theorem for general compact Lie groups to deduce the result from the corresponding result for semisimple and abelian Lie groups. We only sketch the details. If G is a compact connected Lie group then there is a covering homeomorphism $\pi\colon \tilde{G} \longrightarrow G$ where $\tilde{G}$ is the direct product of a torus $\mathbb{T}^r$ and a compact semisimple Lie group Γ.

Define $A = \pi(\mathbb{T}^r \times I_\Gamma)$ and $B = \pi(I_{\mathbb{T}^r} \times \Gamma)$. Then A and B are normal subgroups of G. Furthermore A is abelian (in fact $A \simeq \mathbb{T}^r$) and B is semisimple. Let $\pi_A : G \longrightarrow G/A$ and $\pi_B : G \longrightarrow G/B$ be the quotient homeomorphisms. If we denote $G/B = \tilde{A}$ and $G/A = \tilde{B}$ then $\tilde{A} \simeq \mathbb{T}^r$ and $\tilde{B}$ is semisimple. We consider only the case of Lasota-Yorke maps as the case of uniformly hyperbolic toral maps is identical. If $g\colon X \longrightarrow G$ is a C^1 cocycle we define $g_A = \pi_A g$ and $g_B = \pi_B g$. Note that $g_A($ resp. $g_B)\colon X \longrightarrow \tilde{B}($ resp. $g_A)$ is a C^1 cocycle and hence defines a $\tilde{B}($ resp. $\tilde{A})$ extension. For a prevalent set P of cocycles g both extensions are Bernoulli. Let Σ be the isotropy subgroup of an ergodic component of the g-extension $T(x, g) = (fx, g(x)g)$ and assume that $g \in P$. Then $\pi_A(\Sigma) = \tilde{A}$ and $\pi_B(\Sigma) = \tilde{B}$ which implies $\Sigma = G$. Hence $T(x, g) = (fx, g(x)g)$ is ergodic. Recall that if T is ergodic then T is weak-mixing if and only if for any $e^{i\alpha} \neq 1$ and any non-trivial one-dimensional representation χ of G the equation

$$e^{i\alpha}h(fx) = e^{i\alpha}\chi(g(x))h(x) \ m \text{ a.e.}$$

has no nontrivial measurable solution $h\colon X \to \mathbb{C}$. It follows from the measurable Livšic theorem we have proved that we may assume h is Lipschitz. Since B is semisimple $\chi|B$ is trivial and hence χ determines a character $\chi_A\colon \tilde{A} \longrightarrow S^1$ by $\chi_A(gB) = \chi(g)$,

$g \in A$. Since the g_A extension is weak-mixing χ_A is trivial and $\alpha = 1$, and hence T is weak-mixing and by Rudolph's Theorem [28, Theorem 1] also Bernoulli. The argument in the case of uniformly hyperbolic toral maps with discontinuities is identical.

3 Discussion.

The method of proof of Theorems 1, 3, 4, 7 and 8 is simple and robust as it is based upon the Hopf method. It allows generalization to a wider class of base transformations but the statements of analogous results in a general setting are complicated and certain details of the proofs, particularly those relating to the geometry of the mappings, may need to be checked on a case by case basis.

In the context of one-dimensional mappings it would be interesting to know if measurable solutions to cohomological equations for Lasota-Yorke maps have regular solutions on X, as in the case of Markov maps and β-transformations. We were unable to prove this. A natural question to ask is what can be said about the regularity of measurable solutions to cohomological equations involving nonuniformly expanding maps or more general invariant measures than smooth measures.

Our results in two-dimensions are limited to discontinuous uniformly hyperbolic systems with Lipschitz foliations preserving volume measures and it would be interesting to see whether similar regularity theorems hold for non-uniformly hyperbolic systems with discontinuities. Two crucial features of our proof would breakdown in these settings. The first is that if contraction (expansion) is not uniform then we would lose control over the uniformity of the Lipchitz constant for h restricted to stable (unstable) manifolds. Second we use a strong form of local accessibility. i.e. m a.e. point p has a neighbourhood $U(p)$ such that a.e. pair of points $x, y \in U(p)$ may be joined by a $W^{u,s}$ path (we assume no uniformity over the number n of legs needed) but we do assume we may choose the path so that $\sum_{i=1}^{n} d(w_i, w_{i+1}) < Kd(x, y)$ where K is constant. A weaker assumption such as $\sum_{i=1}^{N-1} d(x_i, x_{i+1}) \leq Kd(y, z)^{\alpha}$ for $\alpha > 0$ would yield Hölder regularity but it is not clear to us for which discontinuous systems this assumption would hold. Another natural question in this context would be to consider whether a priori measurable solutions have smooth versions.

A different approach would be to try to establish measurable Livšic theorems for systems for which a Markov tower construction holds, thereby reducing to a base transformation on which we have uniform estimates.

Appendix

Krüger and Troubetskoy have shown that for a class of non-uniformly hyperbolic systems with singularities (described axiomatically in [12, Section 1]) periodic orbits are dense [12, Lemma 5] and there exists a countable Markov partition for the system. By assumption the two-dimensional systems we consider fall within this class and hence automatically periodic points are dense. In this appendix we show that the existence of a Markov partition with the properties Krüger and Troubetskoy establish is enough to prove our stable ergodicity results.

More precisely Krüger and Troubetskoy have shown that for m a.e. $x \in M := \mathbb{T}^2 \setminus (\cup_{i \geq 0} f^{-i}R^+ \cup f^-R^-)$ there exists a unique corresponding symbol sequence from at most a countable alphabet given by the Markov partition. Furthermore (M, f) is conjugate to a topological Markov chain (Σ, σ) in the sense that there is a subset $B \subset M$ of measure one such that the conjugating map $\phi\colon (B, f) \longrightarrow (\phi B, \sigma)$ satisfies ϕ^{-1} is continuous. We will denote ϕB by $\tilde{B}$.

Choose a periodic point $\omega \in \Sigma$ (not necessarily in $\tilde{B}$) which lies in a neighbourhood $U \subset \Sigma$ such that ϕ^{-1} restricted to $\tilde{B} \cap U$ is continuous. Since $\tilde{B}$ is dense in Σ we may define $p = \phi^{-1}(\omega)$ by continuity if $\omega \notin \tilde{B}$. Suppose for the moment that $\omega \in \tilde{B}$ and hence $\phi^{-1}(\omega) \in M$. Without loss of generality we suppose that ω is a fixed point for σ. Since m a.e. point x in $\phi^{-1}U$ satisfies $\phi f^j(x) = \sigma^j \phi(x)\ \forall j \in \mathbb{Z}$ by using the continuity of ϕ^{-1} and the symbolic dynamics of (B, σ) we may find a sequence of points $\omega_j \in \tilde{B}$ satisfying $\phi^{-1}\sigma^i w_j = f^i \phi^{-1} w_j$ for all $i \in \mathbb{Z}$, $j \geq 0$ and which have the property that $\lim_{j \longrightarrow \infty} f\phi^{-1}(\omega_j) = f\phi^{-1}(\omega)$ and $\lim_{j \to \infty} \phi^{-1}(\omega_j) = \phi^{-1}(\omega)$. Furthermore by continuity of ϕ^{-1} and f it follows that $\phi^{-1}(\omega) = \lim_{j \to \infty} \phi^{-1}(\omega_j) = \lim_{j \to \infty} f\phi^{-1}(\omega_j) = f\phi^{-1}(\omega)$. Thus if $\phi^{-1}(\omega) \in M$ then $p = \phi^{-1}(\omega)$ is a periodic orbit under f. Hence by the Livšic theorem a coboundary must satisfy the equation $g(p)h(p) = h(p)$. If $\phi^{-1}(\omega) \notin M$ then we may define $\phi^{-1}(\omega_j) = p_j$ and $f\phi^{-1}(\omega_j) = fp_j$ and note that $h(fp_j) = g(p_j)h(p_j)$. Taking limits as $j \to \infty$ we obtain $h(p) = g(p)h(p)$ as before, since h and g are Lipschitz. The argument to establish that for any nontrivial irreducible representation h must satisfy the equation $R(g(p))h(p) = h(p)$ is identical. Hence, arguing as in Section 2.2 we establish the prevalence and residualness of Bernoulli abelian extensions.

To establish our genericity results for the Bernoulliness of semisimple extensions we choose a point $\tilde{\omega} \in U$ homoclinic to ω under σ. It may not be true that $\phi^{-1}\sigma^j\tilde{\omega} = f^j\phi^{-1}\tilde{\omega}$ for all j so it is not possible to state that $\phi^{-1}(\tilde{\omega})$ is homoclinic to $p = \phi^{-1}(\omega)$ under f. However since $\phi f^j(x) = \sigma^j\phi(x)\ \forall j \in \mathbb{Z}$ holds for m a.e. $x \in \phi^{-1}U$ we may find a sequence of points $\{\tilde{\omega}_r\}$ which have the property that for each $j \in \mathbb{Z}$, $\phi^{-1}\sigma^j\tilde{\omega}_r = f^i\phi^{-1}\tilde{\omega}_r$ and defining $y_i^r = f^i\phi^{-1}\tilde{\omega}_r$:

(a) for each i, $\{y_i^r\}$ is Cauchy in r;

(b) $\lim_{r \to \infty} y_{\pm r}^r \to p$

The Lipschitz function h satisfies for each r the equations $h(y_r^r) = g(y_{r-1}^r) \ldots g(y_0^r)h(y_0^r)$ and $h(y_0^r) = g(y_{-1}^r) \ldots g(y_{-r}^r)h(y_{-r}^r)$. We may choose the rate of convergence in (a) so that $\lim_{r \to \infty} g(y_{-1}^r) \ldots g(y_{-r}^r) := \gamma^-$ and $\lim_{r \to \infty} g(y_1^r) \ldots g(y_r^r) := \gamma^+$ both exist. Thus by continuity $h(p) = \gamma^+\gamma^- h(p)$ since $\lim_{r \to \infty} y_{\pm r}^r \to p$. As we noted before a C^1 open and dense set of cocycles have the property that $\gamma^+\gamma^-$ and $g(p)$ generate G and hence a C^1 open dense set of semisimple extensions are Bernoulli.

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References

[1] A Boyarsky and P Gora. *Laws of Chaos: Invariant measures and dynamical systems in one dimension*, Probability and Its Applications, Birkhauser, 1997.

[2] M J Field. 'Generating sets for compact semisimple Lie groups', *Proc. AMS*, **127** (1999) no. 11, 3361-3365.

[3] M J Field and W Parry. 'Stable ergodicity of skew extensions by compact Lie groups', *Topology*, **38**, (1999), 167-187.

[4] M Grayson, C Pugh and M Shub. 'Stably ergodic diffeomorphisms', *Annal of Math.* **140** (1994), 295–329.

[5] M W Hirsch, C C Pugh and M Shub. *Invariant manifolds*, (SLNM, Vol. 583. Springer-Verlag: New York, Heidelberg, Berlin, 1977)

[6] B Hunt, T Sauer and J Yorke. 'Prevalence: a translation invariant "almost every" on infinite-dimensional spaces', *Bull. Amer. Math. Soc.*, (NS), 27, (1992), (2), 217-238.

[7] J-L. Journé. 'A regularity lemma for functions of several variables', *Rev. Mat. Iberoamericana*, 4 (1988), no. 2, 187–193.

[8] A Katok. 'Cocycles, cohomology and combinatorial contructions in ergodic theory', *Proceedings of Symposia in Pure Mathematics*, **69**, (2001), 107-173.

[9] A Katok and B Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems* (Encyclopedia of Mathematics and its Applications, **54**, Cambridge University Press, 1995.)

[10] A. Katok, J.-M. Strelcyn with the collaboration of F. Ledrappier and F. Przytycki. 'Invariant manifolds, entropy and billiards; smooth maps with singularities.' *SLNM 1222*, Springer, New Yok, 1986.

[11] H B Keynes and D Newton. 'Ergodic measures for nonabelian compact group extensions', *Compositio Mathematica*,**32**, (1976), 53-70.

[12] T Krüger and S Troubetskoy. 'Markov partitions for non-uniformly hyperbolic systems with singularities', *Ergodic Theory and Dynamical Systems*, **12**, (1992), 487-508.

[13] M Kuranishi. 'Two element generations on semisimple Lie groups', *Kodai. math. Sem. report*,(1949), 9-10.

[14] C Liverani C and M Wojtkowski. (1995): Ergodicity in Hamiltonian Systems. *Dynamics Reported*, **4**, C. K. R. T. Jones, U. Kirchgraber and H. O. Walters (eds.), Springer Verlag, 131-200

[15] A N Livšic. 'Cohomology of Dynamical Systems', *Mathematics of the USSR Izvestija*, **6**(6), (1972), 1278-1301.

[16] A N Livšic. 'Homology properties of Y-Systems', *Math. Notes*, **10**, (1971), 758-763.

[17] V Nitica and A Török. 'Regularity of the transfer map for cohomologous cocycles'. *Erg. Th. & Dynam. Syst.*, **18**, (1998), 1187-1209.

[18] M S M Noorani. 'Ergodicity and weak-mixing of homogeneous extensions of measure preserving transformations with application to Markov shifts', *Monatsh. Math.*, **123** (2) (1197), 149-170.

[19] W Parry. 'Skew-products of shifts with a compact Lie group', *J. London Math. Soc.*, **56**(2) (1997), 400–404.

[20] W Parry and M Pollicott. 'The Livsic cocyle equation for compact Lie group extensions of hyperbolic systems', *J. London Math. Soc.* (2) **56** (1997), 405–416.

[21] W Parry and M Pollicott. 'Stability of mixing for toral extensions of hyperbolic systems', *Proc. Steklov Inst.*

[22] Y. Pesin. 'Lyapunov characteristic exponents and smooth ergodic theory', *Russian Mathematical Surveys.* Volume 32 (4) (1977), 55-114.

[23] M Pollicott and C Walkden. ' Livšic theorems for connected lie groups', *Trans. AMS.* (Volume 353) (7) (2001), 2879-2895.

[24] M Pollicott and M Yuri. 'Regularity of solutions to the measurable Livšic equation', *Trans. AMS.* (Volume 351) (2) (1999), 559–568.

[25] D Ruelle and D Sullivan. 'Currents, flows and diffeomorphisms', *Topology*, **14**, (1975), 319-327

[26] C Pugh and M Shub. 'Stably ergodic dynamical systems and partial hyperbolicity', *J. of Complexity* **13** (1997), 125–179.

[27] W Rudin. *Real and Complex Analysis.* (McGraw-Hill, Third Edition, 1987).

[28] D Rudolph. 'Classifying the isometric extensions of Bernoulli shifts', *J. d'Analyse Math.* **34** (1978), 36–60.

[29] Y. Sinai (Ed). *Dynamical Systems II*, Encyclopedia of Mathematical Sciences, Volume 2, Springer Verlag, Berlin Heidelberg, (1989).

[30] P Walters. 'Equilibrium states for β-transformations and related transformations', *Math. Z.* (**159**) (1978), 65–88.

[31] L S Young. 'Statistical properties of dynamical systems with some hyperbolicity', *Ann. Math.* (**147**) (1998) 585–650.

Matthew Nicol, Department of Mathematics and Statistics, University of Surrey, Guildford, GU2 7XH, UK, email:M.Nicol@surrey.ac.uk

Andrew Scott, Department of Mathematics and Statistics, University of Surrey, Guildford, GU2 7XH, UK, email:A.Scott@surrey.ac.uk