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ON THE FINE STRUCTURE OF STATIONARY MEASURES IN SYSTEMS WHICH CONTRACT-ON-AVERAGE

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Abstract. Suppose $\{f_1, \dots, f_m\}$ is a set of Lipschitz maps of $\mathbb{R}^d$. We form the iterated function system (IFS) by independently choosing the maps so that the map f_i is chosen with probability p_i ($\sum_{i=1}^m p_i = 1$). We assume that the IFS contracts on average. We give an upper bound for the upper Hausdorff dimension of the invariant measure induced on $\mathbb{R}^d$ and as a corollary show that the measure will be singular if the modulus of the entropy $\sum_i p_i \log p_i$ is less than d times the modulus of the Lyapunov exponent of the system. Using a version of Shannon's Theorem for random walks on semigroups we improve this estimate and show that it is actually attainable for certain cases of affine mappings of $\mathbb{R}$.

1. Introduction

Suppose $\{f_1, \dots, f_m\}$ is a set of Lipschitz maps of $\mathbb{R}^d$. We may form an iterated function system (IFS) by independently choosing the maps so that the map f_i is chosen with probability p_i ($\sum_{i=1}^m p_i = 1$). We denote the the probability vector by $\overline{p} := (p_1, \dots, p_m)$, and the IFS itself will be denoted by Φ.

More precisely, let $\Omega = \prod_0^\infty \{1, \dots, m\}$ and equip Ω with the product probability measure ν induced in the standard way on cylinder sets by the probability vector $\overline{p}$. Let $x_0 \in \mathbb{R}^d$. For any $\omega \in \Omega$ and any $n \in \mathbb{N}$ we define the point

$$x_n(\omega) := f_{\omega_0} \dots f_{\omega_{n-1}}(x_0).$$

If $\lim_{n\to\infty} x_n(\omega)$ exists, then we define

$$\phi(\omega) = \lim_{n\to\infty} x_n(\omega). \tag{1.1}$$

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We are going to formulate the hypothesis on Φ such that the mapping $\phi : \Omega \to \mathbb{R}^d$ will be defined ν-a.e. Let hereinafter log denote natural logarithm. Define

$$h(\overline{p}) := \sum_{i=1}^{m} p_i \log p_i.$$

Note that $-h(\overline{p})$ is the measure-theoretic entropy of the Bernoulli shift $\sigma : \Omega \to \Omega$ with the probabilities $(p_1, \dots, p_m)$.

For any Lipschitz map g of $\mathbb{R}^d$ we let $\|g\|$ denote the Lipschitz constant of g. We assume a *contraction on average* (sometimes called *logarithmic average contractivity*) condition to hold: for ν-a.e. $\omega \in \Omega$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|f_{\omega_0} \dots f_{\omega_{n-1}}\| = \chi(\Phi) < 0. \tag{1.2}$$

We will call $\chi(\Phi)$ the *Lyapunov exponent* of the system.

Note that the condition (1.2) is implied by the condition

$$\sum_{i=1}^{m} p_i \log \|f_i\| < 0.$$

It is well known (see P. Diaconis and D. Freedman [8]) that for any such IFS there exists a unique *stationary probability measure* μ on $\mathbb{R}^d$ independent of the choice of initial point, i.e, such that $L^*\mu = \mu$, where L is the Perron-Frobenius operator for the IFS Φ:

$$L\psi(x) = \sum_{i=1}^{m} p_i \psi(f_i x).$$

In fact the measurable mapping ϕ induces μ on Borel sets of $\mathbb{R}^d$ by $\mu(B) = \nu \circ \phi^{-1}(B)$. Sometimes we will also call μ the *invariant measure*.

By results of L. Dubins and D. Freedman [9] (see also M. Barnsley and J. Elton [3, Proposition 1]) on Markov operators, μ must be of *pure type*, i.e., either absolutely continuous or purely singular with respect to Lebesgue measure on $\mathbb{R}^d$. There are also results due to M. Barnsley and J. Elton [3, Theorem 3] about the structure of the support of μ when $d = 1$ and the maps $\{f_i\}$ are affine i.e. $f_i(x) = \lambda_i x + \alpha_i$. In particular they show that if $|\lambda_i| \ge 1$ for at least one i then the topological support of μ is either $\mathbb{R}$ or of form $[a, \infty)$ or $(-\infty, b]$ for $a, b \in \mathbb{R}$. We exploit this observation in Section 3.

An important classical example of an IFS is the one-parameter family

$$f_0(x) = \lambda^{-1} x,$$
$$f_1(x) = \lambda^{-1} x + 1 - \lambda^{-1}$$

with $p_1 \in (0, 1)$ and $|\lambda| > 1$. It has been extensively studied since the 1930's. In recent work by B. Solomyak [26] it was shown that if $p_1 = p_2 = \frac{1}{2}$, then a.e. $\lambda \in (1, 2)$

induces an absolutely continuous measure μ on the interval $[0,1]$. A similar result was later obtained by the same authors for $p_1 \in [1/3, 2/3]$ (see Section 3). However the problem of whether the invariant measure (usually called the *Bernoulli convolution* or the *Erdös measure*) for this system is absolutely continuous or singular for a **given** value of λ (known as the Erdös Problem), is very hard and only few concrete results are known (see [20] for a nice review and collection of references).

The purpose of this paper is to investigate conditions on IFS which contract on average under which their invariant measure is known to be singular or absolutely continuous. The *(upper) Hausdorff dimension* of a probability measure μ is the infimum of the Hausdorff dimensions of Borel sets B such that $\mu(B) = 1$. We denote the upper Hausdorff dimension of μ by $\dim_H(\mu)$.

The structure of the paper is as follows: in Section 2 we present an upper bound for the upper Hausdorff dimension of the invariant measure μ and describe sufficient conditions for μ to be singular in terms of $\chi(\Phi)$, $h(\overline{p})$ and the expansion rate of the semigroup generated by $\{f_i\}$. In Section 3 we present several examples showing how to apply the main theorem. Thus, we have reduced the problem of estimating $\dim_H(\mu)$ to certain combinatorial and algebraic issues concerning the semigroup in question.

We would like to emphasize that although our results apply to a general IFS which contracts-on-average, the most interesting case for us will be the systems in which **not all** of f_i are uniformly contracting, i.e., such that the support of μ is unbounded. One of the reasons for doing so is that there are some indications that $\mathrm{supp}(\mu)$ in this case will be "less fractal" than for uniformly contracting systems (see examples below).

2. Sufficient conditions for singularity of the invariant measure

Let G^+ denote the semigroup generated by the maps $\{f_1, \ldots, f_m\}$. Its elements are all compositions $f_{\omega_0} \circ \cdots \circ f_{\omega_{n-1}}$ for any $n \in \mathbb{N}$ and $\omega_k \in \{1, \ldots, m\}$. It is clear that G^+ can be either the free semigroup $\mathcal{F}_m^+$ (if all such compositions are different) or a proper subsemigroup of $\mathcal{F}_m^+$. Both possibilities can occur (see Section 3).

Let D_n denote the set of all words of length n in G^+. In other words, D_n is the set of equivalence classes in $\prod_0^{n-1}\{1, \ldots, m\}$, namely:

$$(\omega_0^*, \ldots, \omega_{n-1}^*) \sim (\omega_0', \ldots, \omega_{n-1}') \text{ if } f_{\omega_0^*} \circ \cdots \circ f_{\omega_{n-1}^*} = f_{\omega_0'} \circ \cdots \circ f_{\omega_{n-1}'}.$$

From general considerations the growth of G^+ is exponential, i.e., there exists $\theta \in [1, m]$ such that

$$\theta = \lim_{n \to +\infty} \sqrt[n]{\# D_n}. \tag{2.3}$$

Indeed, let $d_n = \# D_n$; then $d_{n+k} \leq d_n d_k$, because $(\omega_0, \ldots, \omega_{n-1}) \sim (\omega_0', \ldots, \omega_{n-1}')$ and $(\omega_n, \ldots, \omega_{n+k-1}) \sim (\omega_n', \ldots, \omega_{n+k-1}')$ imply $(\omega_0, \ldots, \omega_{n+k-1}) \sim (\omega_0', \ldots, \omega_{n+k-1}')$.

Hence, with a little work, (2.3) follows. Obviously, $\theta \geq 1$ and if G^+ is abelian, then $\theta = 1$.

Let H_μ denote the entropy of the random walk on the semigroup G^+ with probabilities $\{p_1, \ldots, p_m\}$. It is defined as follows: let μ_n be the n'th convolution of $(p_1, \ldots, p_m)$ on D_n, i.e.,

$$\mu_n([\omega_0^*, \ldots, \omega_{n-1}^*]) = \sum_{(\omega_0', \ldots, \omega_{n-1}') \sim (\omega_0^*, \ldots, \omega_{n-1}^*)} \nu(\omega_0 = \omega_0', \ldots, \omega_{n-1} = \omega_{n-1}'),$$

where $[\cdot]$ denotes the equivalence class.

We define

$$H_n := - \sum_{[y] \in D_n} \mu_n([y]) \log \mu_n([y]),$$

$$H_\mu := \lim_{n \to \infty} \frac{H_n}{n} \tag{2.4}$$

and finally, for $\theta > 1$,

$$H_\mu' := \frac{H_n}{\log \theta}$$

(it is a standard argument that such a limit exists and equals the infimum of the corresponding sequence). Note that for $\theta = 1$ we have $H_\mu = 0$, because $H_n \leq \log \# D_n$, whence $\lim_n H_n/n = 0$.

By the definition of H_μ' we have

$$0 \leq H_\mu' \leq -\frac{h(\overline{p})}{\log m} \leq 1. \tag{2.5}$$

We will need a version of Shannon's Theorem for random walks. In the case of discrete groups it was proved independently by Y. Derriennic [7] and V. Kaimanovich and A. Vershik [16]. We will adapt the proof from [7] to our "semigroup" context (see also [14, Theorem 1.6.4]).

Lemma 2.1. *Let $\omega \in \Omega$ and*

$$\mathcal{E}_n(\omega) = \{\omega' \in \Omega \mid (\omega_0, \ldots, \omega_{n-1}) \sim (\omega_0', \ldots, \omega_{n-1}')\}.$$

Then for ν-a.e. ω,

$$\lim_{n \to +\infty} \frac{\log \nu \mathcal{E}_n(\omega)}{n} = -H_\mu.$$

Proof: Let $[\omega]_n$ denote the set of all words of length n equivalent to $(\omega_0, \ldots, \omega_{n-1})$. We will identify $[\omega]_n$ with $\mathcal{E}_n(\omega)$. Hence $\nu(\mathcal{E}_n(\omega)) = \mu_n([\omega]_n)$.

Let $f_n(\omega) := -\log \mu_n([\omega]_n)$. By the same reason as in the proof of formula (2.3),

$$\mu_{n+k}([\omega]_{n+k}) \geq \mu_n([\omega]_n) \cdot \mu_k([\omega_n, \ldots, \omega_{n+k-1}])$$

for any $\omega \in \Omega, n \geq 1, k \geq 1$. Hence

$$f_{n+k}(\omega) \leq f_n(\omega) + f_k(\sigma^n \omega), \quad n, k \geq 1,$$

and by Kingman's Subadditive Ergodic Theorem, there exists the limit $\overline{f}(\omega) = \lim_n \frac{1}{n} f_n(\omega)$ for ν-a.e. ω and

$$\frac{1}{n} \int_{D_n} f_n \, d\mu_n \to \overline{f}, \quad n \to +\infty$$

as well. It suffices to note that $\int_{D_n} f_n \, d\mu_n = -\sum_{[y] \in D_n} \mu_n([y]) \log \mu_n([y]) = -H_n$ and apply (2.4). □

Now we are ready to formulate the main result of this paper.

Theorem 2.2. *Suppose Φ is an IFS on $\mathbb{R}^d$ which contracts on average. Then*

$$\dim_H(\mu) \leq -\frac{H_\mu}{\chi(\Phi)}. \tag{2.6}$$

This has three immediate corollaries:

Corollary 2.3. *If $\theta = 1$ (i.e., the semigroup G^+ grows subexponentially), then $\dim_H(\mu) = 0$.*

Corollary 2.4. *If*

$$\frac{h(\overline{p}) \log \theta}{\chi(\Phi)} < d \log m,$$

then μ is singular, and

$$\dim_H(\mu) \leq \frac{h(\overline{p}) \log \theta}{\chi(\Phi) \log m} < d.$$

In particular, if $p_1 = \cdots = p_m = \frac{1}{m}$, then

$$\dim_H(\mu) \leq \frac{\log \theta}{|\chi(\Phi)|}. \tag{2.7}$$

Proof: follows from (2.5) and the fact that if $\dim_H(\mu) < d$, then μ is singular. □

Corollary 2.5. *The measure μ is singular for any Φ such that*

$$d|\chi(\Phi)| > |h(\overline{p})|. \tag{2.8}$$

Besides, if (2.8) is satisfied, then

$$\dim_H(\mu) \leq \frac{h(\overline{p})}{\chi(\Phi)} < d.$$

Remark 2.6. As far as we know, there have been no analogs of Theorem 2.2 in such a general framework. However, F. Przytycki and M. Urbański [23] proved the inequality (2.6) for the case of the Erdös measure μ and Pisot number λ (and the equality in (2.6) was shown by S. Lalley [17] – see Example 3.1 below).

V. Kaimanovich [15] obtained a similar result for the Hausdorff dimension of the harmonic measure on trees with applications to certain classes of random walks. R. Lyons [19] has a result analogous to Corollary 2.5 in the context of random continued fractions. S. Pincus [22] has related results in the context of mappings on the line and 2×2 matrices in the plane. K. Simon, B. Solomyak and M. Urbański [25] have a theorem similar to Corollary 2.5 in the context of parabolic iterated function systems on the real line. Moreover, they were able to establish certain parameter values of their system for which the measure μ is absolutely continuous a.e.

Example 2.7. Let us give a simple example. Suppose $f_1(x) = 2x+1, f_2(x) = \frac{1}{16}x+1$ chosen with probabilities $p_1 = p_2 = \frac{1}{2}$. Then $\chi(\Phi) = -\frac{3}{2}\log 2 < h(\overline{p}) = -\log 2$ and hence by Corollary 2.5, the invariant measure μ is singular with respect to Lebesgue measure, and $\dim_H(\mu) \le \frac{2}{3}$. However it is easy to show that the topological support of the invariant measure is the interval $[1, \infty)$. Note that a more detailed analysis shows that since $f_1 f_2 f_1^3 f_2 f_1 = f_2 f_1^5 f_2$, we have $\theta < 1.9836$, whence by the formula (2.7), $\dim_H(\mu) \le \frac{2}{3}\log_2 \theta < 0.6588$. For more examples see Section 3.

Proof of Theorem 2.2:

We begin this proof with a simple lemma. Let $B(x, r)$ denote the ball of radius r about the point $x \in \mathbb{R}^d$.

Lemma 2.8. *Suppose that for each $n \ge 0$ there exists a finite union*

$$K_n := \bigcup_{i=1}^{m(n)} B(x_{i,n}, r_{i,n})$$

of balls $B(x_{i,n}, r_{i,n})$ such that

(a) $\lim_{n\to\infty} \mu(K_n) = 1$,
(b) $\lim_{n\to\infty} \max\{r_{i,n}; i = 1, ..., m(n)\} = 0$,
(c) $\lim_{n\to\infty} \sum_{i=1}^{m(n)} r_{i,n}^{\alpha} = 0$.

Then there exists a Borel set K such that $\mu(K) = 1$ and whose α-dimensional Hausdorff measure is zero. In particular $\dim_H(\mu) \le \alpha$.

Proof of Lemma 2.8: By taking a subsequence we may asssume that $\mu(K_n) \ge 1 - 2^{-n}$. Let $L_t = \cap_{n\ge t} K_n$. In light of condition (c), the α-dimensional Hausdorff measure of L_t is zero. Since $L_t = K_t \setminus \cup_{n>t}(\mathbb{R}^d \setminus K_n)$ we have $\mu(L_t) \ge \mu(K_t) - \sum_{n>t}[1 - \mu(K_n)] \ge 1 - 2^{-t}$. Hence if we define $K = \cup_{t\ge 0} L_t$ then $\mu(K) = 1$ and $\dim_H(\mu) \le \alpha$. $\square$

Now we use Lemma 2.8 to prove Theorem 2.2. Fix $\delta > 0$ such that $\delta + \exp\chi(\Phi) < 1$ and let B be a ball such that $\mu(B) > 1 - \delta$. Define $B_n(\omega) := f_{\omega_0} f_{\omega_1} \dots f_{\omega_{n-1}}(B)$. Let

$$K_n' := \bigcup_{\omega \in \Omega} B_n(\omega).$$

Since μ is invariant under our IFS, we have $\mu(K_n') = \mu(B) > 1 - \delta$ for each n. Let $\gamma := \exp\chi(\Phi)$ and define

$$J_n := \{\omega \in \Omega : \operatorname{diam} B_n(\omega) \le (\gamma + \delta)^n \text{ and } \mu_n([\omega]_n) \ge e^{-n(H_\mu + \delta)}\}$$

(where μ_n and $[\omega]_n$ is as in Lemma 2.1) and

$$K_n := \bigcup_{\omega \in J_n} B_n(\omega).$$

We have to check all three conditions of Lemma 2.8. First by the definition of the Lyapunov exponent and Lemma 2.1, we conclude that both extra conditions imposed in the definition of J_n will not seriously decrease the measure of K_n compared to K_n' for large n.

More precisely, for sufficiently large n we have $\mu(K_n' \setminus K_n) < \delta$, whence

$$\mu(K_n) > 1 - 2\delta.$$

Furthermore, $\operatorname{diam} B_n(\omega) \to 0$ because $\gamma + \delta < 1$. Thus, it suffices to check that

$$\sum_{\omega \in J_n} (\operatorname{diam} B_n(\omega))^\alpha \to 0.$$

Note first that in the notation of Lemma 2.8, we have $m(n) \le e^{n(H_\mu + \delta)}$ because $\mu_n([\omega]_n) \ge e^{-n(H_\mu + \delta)}$.

Therefore, we have to check that for sufficiently small $\delta > 0$,

$$(\gamma + \delta)^{n\alpha} \cdot e^{n(H_\mu + \delta)} \to 0,$$

which is equivalent to $\gamma^\alpha e^{H_\mu} < 1$, which in turn is equivalent to the inequality

$$\alpha < -\frac{H_\mu}{\chi(\Phi)}. \tag{2.9}$$

Therefore, $\dim_H(\mu) \le \alpha$ for any α satisfying (2.9). This proves the theorem. $\square$

3. Examples

We are going to consider several examples, all of which are affine IFS.

Example 3.1. (Bernoulli convolutions). Put $\Omega := \prod_0^\infty \{0,1\}$ and let $\lambda > 1,\ f_0(x) = \lambda^{-1}x, f_1(x) = \lambda^{-1}x + 1 - \lambda^{-1}, p_1 = p_2 = \frac{1}{2}$ (see Introduction). In this case $\chi(\Phi) = -\log\lambda$, and

$$f_{\omega_0} \circ \cdots \circ f_{\omega_{n-1}}(0) = (1-\lambda^{-1}) \sum_{k=0}^{n-1} \omega_k \lambda^{-k},$$

thus,

$$\phi(\omega) = (1-\lambda^{-1}) \sum_{k=0}^{\infty} \omega_k \lambda^{-k}.$$

Hence $(\omega_0, \dots, \omega_{n-1}) \sim (\omega'_0, \dots, \omega'_{n-1})$ iff $\sum_0^{n-1} \omega_k \lambda^{-k} = \sum_0^{n-1} \omega'_k \lambda^{-k}$. Assume first $\lambda > 2$. Then $\mathrm{supp}(\mu)$ is known to be a Cantor set, the semigroup G^+ is obviously free (as $f_0([0,1]) \cap f_1([0,1]) = \emptyset$), and from Corollary 2.5 it follows

$$\dim_H(\mu) \le \frac{\log 2}{\log \lambda} < 1. \tag{3.10}$$

Note that in this case it is well known that $\dim_H(\mu) = \dim(\mathrm{supp}\,\mu) = \frac{\log 2}{\log \lambda}$.

If $1 < \lambda < 2$, then $\mathrm{supp}(\mu) = [0,1]$; if λ is transcendental, it is easy to see that $G^+ = \mathcal{F}_2^+$, whence $\theta = 2$ and $H'_\mu = 1$. Hence Corollary 2.5 again gives us the estimate (3.10), which is unfortunately useless, as $\lambda < 2$. However, in some cases of algebraic λ Theorem 2.2 can be used more efficiently.

More specifically, assume λ to be a *Pisot number*, i.e., an algebraic integer greater than 1 whose conjugates have moduli less than 1. The famous Separation Lemma due to A. Garsia [13] states that there exists a constant $C = C(\lambda) > 0$ such that if $\sum_0^n \omega_k \lambda^{-k} \ne \sum_0^n \omega'_k \lambda^{-k}$, then $\left|\sum_0^n (\omega_k - \omega'_k)\lambda^{-k}\right| \ge C\lambda^{-n}$. Hence it is easy to see that $\theta = \lambda$, and from (2.6) follows $\dim_H(\mu) \le H'_\mu$.

In work by S. Lalley [17] the Separation Lemma was used to show that in fact

$$\dim_H(\mu) = H'_\mu < 1.$$

The most transparent subcase is $\lambda = \frac{1+\sqrt{5}}{2}$. It was studied in several papers (see references in [24]); in particular, for this λ we have $G^+ = \langle a, b \mid ab^2 = ba^2 \rangle$ and $\dim_H(\mu) = H'_\mu = 0.995713\ldots$ (this numerical result is due to J. C. Alexander and D. Zagier [1]). Besides, the measure μ was shown to be quasi-invariant under the *β-shift* (for $\beta = \lambda$) $\tau_\lambda : [0,1) \to [0,1)$ defined by the formula $\tau_\lambda x = \{\lambda x\}$ and the corresponding density is also known (see [24]).

Similar results hold for a more general Bernoulli convolution $\mu = B_\lambda(p, 1-p)$, i.e., the one for which the probability of taking f_0 is $p \in (0,1)$.

We believe the techniques of [17, Proposition 4] can be used to show that the equality holds in a more general situation. Let us formulate the corresponding conjecture; put as above, $x_n(\omega) := f_{\omega_0} \ldots f_{\omega_{n-1}}(x_0)$ and $\gamma = \exp \chi(\Phi)$.

Conjecture. Suppose we have an affine IFS on the real line (i.e., $f_i(x) = \lambda_i x + b_i$) and $|\lambda_i| \geq 1$ for at least one $i \in \{1, \dots, m\}$; assume the following Weak Separation Condition to be satisfied (we borrow this term from [18]): for ν^2-a.e. pair $(\omega, \omega') \in \Omega^2$ and arbitrary $\delta > 0$,

$$|x_n(\omega) - x_n(\omega')| \geq \text{const} \cdot (\gamma - \delta)^n \tag{3.11}$$

whenever $x_n(\omega) \neq x_n(\omega')$. We *conjecture* that the inequality (2.6) in this context is actually an equality. We state without proof that (3.11) does hold in the framework of Example 3.2 with λ being a Pisot number (see below).

Suppose Φ is an affine IFS in $\mathbb{R}$. If Φ is not uniformly contracting (i.e., there exists i such that $|\lambda_i| \geq 1$), then by the result from [3] mentioned above, the support of the invariant measure in this case is either a single point or $\mathbb{R}$ or $[d, +\infty)$ or finally $(-\infty, d]$ for some $d \in \mathbb{R}$. We may rule out the first case. The fact that $\mathrm{supp}(\mu)$ is connected makes the problem about the fine structure of μ nontrivial.

Example 3.2. Let $\lambda > 1$ and

$$f_1(x) = \lambda^{-1}x, \;\; f_2(x) = x + 1 \text{ and } p_1 = p_2 = \frac{1}{2}. \tag{3.12}$$

The support of $\mu = \mu(\lambda)$ is $[0, +\infty)$, and $\chi(\Phi) = -\frac{1}{2}\log\lambda$. Hence by Corollary 2.5, for $\lambda > 4$ the measure μ is singular, and $\dim_H(\mu) \leq \frac{2\log 2}{\log\lambda} < 1$.

We claim that for any transcendental λ the semigroup G^+ is free. A trivial induction argument shows that

$$f_1^{n_1} f_2^{k_1} \dots f_1^{n_s} f_2^{k_s}(x) = \lambda^{-\sum_1^s n_j} x + \sum_{j=1}^{s} k_j \lambda^{-\sum_{i=1}^{j} n_i},$$

whence if λ is not algebraic,

$$f_1^{n_1} f_2^{k_1} \dots f_1^{n_s} f_2^{k_s} = f_1^{n'_1} f_2^{k'_1} \dots f_1^{n'_s} f_2^{k'_s}$$

implies $n_j \equiv n'_j, \; k_j \equiv k'_j, \; j = 1, \dots, s$. Hence $\#D_n = 2^n, \; \theta = 2$ and $G^+ = \mathcal{F}_2^+$.

It is worth noting that for certain particular values of $\lambda \in (1, 4)$ the measure μ is nonetheless singular (similarly to the Bernoulli convolutions). Namely, since μ is invariant under the IFS Φ, we have the following self-similar relation for its Fourier transform:

$$\widehat{\mu}(x) = \frac{1}{2}\,\widehat{\mu}(\lambda^{-1}x) + \frac{1}{2}\,e^{ix}\widehat{\mu}(x),$$

whence

$$\widehat{\mu}(x) = \prod_{n=0}^{\infty} \frac{1}{2 - \exp(i\lambda^{-n}x)}. \tag{3.13}$$

Assume again λ to be a Pisot number. Then as is well known, the distance to the nearest integer for λ^n tends to 0 at exponential rate. Following the line of the proof of the classical work [11] (see also [3] for the case $\lambda = 2$), we can consider the subsequence $x_n = 2\pi\lambda^n$ and show that $\widehat{\mu}(x_n) \not\to 0$ as $n \to +\infty$, which implies that the Riemann-Lebesgue Lemma is not satisfied, whence μ cannot be absolutely continuous. Therefore, it is singular by the Law of Pure Types.

Thus, we proved that μ is singular for $\lambda \geq 4$ (as 4 is a Pisot number); at the same time it is singular for an infinite number of parameters $\lambda \in (1, 4)$ as well. It is an open question whether its Hausdorff dimension is less than 1 for a Pisot number λ (for the Bernoulli convolutions it is true, see above).

It is worth mentioning that in this example the stationary measure has an "arithmetic" interpretation as well. Namely, let $\Sigma = \prod_1^\infty \mathbb{Z}_+$ and ξ denote the stationary product measure on Σ with the following geometric distribution: $\xi(\varepsilon_n = k) = 2^{-k-1}$, $k = 0, 1, \ldots$ Let $L_\lambda : \Sigma \to \mathbb{R}_+$ be defined as follows:

$$L_\lambda(\varepsilon_1, \varepsilon_2, \ldots) = \sum_{n=1}^\infty \varepsilon_n \lambda^{-n}$$

(it is obvious that L_λ is well defined for ξ-a.e. $\varepsilon \in \Sigma$). Then from (3.13) it follows

$$\mu = \xi \circ L_\lambda^{-1},$$

as $\widehat{\xi}(x) = \frac{1}{2} + \frac{1}{4}e^{ix} + \frac{1}{8}e^{2ix} + \cdots = \frac{1}{2-e^{ix}}$. Thus, the essential difference with the case of Bernoulli convolutions is that the set of "digits" here is infinite.

When this paper was in preparation, Y. Peres suggested the following claim.

Lemma 3.3. *For $\mathcal{L}_1$-a.e. $\lambda \in (1, 3 \cdot 2^{-2/3})$ the measure μ is absolutely continuous.*

Proof. Since

$$\frac{1}{2 - e^{ix}} = \frac{2 + e^{ix}}{4 - e^{2ix}} = \frac{\frac{2}{3} + \frac{1}{3}e^{ix}}{\frac{4}{3} - \frac{1}{3}e^{2ix}},$$

by (3.13) the measure μ is the convolution of the Bernoulli measure $B_\lambda(2/3, 1/3)$ and the stationary measure for the IFS

$$g_1(x) = \lambda^{-2}x,$$
$$g_2(x) = x + 1$$

with $p_1 = \frac{1}{4}$, $p_2 = \frac{3}{4}$. In [21] it was shown that for any $p \in [1/3, 2/3]$ the Bernoulli measure is absolutely continuous for a.e. $\lambda \in (1, p^{-p}(1-p)^{-1+p})$. Hence the measure $B_\lambda(2/3, 1/3)$ will be absolutely continuous for a.e. $\lambda \in (1, 3 \cdot 2^{-2/3})$ and so will be μ. □

Remark 3.4. Since

$$\frac{1}{1 - z} = \prod_{n=0}^\infty \left(1 + z^{2^n}\right), \quad |z| < 1,$$

it is easy to deduce that

$$\frac{1}{2-e^{ix}} = \prod_{n=0}^{\infty}\left(\frac{2^{2^n}}{2^{2^n}+1} + \frac{1}{2^{2^n}+1}\exp\left(2^n ix\right)\right),$$

and by (3.13)

$$\mu = B_\lambda\left(\frac{2}{3},\frac{1}{3}\right) * B_{\lambda,2}\left(\frac{4}{5},\frac{1}{5}\right) * \cdots * B_{\lambda,2^n}\left(\frac{2^{2^n}}{2^{2^n}+1},\frac{1}{2^{2^n}+1}\right) * \ldots, \tag{3.14}$$

where $B_{\lambda,t}(p,1-p)$ is the infinite convolution of two-point discrete measures whose "basic" distribution is supported on the points 0 and t with probabilities p and $1-p$ respectively (hence $B_\lambda = B_{\lambda,1}$).

Summing up, we have the following

Proposition 3.5. *For the IFS (3.12) the following properties are satisfied:*

1. *For* $\lambda \ge 4$ *the measure* μ *is singular, and* $\dim_H(\mu) \le \frac{2\log 2}{\log\lambda}$*;*
2. *for* $\mathcal{L}_1$*-a.e.* $\lambda \in (1, 3\cdot 2^{-2/3})$ *it is absolutely continuous;*
3. *for any Pisot* λ *it is singular.*

A natural question to ask is what happens between $3\cdot 2^{-2/3} \approx 1.8899$ and 4. Note that if $\lambda > 3\cdot 2^{-2/3}$, then *all* convolutions in (3.14) are singular. Nonetheless, we *conjecture* that for $\mathcal{L}_1$-a.e. $\lambda \in (3\cdot 2^{-2/3}, 4)$ the measure μ will be absolutely continuous as well.

Example 3.6. Let $\lambda > 1$, $f_1(x) = \lambda x + 1$, $f_2(x) = \lambda^{-2}x$, and $p_1 = p_2 = \frac{1}{2}$. Here $\chi(\Phi) = -\frac{1}{2}\log\lambda$. Let for the simplicity of notation $a = f_1$, $b = f_2$. Since

$$ababa = ba^3b, \tag{3.15}$$

we have $d_{n+1} \le 2d_n - d_{n-4}$, whence $\theta < 1.9277$. Thus, from the estimate (2.7) it follows that the measure μ is singular at least for $\lambda > 1.9277^2 \approx 3.716$. However, it is clear that the actual estimate must be even sharper, because in fact there are infinitely many relations between a and b.

Namely, from general considerations it follows that G^+ will be one and the same for any transcendental λ. We do not know whether G^+ is finitely presented but at least written in the generators a, b it is not. For example, for any $n = 2, 3, \ldots$ and any $k = 1, 2, \ldots, n-1$ we have in addition to the relation (3.15),

$$a^{2k}b^n a^{2(n-k)} = b^{n-k}a^{2n}b^k$$

(these are a direct consequence of the fact that $a^{2k}b^k$ and $a^{2n}b^n$ always commute). These relations are independent, i.e., none of them is a consequence of any other one. At the same time, there are relations that cannot be deduced from any of those described above; for instance, $abab^2a^3 = b^2a^5b$. There are indications that actually θ is at least less than 1.7.

As far as we are concerned, there are no general results on the structure of $\mathrm{supp}(\mu)$ in the case of higher dimensions. We are going to present an example of a family of IFS for $d = 2$ such that $\mathrm{supp}(\mu) = \mathbb{R}^2$, whereas μ is singular; at the same time this system does not "split" into one-dimensional actions. In a certain sense the following example is a two-dimensional generalization of Example 3.2.

Example 3.7. Let α be a real number such that α/π is irrational and let R_α denote the rotation of $\mathbb{R}^2$ by the angle α, i.e.,

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

Let $\lambda > 1$ and the one-parameter family of IFS Φ_λ be defined as follows:

$$f_1(x) = A_\lambda^{-1}x,$$
$$f_2(x) = x + \overline{e},$$

where $A_\lambda = \lambda R_\alpha$ and $\overline{e} \in S^1$ is fixed. As above, we assume $p_1 = p_2 = \frac{1}{2}$.

Proposition 3.8. 1. *For any $\lambda > 1$ the IFS Φ_λ contracts on average and the support of the invariant measure μ_λ is full, i.e.,*

$$\mathit{supp}(\mu_\lambda) = \mathbb{R}^2. \tag{3.16}$$

2. *For $\lambda > 2$ the measure μ_λ is singular, and*

$$\dim_H(\mu_\lambda) < \frac{2\log 2}{\log\lambda} < 2. \tag{3.17}$$

Proof: By the same reason as in the previous examples, we have $\chi(\Phi_\lambda) = -\frac{1}{2}\log\lambda$, whence for any $\lambda > 1$ the system contracts on average. From Corollary 2.5 it follows that $\lambda > 2$ implies the singularity of μ_λ together with (3.17).

The most delicate part of the proposition is the relation (3.16). Let us prove it. Assume $\mathcal{M} := \mathrm{supp}(\mu_\lambda) \neq \mathbb{R}^2$; then there exists a disc $B(x,\delta)$ whose intersection with $\mathcal{M}$ is empty. Hence by definition, $f_1^{-n}B(x,\delta) \cap \mathcal{M} = \emptyset$ for any $n \geq 1$. We have

$$f_1^{-n}B(x,\delta) = B(y_n, \lambda^n\delta),$$

where $y_n = f_1^{-n}(x) = \lambda^n R_\alpha^n(x)$. Since $\alpha/2\pi$ is irrational, the rotation $R_\alpha : S^1 \to S^1$ is *minimal*, i.e., the orbit of every point is dense in S^1 (see, e.g., [6]). We apply this claim to the circle of radius $\|x\|$. Thus, for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that

$$\|R_\alpha^n(x) - \|x\|\overline{e}\,\| < \varepsilon. \tag{3.18}$$

Fix $r > 1$, $\varepsilon = \delta/2$ and n large enough to satisfy $\lambda^n \geq 2r/\delta$ together with (3.18). Let $z = \lambda^n\|x\|\overline{e}$; we claim that

$$B(z,r) \subset B(y_n, \lambda^n\delta).$$

Indeed, let $y \in B(z,r)$, i.e., $\|y-z\| \le r$. Hence

$$\begin{aligned}\|y-y_n\| &\le \|y-z\| + \|z-y_n\| \\ &\le r + \lambda^n \varepsilon = r + \frac{1}{2}\lambda^n \delta < \lambda^n \delta.\end{aligned}$$

Hence $B(z,r) \cap \mathcal{M} = \emptyset$, and $f_2^{-k} B(z,r) \cap \mathcal{M} = \emptyset$ as well for any $k \ge 0$. Since $\|\overline{e}\| = 1$ and z belongs to the half-line $\{t\overline{e}, t \ge 0\}$, there exists $k \ge 0$ such that $f_2^{-k} B(z,r) \supset B(0, r-1)$. Hence for any $r > 1$, $B(0,r-1) \cap \mathcal{M} = \emptyset$, which means $\mathcal{M} = \emptyset$. The proposition is proven. $\square$

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