Drift for Euclidean extensions of dynamical systems

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Abstract

We consider the behaviour of generic special Euclidean ($\mathbf{SE}(n)$) group extensions of dynamical systems that are chaotic or quasiperiodic. Results of Nicol *et al* (1999) [13] show that for a generic extension of a chaotic base dynamics, one will see a Brownian-like random walk if $n > 1$ is odd or if $n = 2$. For $\mathbf{SE}(2)$-extensions of quasiperiodic dynamics, there is bounded motion for almost all smooth enough extensions.

1 Euclidean extensions of base dynamics

There has recently been progress in understanding the generic qualitative dynamics of spatio-temporal dynamics of patterns on Euclidean space $\mathbb{R}^n$ that are invariant under the Euclidean group $\mathbf{E}(n)$ and special Euclidean group $\mathbf{SE}(n)$, see for example [1, 2, 3, 4, 6, 7, 9, 10, 14, 15, 16]. Because $\mathbf{E}(n)$ is not compact one can obtain unbounded motion or 'unbounded drift' along group orbits. In particular, the generic drifting behaviour and bifurcations of relative equilibria and periodic orbits is now well understood.

For more complicated dynamics on orbit space there is less known. Note that Fiedler and Turaev [7] have observed Brownian-like dynamics near relative homoclinic orbits in $\mathbf{SE}(2)$ systems. Biktashev and Holden [4] consider 'hypermeander' and observe Brownian motion of a spiral tip in an $\mathbf{E}(2)$ equivariant system with a relative chaotic attractor. With such applications in mind, this paper considers the abstract problem of drifting behaviour for noncompact extensions of specified dynamics. Let $\Gamma = \mathbf{SE}(n) = \mathbf{SO}(n) \ltimes \mathbb{R}^n$ be the special Euclidean group of transformations $x \mapsto Mx + b$ with $b \in \mathbb{R}^n$ and $M \in \mathbf{SO}(n)$. Observe that $\Gamma = G \ltimes \mathbb{R}^n$ where G is compact and $\mathbb{R}^n$ is an abelian normal subgroup. Let X be a compact space (*base space*) and let $f : X \to X$ generate a dynamical system on X. We consider Γ-extensions of the form

$$T(x, \gamma) = (f(x), \gamma\zeta(x)), \tag{1.1}$$

i.e. a map from $X \times \Gamma$ to itself, where $\zeta \in C^r(X, \Gamma)$. Note that ζ acts on the right, and so this map is equivariant under $\delta \in \Gamma$ acting on the left by $(x, \gamma) \mapsto (x, \delta\gamma)$. Given the structure $\Gamma = G \ltimes \mathbb{R}^n$ we write

$$T(x, g, v) = (f(x), gh(x), v + \rho_g k(x)),$$

where $g \in G$, $v \in \mathbb{R}^n$, $h \in C^r(X, G)$ and $k \in C^r(X, \mathbb{R}^n)$. Note that T can be viewed as an $\mathbb{R}^n$-extension of a G-extension $S(x, g) = (f(x), gh(x))$. (Any transformation of the form (1.1) generates cocycles $h_j(x)$ and $\rho_g k_j(x)$ such that we can write $T^j(x, g, v) = (f^j(x), gh_j(x), v + \rho_g k_j(x))$ where ρ_g is a representation of G on $\mathbb{R}^n$.) Similar extensions can be defined for flows.

We are interested in unboundedness or boundedness of *most initial conditions* (i.e. almost all with respect to some measure) for *most extensions* T (i.e. for generic sets of extensions). To this end we assume that the base dynamics has an ergodic invariant measure μ. The generic behaviour of a Γ-extension can then be understood as follows. Firstly, what is the generic behaviour of the G-extension? Secondly, what is the generic behaviour of the $\mathbb{R}^n$-extension of the G-extension? For the first question we appeal to

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recent results on compact group extensions of ergodic dynamics while for the second we use central limit theorem results (in the chaotic case).

Since unboundedness can only occur in the $\mathbb{R}^n$ component, we are most interested in the boundedness of the cocycle $\rho_g k_j(x)$ over the compact group extension. Suppose that we have an ergodic invariant measure m for the G-extension and a trajectory $(x_j, g_j, v_j) = T^j(x_0, g_0, v_0)$. By applying the ergodic theorem in v, we can define the average *linear growth rate* of v_j by

$$\bar{v} = \lim_{N\to\infty} \frac{1}{N} v(N) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \rho_{g_j} k(x_j)$$

which is constant for m-almost all initial conditions $(x_0, g_0) \in X \times G$. We define the *deviation from linear growth* ϵ_{x_0,v_0} by

$$v_{x_0,g_0}(N) = \bar{v}N + \epsilon_{x_0,g_0}(N),$$

and so for m-almost all initial (x_0, g_0) we have $\epsilon_{x_0,g_0}(N) = o(N)$ as $N \to \infty$.

2 Drift with chaotic forcing

The article [13] considers a number of assumptions on base dynamics that give rise to chaotic systems; we limit ourselves to the specific situation of $\mathbf{SE}(n)$-extensions of a chaotic map.

Theorem 2.1 *([13]) Suppose that $f : X \to X$ is Axiom A when restricted to a topologically mixing hyperbolic basic set $\Lambda \subset X$ equipped with a Gibbs measure μ. Suppose further either that n is odd, or that $n = 2$ and f is Anosov. Then generically (in the C^r topology for any $r \geq 1$) an $\mathbf{SE}(n)$-extension of f will be such that*

(i) $m = \mu \times \nu_{\mathbf{SO}(n)}$ is an ergodic measure for the associated $\mathbf{SO}(n)$-extension.

(ii) $\frac{1}{\sqrt{N}}\epsilon(N)$ converges in distribution to a nondegenerate standard n-dimensional normal distribution with mean zero and covariance matrix $\sigma^2 I_n$ with $\sigma > 0$.

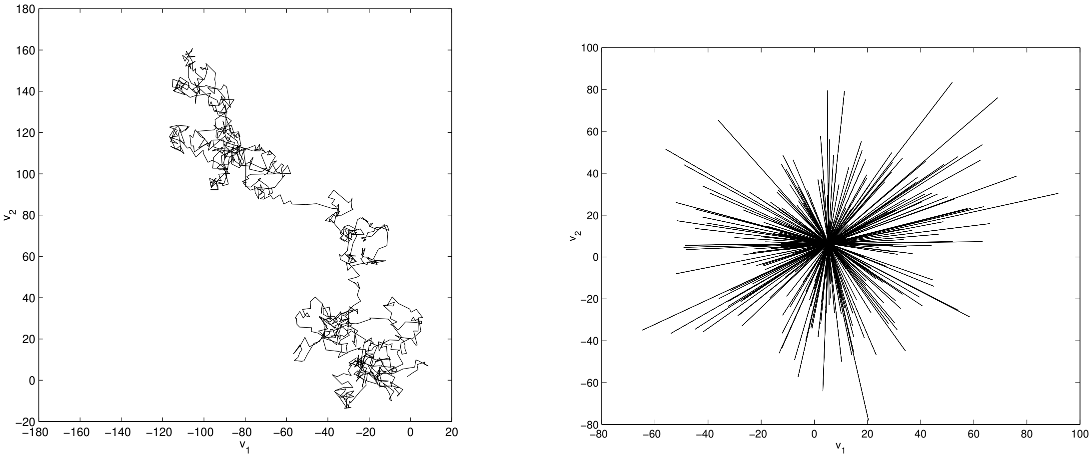
Result (i) follows directly from results of [8] on lifting ergodicity to compact group extensions. (ii) is proven by adapting central limit theorems for partially hyperbolic systems from [5, 12]. The convergence in (ii) means that for any cube $I \subset \mathbb{R}^n$,

$$\lim_{N\to\infty} m\left\{(x, g) \in X \times G : \tfrac{1}{\sqrt{N}}\epsilon_{x,g}(N) \in I\right\} = \frac{1}{(2\pi\sigma^2)^{n/2}} \int_I e^{-\frac{x_1^2+\cdots+x_n^2}{2\sigma^2}} dx_1 \cdots dx_n\,. \tag{2.1}$$

An ensemble of initial conditions released at the same point in x will therefore evolve asymptotically to resemble a Brownian motion with variance growing as σ per unit time. Moreover, generically $\sigma \neq 0$ and so the distribution is nondegenerate. Biktashev and Holden [4] have a related result for the case $n = 2$ using an assumption of decay of correlations. In fact, we conjecture that the extra condition that f is Anosov can be weakened for the result above.

Example of an SE(2)-extension. To see that the results of Theorem 2.1 for $\mathbf{SE}(2)$-extensions may hold even if f is not necessarily Anosov, we present some *dstool* [11] numerical simulations of trajectories of the map

$$\begin{aligned} x' &= 4x(1-x) \\ \theta' &= \theta + \omega(x) \\ v_1' &= v_1 + b(x)\cos\theta \\ v_2' &= v_2 + b(x)\sin\theta \end{aligned} \tag{2.2}$$



(a) (b)

Figure 1: (a) Path of a single initial condition plotted every 100 iterations of the extension with chaotic base transformation (2.2) in the (v_1, v_2) plane. (b) Paths of an ensemble of 300 initial conditions in x started at centre of diagram after 10,000 timesteps. Observe that the drift after the time is in an apparently random direction; in fact the distribution converges to a suitably scaled normal distribution.

where

$$\omega(x) = \omega_1 + \omega_2 \cos x + \omega_3 \cos^2 x + \omega_4 \sin x + \omega_5 \sin^2 x + \omega_6 \cos x \sin^2 x$$

and

$$b(x) = b_1 + b_2 \cos x + b_3 \cos^2 x + b_4 \sin x.$$

Note that the base transformation is strongly chaotic, and we pick the coefficients arbitrarily as

$$\begin{array}{l} (\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6) = (0.1, 0.2, 0.3, 0.4, 0.5, 0.6) \\ (b_1, b_2, b_3, b_4) = (0.1, 0.2, 0.5, -0.2). \end{array} \tag{2.3}$$

Note that x is the base transformation, θ is a coordinate in $\mathbf{SO}(2)$ acting in the standard way on $(v_1, v_2) \in \mathbb{R}^2$. Taking 300 initial conditions identical in v_1, v_2 and θ but differing in x, a behaviour typical of a random walk is observed, as shown in Figure 1. The ensemble mean motion in (v_1, v_2) is zero and the variance grows linearly with time as shown in Figure 2.

Example of an $\mathbf{SE}(3)$-extension By contrast, consider the $\mathbf{SE}(3)$-extension defined by

$$\begin{array}{rcl} x' & = & 4x(1-x) \\ \mathbf{A}' & = & \mathbf{A}\mathbf{R}(\phi(x), \mathbf{n}(x)) \\ \mathbf{v}' & = & \mathbf{v} + \mathbf{A}\mathbf{b}(x) \end{array} \tag{2.4}$$

where $\mathbf{A} \in \mathbb{R}^{3\times 3}$, $\mathbf{v}, \mathbf{b} \in \mathbb{R}^3$. $\mathbf{R}$ is a family of transformations in $\mathbf{SO}(3)$ parametrized by the rotation angle $\phi \in \mathbb{R}$ and unit vector axis $\mathbf{n} \in \mathbb{R}^3$, namely $\mathbf{R}(\phi, \mathbf{n}) = \mathrm{Id} + \sin\phi N + (1 - \cos\phi)N^2$ where

$$N = \left(\begin{array}{ccc} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{array} \right).$$

Setting $\mathbf{b} = (1, 0, 0)$, $\mathbf{n}(x)$ is the unit vector in the direction $\mathbf{m} = (1, 0.2 \sin 2\omega(x), 0.2 \cos\omega(x))$, $\omega(x)$ as in (2.3) and $\phi = 2$, we numerically iterated the map (2.4) starting from a random initial condition in x, $\mathbf{A} = \mathrm{Id}$ and $\mathbf{v} = \mathbf{0}$. The expected random walk behaviour of a typical trajectory is shown in Figure 3. Observe that the extension to $\mathbf{SO}(3)$ seems to be ergodic, implying that the mean drift is zero.

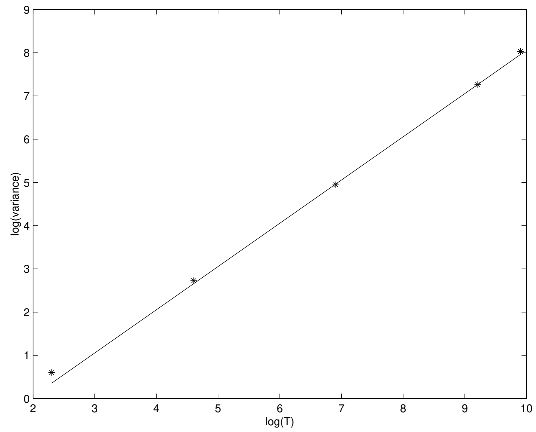
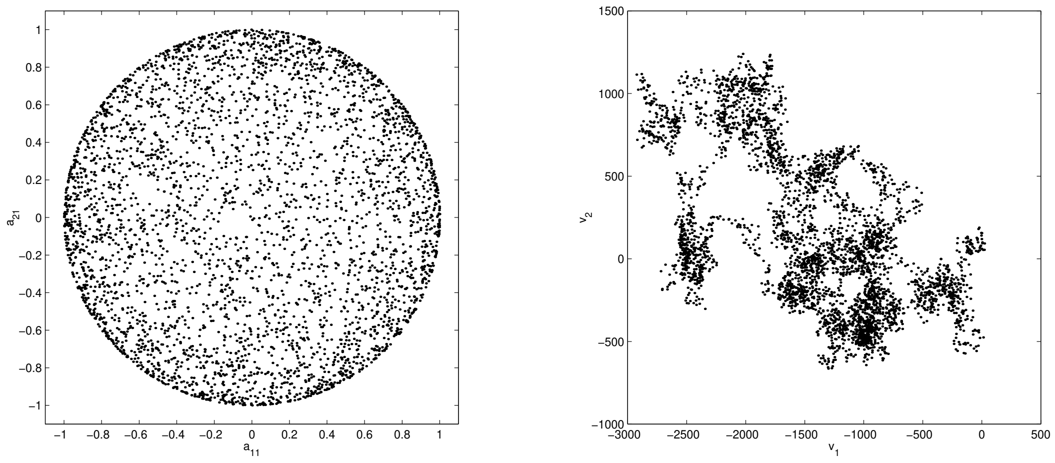


Figure 2: Variance of distance travelled versus time for an ensemble of paths for the extension with chaotic base transformation (2.2) after 10,000 timesteps. As expected, the variance grows linearly with time as in a standard random walk.



(a) (b)

Figure 3: Projection of a single trajectory of the $\mathbf{SE}(3)$-extension (2.4) plotted every 1000 iterates (a) in the (a_{11}, a_{21}) plane and (b) in the (v_1, v_2) plane. The first figure suggests that the $\mathbf{SO}(3)$-extension is ergodic; the second figure illustrates the Brownian motion behaviour in $\mathbb{R}^3$.

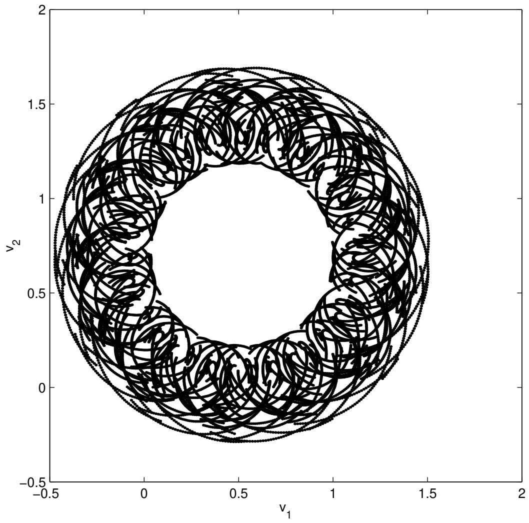


Figure 4: The paths in (v_1, v_2) of a typical initial condition for 20,000 iterates of the extension with quasiperiodic base transformation (3.1). For these parameter values, we have the typical behaviour for a smooth extension of a quasiperiodic base transformation. For all initial conditions, trajectories remain bounded.

3 Euclidean extension with quasiperiodic forcing

In [13] we also consider the behaviour of generic Euclidean extensions of quasiperiodic dynamics. The behaviour appears to depend on (a) the Diophantine properties of the quasiperiodicity and (b) the smoothness of the extension. Suppose that $X = T^m$ is a torus; we consider flows $\dot{\theta} = \alpha$ with $\theta \in T^m$. A typical result is the following:

Theorem 3.1 *([13]) Suppose $\dot{\theta} = \alpha$ is a flow on a torus T^m. For almost every α and almost every smooth* $\mathbf{SE}(2)$*-extension the dynamics is bounded. More precisely, let b_0 be the 0^{th} coefficient of the Fourier expansion of the skewing function into* $\mathbf{SO}(2)$*. If α and b_0 satisfy certain Diophantine conditions (which they do for a full measure set in $\mathbb{R}^2$) then all trajectories are bounded.*

In the non-Diophantine (and also in the non-smooth) case, one can show that that trajectories can be unbounded for generic sets of extensions. The results on unboundedness in the non-Diophantine case hold more generally for $\mathbf{SE}(n)$-extensions of flows and maps, provided $n = 2$ or n is odd. However, the boundedness results are presently restricted to $\mathbf{SE}(2)$-extensions of flows.

(a) Numerical example with quasiperiodic base

As an example of a $\mathbf{SE}(2)$-extension of a quasiperiodic system we consider the case where the logistic map in (3.1) is replaced by an irrational rotation:

$$\begin{array}{rcl} x' &=& x + \alpha \ (\mathrm{mod}\, 2\pi) \\ \theta' &=& \theta + \omega(x) \\ v_1' &=& v_1 + b(x)\cos\theta \\ v_2' &=& v_2 + b(x)\sin\theta \end{array} \tag{3.1}$$

The same parameter values (2.3) were used and $\alpha = 2\pi\gamma$ where γ is the Golden mean. Observe in Figure 4 that bounded motion is seen for all initial conditions.

SE(3)**-extensions of quasiperiodic base.** Using the same family of skewing functions we have looked at **SE**(3)-extensions of $x \mapsto x + \alpha$ with $\alpha = 2\pi\gamma$ for a variety of parameter values ϕ, ω and b. It is remarkable that for every extension we explored the **SO**(3)-extension was non-ergodic, typically leading to a linear drift.

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