
Volume 21, Number 8, August 2008
ISSN 0893-9659
ELSEVIER
Applied Mathematics Letters
an international journal of rapid publication
Editorial Board
Ravi P. Agarwal
R. Ahlswede
Wolfgang Alt
A. Ambrosetti
H. T. Banks
Claude Bardos
A. Bensoussan
Paul Boggs
F. Brezzi
P. L. Butzer
C. Cergignani
Nick Cercone
S. Bart Childs
Bruno Codenotti
Thomas F. Coleman
Jerome R. Cox Jr
R. Dautray
A. W. M. Dress
I. Ekeland
B. Fuchssteiner
S. Gelfand
James Glimm
K. Glover
Roland Glowinski
I. Gohberg
Andrew Goldberg
R. L. Graham
G. J. Habetler
M. Iri
Rudolf E. Kalman
Donald E. Knuth
H. K. Kuiken
A. Kurzhanski
T. J. Laffey
Leslie Lamport
Peter D. Lax
G. Lebon
Mark Levi
Pierre Louis Lions
William F. Lucas
P. Markowich
Bernard J. Matkowsky
Gordon McCalla
A. Nachman
J.R. Ockendon
J. Tinsley Oden
Steven A. Orszag
Eduardo L. Ortiz
L. Pandolfi
Jong-Shi Pang
N. Papamichael
L. A. Peletier
Ervin Y. Rodin
Donald J. Rose
Thomas L. Saaty
Jean Claude Saut
Martin H. Schultz
Denis Serre
R. P. Srivastav
R. Temam
Samuel C. C. Ting
Joseph F. Traub
Alan Tucker
Paul Michael Van Dooren
Graeme C. Wake
Tarynn M. Witten
Lotfi A. Zadeh
NOW included in your subscription:
ELECTRONIC ACCESS
www.elsevier.com/locate/elecacc

Available online at www.sciencedirect.com

ScienceDirect

Applied Mathematics Letters 21 (2008) 861–865

Applied Mathematics Letters

www.elsevier.com/locate/aml

A controllability approach to shape identification

R. Azencott[a], R. Glowinski[a], A.M. Ramos[b,*]

[a] *Department of Mathematics, University of Houston, Houston, TX 77204-3476, USA*

[b] *Departmento de Matemática Aplicada, Universidad Complutense de Madrid, Plaza de Ciencias, 3, 28040, Spain*

Received 16 August 2007; accepted 16 August 2007

Abstract

The main goal of this work is to discuss a controllability approach to the image matching/shape identification problem, an important issue in many applications, medical ones in particular. The matching problem is formulated as an approximate controllability problem involving a cost functional whose gradient is computed using an adjoint equation based methodology. The time discrete version of the image matching problem is also discussed in this work.

Keywords: Shape identification; Image matching; Group of diffeomorphisms; Controllability; Adjoint equation; Time discrete controllability

1. Introduction

Automatic registration of image pairs pervades artificial vision, starting with stereovision, and optical flow in video sequences. In biomedical imaging, automatic matching for 3D images of soft organs across different subjects is an important step, and quantifying dissimilarities between organ shapes impacts clinical diagnosis. Nonlinear image registration relies on minimizing cost functionals combining two terms, smoothness and disparity. Similar strategies for "space warping" of 3D images yield robust comparisons of soft shapes. A powerful mathematical approach, linked to geodesics in infinite dimensional Lie groups of diffeomorphisms of $\mathbb{R}^3$, has been successfully explored for soft shape matching [1–4].

2. Optimal flows of infinitesimal deformations

For incompressible fluids, Arnold [5] showed that, if $\mathbf{f}_t(x)$ is the position at time t of a fluid particle starting at x, then the map $t \to \mathbf{f}_t(x)$ defines a geodesic in the group of diffeomorphisms of $\mathbb{R}^3$, for the metric defined by the integral in time and space of the fluid kinetic energy. In the past 10 years, geodesics in groups of diffeomorphisms have provided a fertile framework for optimal matching of curves and surfaces by diffeomorphisms with minimal "energy" [1–4]. Call $\mathcal{U}$ the space of "infinitesimal deformation" flows $\mathbf{v} : t \to \mathbf{v}_t$, $0 \le t \le 1$, where each $\mathbf{v}_t$ is a

* Corresponding author. Tel.: +34 91 394 44 80; fax: +34 91 394 46 13.
E-mail addresses: razencot@math.uh.edu (R. Azencott), roland@math.uh.edu (R. Glowinski), angel@mat.ucm.es (A.M. Ramos).

0893-9659/$ - see front matter © 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2007.08.013

vector field null at infinity in $\mathbb{R}^3$, belonging to a Hilbert space H (e.g. a Sobolev space H^s with $s > 5/2$), with strong Lipschitz continuity in t, and finite "kinetic energy" $J(v)$ defined by

$$J(\mathbf{v}) = \frac{1}{2}\int_0^1 |\mathbf{v}_t|^2 \mathrm{d}t, \quad \text{where } |\mathbf{v}_t| \text{ is the norm of } \mathbf{v}_t \text{ in } H.$$

Each flow $\mathbf{v}$ in $\mathcal{U}$ generates a map $t \to \mathbf{f}_t$, $0 \le t \le 1$, into the group of diffeomorphisms of $\mathbb{R}^3$, by integration over t of the O.D.E.

$$\partial_t \mathbf{f}_t(x) = \mathbf{v}_t(\mathbf{f}_t(x)), \quad \text{with the initial condition } \mathbf{f}_0(x) = x, \text{ for all } x \in \mathbb{R}^3. \tag{1}$$

As shown in [4], the set $\mathbf{G}$ of all $g = \mathbf{f}_1 : \mathbb{R}^3 \to \mathbb{R}^3$ thus reachable at $t = 1$ by integrating arbitrary flows $\mathbf{v}$ in $\mathcal{U}$ is a group $\mathbf{G}$ of diffeomorphisms of $\mathbb{R}^3$. Let $\mathbf{e}$ be the identity map of $\mathbb{R}^3$, and define for all g in $\mathbf{G}$

$$D^2(\mathbf{e}, \mathbf{g}) = \text{ minimal kinetic energy } J(\mathbf{v}) \text{ over all flows } \mathbf{v} \text{ in } \mathcal{U} \text{ such that } \mathbf{f}_1 = \mathbf{g}.$$

Then, for each $\mathbf{g} \in \mathbf{G}$, there exists a minimizing flow $\mathbf{v}$ in $\mathcal{U}$ for which $\mathbf{f}_1 = \mathbf{g}$ and $J(\mathbf{v}) = D^2(\mathbf{e}, \mathbf{g})$. The corresponding map $t \to \mathbf{f}_t$, $0 \le t \le 1$, is a geodesic linking $\mathbf{e}$ to $\mathbf{g}$ in the group $\mathbf{G}$, for the metric defined by the left-invariant distance $D(\mathbf{g}, \mathbf{h})$ given on G by

$$D(\mathbf{g}, \mathbf{h}) = D(\mathbf{e}, \mathbf{g}^{-1}\mathbf{h}) \quad \text{for all } \mathbf{g}, \mathbf{h} \text{ in } \mathbf{G}.$$

3. Optimal matching of soft shapes

Call $\mathbf{S}$ the set of "soft shapes" modelled by piecewise smooth compact surfaces Γ in $\mathbb{R}^3$. As in [2,3], consider each Γ in $\mathbf{S}$ as the support of a natural positive measure $M(\Gamma)$ of mass 1, well approximated by linear combinations of Dirac measures located at the nodes of increasingly finer meshes on Γ. Let $Q(y - x)$ be a positive definite kernel of class C^2 on $\mathbb{R}^3 \times \mathbb{R}^3$, null at infinity, as well as its derivatives of orders 1 and 2. Endow the space of bounded measures BM with the Hilbert scalar product

$$\langle \mu_1, \mu_2 \rangle_Q = \int\int Q(y - x)\mathrm{d}\mu_1(x)\mathrm{d}\mu_2(y), \quad \text{and associated norm } \|\mu\|_Q.$$

Diffeomorphisms $\mathbf{f}$ of $\mathbb{R}^3$ act naturally on BM by standard transport of measures. For optimal matching of two given shapes Γ_0 and Γ_{R} in $\mathbb{R}^3$, the geometric disparity between $\mathbf{f}(\Gamma_0)$ and Γ_{R} can be defined as

$$\mathrm{Disp}(\mathbf{f}) = \|\mathbf{f}(M_0) - M_{\mathrm{R}}\|_Q^2, \quad \text{where } M_0 = M(\Gamma_0) \text{ and } M_{\mathrm{R}} = M(\Gamma_{\mathrm{R}}). \tag{2}$$

Then there exists [2,3] a flow $\mathbf{v}$ in $\mathcal{U}$ solving the minimizing problem

$$\text{minimize } \mathrm{Cost}(\mathbf{v}) = J(\mathbf{v}) + \mathrm{Disp}(\mathbf{f}_1) = \text{ kinetic energy } + \text{ final disparity}, \quad \text{for } \mathbf{v} \text{ in } \mathcal{U}.$$

The final diffeomorphism $\mathbf{f}_1$ of $\mathbb{R}^3$ reached at time 1 by integration of a minimizing $\mathbf{v}$ is an optimal matching between shapes Γ_0 and Γ_{R}. Replacing the kernel Q by $\varepsilon^{-1}Q$ with ε tending to 0 shows [2–4] the existence of a flow $\mathbf{v}$ in $\mathcal{U}$ solving the constrained minimization problem

$$\text{minimize } J(\mathbf{v}) \text{ for } \mathbf{v} \text{ in } \mathcal{U}, \text{ under the constraint } \mathbf{f}_1(\Gamma_0) = \Gamma_{\mathrm{R}}. \tag{3}$$

The minimal value of $J(\mathbf{v})$ then defines the square of a natural distance $D(\Gamma_0, \Gamma_{\mathrm{R}})$ between shapes Γ_0 and Γ_{R}. These approaches have been efficiently applied [1,3] to databases of 3D MRI images of human brains, in the context of non-invasive medical diagnosis.

4. A controllability approach

The variational problems just presented above are approximate controllability problems in Hilbert spaces, in the sense of J.L. Lions. We apply to their numerical solutions the rich computational methodology developed for classical controllability problems for systems governed by partial differential equations (see [6]). We show how these controllability techniques can address the solution of the shape comparison problems. For simplicity we focus our discussion on curves in $\mathbb{R}^2$ but our approach is easily extended to piecewise smooth surfaces in $\mathbb{R}^3$.

Define now the space $\mathcal{U}$ of vector field flows $\mathbf{v}$ on $\mathbb{R}^2$ by $\mathcal{U} = L^2(0,1;\mathbf{V}_\alpha)$ where $\mathbf{V}_\alpha$ is the Sobolev space $(H^\alpha(\mathbb{R}^2))^2$, endowed with the norm $\|\cdot\|_\alpha$ with $\alpha \geqslant 2$. Let Γ_0 and Γ_{R} be piecewise smooth bounded curves in $\mathbb{R}^2$. As seen above, one can define a natural distance $D(\Gamma_0, \Gamma_{\mathrm{R}})$ between Γ_0 and Γ_{R} by $D(\Gamma_0,\Gamma_{\mathrm{R}}) = [J(\mathbf{u})]^{1/2}$, where $\mathbf{u} \in \mathcal{U}$ solves the minimization problem (3), which we restate as

$$\text{minimize } J(\mathbf{v}) = \frac{1}{2}\int_0^2 |\mathbf{v}_t|_\alpha^2 \mathrm{d}t \quad \text{for } \mathbf{v} \text{ in } \mathcal{U}, \text{ under the constraint } \mathbf{f}_1(\Gamma_0) = \Gamma_{\mathrm{R}}, \tag{4}$$

where $\mathbf{f}_t$ is solution of the ODE (1) associated with the vector field flow $\mathbf{v}$. The minimization problem (4) is an exact controllability problem in the sense of [6], with $\mathbf{v}$ being the control variable, $\mathcal{U}$ the control set, $\mathbf{f}$ the state variable and (1) the state equation.

Following [6] we relax the condition $\mathbf{f}_1(\Gamma_0) = \Gamma_{\mathrm{R}}$ and approximate the constrained minimization problem (4) by the *penalized unconstrained problem*

$$\text{minimize } J_\varepsilon(\mathbf{v}) = J(\mathbf{v}) + \frac{1}{2\varepsilon}\mathrm{Disp}(\mathbf{f}_1) \quad \text{for } \mathbf{v} \text{ in } \mathcal{U}, \tag{5}$$

where $\varepsilon > 0$ is a small parameter. The disparity term $\mathrm{Disp}(\mathbf{f}_1)$ is analogous to (2) above and is computed as follows. Let the self-adjoint operator S be a duality isomorphism between $H^2(\mathbb{R}^2)$ and $H^{-2}(\mathbb{R}^2)$. Call $\langle\cdot,\cdot\rangle$ the duality pairing between these two spaces. Define $\Gamma_1 = \mathbf{f}_1(\Gamma_0)$. Let ψ_1, ψ_R be the unique solutions in $H^2(\mathbb{R}^2)$ of the following *linear variational problems*:

$$\langle S\psi_1, \varphi\rangle = \int_{\Gamma_1} \varphi \, \mathrm{d}\Gamma_1, \qquad \langle S\psi_R, \varphi\rangle = \int_{\Gamma_R} \varphi \, \mathrm{d}\Gamma_R, \quad \forall \varphi \in H^2(\mathbb{R}^2).$$

We then compute $\mathrm{Disp}(\mathbf{f}_1)$ as

$$\mathrm{Disp}(\mathbf{f}_1) = \frac{1}{2\varepsilon}\langle S(\psi_1 - \psi_R), \psi_1 - \psi_R\rangle \quad \text{for } \mathbf{v} \text{ in } \mathcal{U}.$$

Proofs analogous to those of [2,4] show the existence of $\mathbf{u}_\varepsilon$ in $\mathcal{U}$ minimizing (5) and of the differential $DJ_\varepsilon(\mathbf{u}_\varepsilon)$ of $J_\varepsilon(\cdot)$ at $\mathbf{u}_\varepsilon$, so that $\mathbf{u}_\varepsilon$ verifies the optimality condition

$$DJ_\varepsilon(\mathbf{u}_\varepsilon) = 0.$$

To solve by iterative methods the approximate problem (5), one needs to compute $DJ_\varepsilon(\mathbf{v})$ for $\mathbf{v}$ in $\mathcal{U}$ (other approaches are possible, like those based on automatic differentiation). The computation of $DJ_\varepsilon(\mathbf{v})$ is sketched in Section 5.

5. On the computation of $DJ_\varepsilon(\mathbf{v})$

In the following, $\langle\cdot,\cdot\rangle$ denotes various duality pairings (details omitted here). For an arbitrary initial point x in $\mathbb{R}^2$ we define $\mathbf{y}(t) = \mathbf{y}(x,t) = \mathbf{f}_t(x)$. Using classical perturbation techniques (see [6]), we can show with obvious notation that

$$\langle DJ_\varepsilon(\mathbf{v}), \mathbf{w}\rangle = \int_0^1 (\mathbf{v}_t, \mathbf{w}_t)_\alpha \mathrm{d}t + \int_0^1 \langle \mathbf{p}_t, \mathbf{w}_t(\mathbf{y}_t)\rangle \mathrm{d}t, \quad \forall \mathbf{v}, \mathbf{w} \in \mathcal{U}, \tag{6}$$

where the vector-valued function $\mathbf{p} = \mathbf{p}_t(x)$ is solution of the adjoint equation

$$-\frac{\partial \mathbf{p}}{\partial t} - \nabla_x \mathbf{v}(\mathbf{y}(t), t)^{\mathrm{t}}\mathbf{p} = 0 \quad \text{in } \mathbb{R}^2 \times (0,1) \quad \text{and} \quad \mathbf{p}_1 = \mathbf{q}, \tag{7}$$

with the distribution $\mathbf{q}$ defined by

$$\begin{aligned}\langle \mathbf{q}, \mathbf{z}\rangle = \frac{1}{\varepsilon}\Bigg[&\int_0^L \nabla(\psi_1 - \psi_R)\,(\mathbf{y}(\mathbf{x}_0(s),1)) \cdot \mathbf{z}(\mathbf{x}_0(s))\,|\mathbf{T}_1(s)|\,\mathrm{d}s \\ &+ \int_0^L (\psi_1 - \psi_R)\,(\mathbf{y}(\mathbf{x}_0(s),1))\,\mathbf{T}_1(s) \cdot \nabla_x \mathbf{z}(\mathbf{x}_0(s))\frac{\mathrm{d}\mathbf{x}_0}{\mathrm{d}s}(s)\,|\mathbf{T}_1(s)|^{-1}\,\mathrm{d}s\Bigg], \quad \forall \mathbf{z} \in (\mathcal{D}(\mathbb{R}^2))^2,\end{aligned} \tag{8}$$

where $s \to x_0(s)$, $0 < s < L$, is the arc length parametrization of Γ_0, and each vector $\mathbf{T}_1(s)$ is tangent to Γ_1 at the point $\mathbf{y}(\mathbf{x}_0(s), 1)$, and defined by

$$\mathbf{T}_1(s) = \nabla_x \mathbf{y}(\mathbf{x}_0(s), 1)\frac{\mathrm{d}\mathbf{x}_0}{\mathrm{d}s}(s).$$

After appropriate space–time discretization, the discrete analogues of (6)–(8) can be used to compute an approximation of $\mathbf{u}_\varepsilon$, via conjugate gradient or BFGS algorithms. In the following section we develop the time discretization, which is usually the most delicate part.

6. Time discretization

We consider the time discretization step Δt, defined by $\Delta t = 1/N$, where N is a positive integer. Then, if we denote $n\Delta t$ by t^n, we have $0 < t^1 < t^2 < \cdots < t^N = 1$. Now, we approximate $\mathcal{U}$ by $\mathcal{U}^{\Delta t} = (V_\alpha)^N$. Then, we approximate the minimization problem (5) by

$$\text{minimize } J_\varepsilon^{\Delta t}(\mathbf{v}^{\Delta t}),$$

with $\mathbf{v}^{\Delta t} = (v^n)_{n=1}^N$,

$$J_\varepsilon^{\Delta t}(\mathbf{v}^{\Delta t}) = \frac{\Delta t}{2}\sum_{n=1}^{N} |\mathbf{v}^n|^2 \mathrm{d}t + \frac{1}{2\varepsilon}\langle S(\psi_1^{\Delta t} - \psi_R), \psi_1^{\Delta t} - \psi_R\rangle$$

and $\psi_1^{\Delta t}$ is defined by the unique solution in $H^2(\mathbb{R}^2)$ of the following *linear variational problem*:

$$\langle S\psi_1^{\Delta t}, \varphi\rangle = \int_{\Gamma_1} \varphi \,\mathrm{d}\Gamma_1^{\Delta t}, \quad \forall \varphi \in H^2(\mathbb{R}^2).$$

Here, $\Gamma_1^{\Delta t} = \{\mathbf{y}^N(x) : x \in \Gamma_0\}$ and $\mathbf{y}^{\Delta t} = (\mathbf{y}^n)_{n=0}^N$ is the solution of

$$\begin{cases} \mathbf{y}^0(x) = x, \quad \text{for all } x \in \mathbb{R}^2. \text{ For } n = 1, \ldots, N,\ \mathbf{y}^n \text{ is computed by} \\ \dfrac{\mathbf{y}^n - \mathbf{y}^{n-1}}{\Delta t} = \mathbf{v}^n(\mathbf{y}^{n-1}) \quad \text{in } \mathbb{R}^2. \end{cases}$$

Finally, endowing the space $\mathcal{U}^{\Delta t}$ with the Hilbert scalar product $(\cdot,\cdot)_{\Delta t}$ defined by

$$(\mathbf{v}^{\Delta t}, \mathbf{w}^{\Delta t})_{\Delta t} = \Delta t \sum_{n=1}^{N} (v^n, w^n)_\alpha$$

and using again classical perturbation techniques, we can show with obvious notation that

$$\langle DJ_\varepsilon^{\Delta t}(\mathbf{v}^{\Delta t}), \mathbf{w}^{\Delta t}\rangle = (\mathbf{v}^{\Delta t}, \mathbf{w}^{\Delta t})_{\Delta t} + \Delta t \sum_{n=1}^{N} \langle \mathbf{p}^n, \mathbf{w}^n(\mathbf{y}^{n-1})\rangle, \quad \forall \mathbf{v}^{\Delta t}, \mathbf{w}^{\Delta t} \in \mathcal{U}^{\Delta t},$$

where $\{\mathbf{p}^n\}_{n=1}^{N+1} = \{\mathbf{p}^n(x)\}_{n=1}^{N+1}$ is solution of the *time discrete adjoint equation*

$$\begin{cases} \mathbf{p}^{N+1} = \mathbf{q} \quad \text{and for } n = N, N-1, \ldots, 1, \mathbf{p}^n \text{ is computed by} \\ -\dfrac{\mathbf{p}^{n+1} - \mathbf{p}^n}{\Delta t} - \nabla \mathbf{v}^{n+1}(\mathbf{y}^n)^{\mathrm{t}} \mathbf{p}^{n+1} = 0 \quad \text{in } \mathbb{R}^2, \end{cases}$$

with the distribution $\mathbf{q}$ defined by

$$\begin{aligned} \langle \mathbf{q}, \mathbf{z}\rangle = \frac{1}{\varepsilon}\Bigg[&\int_0^L \nabla(\psi_1^{\Delta t} - \psi_R)\left(\mathbf{y}^N(\mathbf{x}_0(s))\right) \cdot \mathbf{z}(\mathbf{x}_0(s)) \left|\mathbf{T}_1^{\Delta t}(s)\right| \mathrm{d}s \\ &+ \int_0^L \left(\psi_1^{\Delta t} - \psi_R\right)\left(\mathbf{y}^N(\mathbf{x}_0(s))\right) \mathbf{T}_1^{\Delta t}(s) \cdot \nabla \mathbf{z}(\mathbf{x}_0(s)) \frac{\mathrm{d}\mathbf{x}_0}{\mathrm{d}s}(s) \left|\mathbf{T}_1^{\Delta t}(s)\right|^{-1} \Bigg] \mathrm{d}s, \quad \forall \mathbf{z} \in (\mathcal{D}(\mathbb{R}^2))^2, \end{aligned}$$

with

$$\mathbf{T}_1^{\Delta t}(s) = \nabla \mathbf{y}^N(\mathbf{x}_0(s))\frac{\mathrm{d}\mathbf{x}_0}{\mathrm{d}s}(s).$$

Acknowledgements

A.M. Ramos was partially supported by the Spanish "Dirección General de Universidades e Investigación de la Conserjería de Educación de la Comunidad de Madrid" and the "Universidad Complutense de Madrid" in Spain through the CCG06-UCM/ESP-1110 project.

References

[1] P. Dupuis, U. Grenander, M. Miller, Variational problems on flows of diffeomorphisms for image matching, Quart. Appl. Math. 56 (3) (1998) 587–600.
[2] J. Glaunes, A. Trouve, L. Younes, Diffeomorphic matching of distributions: A new approach for unlabelled point-sets and sub-manifolds matching, in: IEEE Conference on Computer Vision and Pattern Recognition, CVPR, Washingtion, D.C., 2004.
[3] M.I. Miller, A. Trouve, L. Younes, Computing large deformations via geodesic flows of diffeomorphisms, Int. J. Comput. Vis. 61 (2) (2005) 139–157.
[4] A. Trouve, Diffeomorphisms groups and pattern matching in image analysis, Int. J. Comput. Vis. 28 (3) (1998) 213–221.
[5] V.I. Arnold, Groupes de Lie de dimension infinie et hydrodynamique des fluides, Ann. Inst. Fourier (Grenoble) 1 (1966) 319–361.
[6] R. Glowinski, J.L. Lions, Exact and approximate controllability for distributed parameter systems, Part I, Acta Numer. (1994) 269–378; Part II, Acta Numer. (1995) 159–333.