EVERY REAL ELLIPSOID IN $\mathbb{C}^2$ ADMITS CR UMBILICAL POINTS

XIAOJUN HUANG AND SHANYU JI

To the memory of Professor S. S. Chern

1. Introduction

By the Cartan-Chern-Moser theory [CM], the germ of a strongly pseudoconvex real analytic hypersurface $M \subset \mathbb{C}^n$ is determined, up to a local biholomorphic map, by a set of complete invariants which can be expressed by the curvatures of a connection or the coefficients in a normal form.

When $n \geq 3$, the fourth-order pseudoconformal curvature tensor $\mathcal{S}$ of Chern-Moser [CM] is of fundamental importance because it generates other invariants by differentiation. It is known that $\mathcal{S} \equiv 0$ if and only if M is locally biholomorphic to the sphere $\partial\mathbb{B}^n$. When $n = 2$, the fourth-order curvature tensor vanishes identically and its role is played by the Cartan six-order invariant curvature tensor $\mathcal{P}$ [Car]. Similarly, $\mathcal{P} \equiv 0$ if and only if M is locally biholomorphic to the 3-sphere $\partial\mathbb{B}^2$. In both cases, a point on M, at which the Chern-Moser tensor $\mathcal{S}$ (or the Cartan curvature tensor $\mathcal{P}$ for the case of $n = 2$) vanishes, is called a CR umbilical point, or briefly, an umbilical point ([CM]). CR umbilical points are biholomorphic differential invariants of M.

The study of CR umbilical points on a compact strongly pseudoconvex hypersurface M provides useful information for the holomorphic structure of its enclosed domain, as well as the intrinsic CR structure of M itself. However, different from the situation in the classical Differential Geometry, except in the trivial spherical case, where $\mathcal{S}$ or $\mathcal{P} \equiv 0$, computing umbilical points seems to be a very difficult problem. This is because the explicit formula for the fundamental Cartan-Chern-Moser curvature tensions is too complicated. Indeed, the situation is already non-trivial even in the simplest non-spherical case— where M is a real ellipsoid. Recently, based on his previous work on the complex dynamics property of real ellipsoids, Webster proved the following: (See §3 for the definitions)

Theorem 1.1. *(Webster [We2]): A generic real ellipsoid in $\mathbb{C}^n$ with $n \geq 3$ does not admit any umbilical point.*

Umbilical points on a certain class of real hypersurfaces of revolution were also studied by Webster [We3].

A natural question arising from [We2] is then to ask whether a generic real ellipsoid in $\mathbb{C}^2$ shares the same property as its analogy in higher dimensions. It is indeed this problem that motivated our present work, and we provide, in this paper, the following:

Theorem 1.2. *Every real ellipsoid $M \subset \mathbb{C}^2$ admits at least four umbilical points.*

The first author was supported in part by NSF-0200689.

Theorem 1.2 resembles the classical result for the umbilical points on the ellipsoids in $\mathbb{R}^3$ [pp222, Spv]. A famous theorem of Hamburger [Ham] states that every compact real analytic convex surface in $\mathbb{R}^3$ admits at least two umbilical points. We do not know if there is a CR version of the Hamburger theorem. More precisely, it is an open question to us if every compact strongly convex hypersurface in $\mathbb{C}^2$ admits at least two CR umbilical points. Notice that only for $n = 2$, the fundamental curvature tension reduces to a function. It may not be surprising that it is more likely to find umbilical points on a hypersurface in $\mathbb{C}^2$ than to find umbilical points for a hypersurface in $\mathbb{C}^n$ $(n \geq 3)$.

The proof of Theorem 1.2 uses Chern's inhomogeneous coordinates for the projective G-structure bundle of the Segre family of a real analytic strongly pseudoconvex hypersurface [Ch] [CJ], and a formula derived in Huang-Ji-Yau [Theorem 3.1, HJY] for the complexified Cartan fundamental curvature tension represented under these coordinates. The formula of [HJY] seems to fit particularly well with the computation here.

In the classical Differential Geometry [Spv], surfaces in $\mathbb{R}^3$ without umbilical points must be diffeomorphic to a torus. The boundary of a small thickening of the unit circle in $\mathbb{R}^2$ provides examples of closed surfaces without any umbilical point. However, this type of examples does not give compact CR manifolds without CR umbilical points. The following theorem gives a precise description for the set of umbilical points for the thickening of a closed real curve. It is not clear to us if there is any embeddable three dimensional compact CR manifold which has no CR umbilical points.

Theorem 1.3. *Let $M_\epsilon \subset \mathbb{C}^2$ be the boundary of the ϵ-thickening of the unit circle $\{|z| = 1, w = 0\}$in $\mathbb{C}^2$, defined by the equation $(1 - |z|)^2 + |w|^2 = \epsilon^2$, where ϵ is a sufficiently small positive number. Then the set of all umbilical points of M_ϵ forms a disjoint union of a closed real analytic curve and two two-dimensional totally real analytic tori.*

Acknowledgment This work was carried out when both authors were enjoying their pleasant and fruitful stay at the School of Mathematical Sciences, Wuhan University, China, in the summer of 2004. The authors are indebted to Professor Hua Chen for his effort and arrangement, which made the visit possible. The authors also would like very much to thank S. Webster for his many stimulating conversations related to the paper.

2. Umbilical Points of Real Hypersurfaces In $\mathbb{C}^2$

In this section, we briefly review the Cartan-Chern-Moser theory (cf. [C][HJ] [Hu]). We restrict ourselves to the case of $n = 2$. Let

$$M = \{(z, w) \in \mathbb{C}^2 : \ r(z, w, \overline{z}, \overline{w}) = 0\} \tag{2.1}$$

be a Levi non-degenerate smooth real analytic hypersurface with $(z_0, w_0) \in M$. Its complexification, called the *Segre family* of M, is then the complex three-fold

$$\mathcal{M} = \{(z, w, \zeta, \eta) \mid r(z, w, \zeta, \eta) = 0\} \subset \mathbb{C}^4.$$

Clearly $(z_0, w_0, \overline{z_0}, \overline{w_0}) \in \mathcal{M}$. Assume that

$$r_w(z_0, w_0, \overline{z_0}, \overline{w_0}) := \frac{\partial r}{\partial w}(z_0, w_0, \overline{z_0}, \overline{w_0}) \neq 0. \tag{2.2}$$

Define

(2.3)
$$S : \mathcal{M} \to \widetilde{\mathcal{M}} := S(\mathcal{M}) \subset \mathbb{C}^2 \times \mathbb{P}^1, \ \ (z, w, \zeta, \eta) \mapsto \left(z, w, \left[\frac{\partial r}{\partial z} : \frac{\partial r}{\partial w} \right] (z, w, \zeta, \eta) \right).$$

S is locally biholomorphic by the Levi non-degeneracy condition. (See Proposition 4.1 of [CJ]). With the assumption in (2.2), we can regard (z, w, ζ) as a local non-homogeneous coordinates system for $\mathcal{M}$, and we can write $S(z, w, \zeta) = (z, w, -\frac{r_z}{r_w})$. Then we use (z, w, p) as a local coordinates system, called the Chern coordinates system, for $\widetilde{\mathcal{M}}$, where

$$p = -\frac{r_z}{r_w}. \tag{2.4}$$

Making use of the implicit function theorem, we can find a unique holomorphic function (in its argument) $h(z, \overline{z}, \overline{w})$ near $(z_0, \overline{z_0}, \overline{w_0})$ with $h(z_0, \overline{z_0}, \overline{w_0}) = w_0$ such that $w = h(z, \overline{z}, \overline{w})$ solves the equation: $r(z, w, \overline{z}, \overline{w}) = 0$. Then for $(z, w, \zeta, \eta) \in \mathcal{M}$, we have

$$p(z, w, \zeta) = \frac{\partial h(z, \zeta, \eta)}{\partial z}. \tag{2.5}$$

We have $dp = p_{11} dz + \hat{p}_1^1 d\zeta + \hat{p}_1^2 d\eta$, where $p_{11} = \frac{\partial^2 h}{\partial z^2}$, $\hat{p}_1^1 = \frac{\partial^2 h}{\partial \zeta \partial z}$ and $\hat{p}_1^2 = \frac{\partial^2 h}{\partial \eta \partial z}$; and we have the following identity:

$$-p dz + dw - \hat{p}^1 d\zeta - \hat{p}^2 d\eta = 0 \tag{2.6}$$

where $p = p_1 = \frac{\partial h}{\partial z}$, $\hat{p}^1 = \frac{\partial h}{\partial \zeta}$ and $\hat{p}^2 = \frac{\partial h}{\partial \eta}$. Therefore, we obtain

$$dp|_{\mathcal{M}} = \left(p_{11} - \frac{\hat{p}_1^2 p_1}{\hat{p}^2} \right) dz + \frac{\hat{p}_1^2}{\hat{p}^2} dw + \left(p_1^{,1} - \frac{p_1^{,2} p^{,1}}{p^{,2}} \right) d\zeta. \tag{2.7}$$

Hence, we have the following holomorphic coframe on $\widetilde{\mathcal{M}}$:

$$\begin{aligned} \theta &= dw - p dz = dw - p_1 dz, \\ \theta^1 &= dz, \\ \theta_1 &= = \frac{\hat{p}_1^2}{\hat{p}^2} \theta + \left(\hat{p}_1^1 - \frac{\hat{p}_1^2 \hat{p}^1}{\hat{p}^2} \right) d\zeta = dp - p_{11} dz. \end{aligned}$$

We emphasize again that p_{11} is a holomorphic function in (z, w, p) near (z_0, w_0, p_0) with $(z_0, w_0) \in M$ and $p_0 = p(z_0, w_0, \overline{z_0})$; and p_{11} is given by the following formula:

$$p_{11} = \frac{\partial^2 h}{\partial z^2} \tag{2.8}$$

Define the holomorphic coframes over $\mathcal{M}$:

$$\omega = u\theta, \ \omega^1 = u^1 \theta + u_1^1 \theta^1, \ \omega_1 = v_1 \theta + v_1^1 \theta_1 \tag{2.9}$$

where u, u_1^1, u^1, v_1 are holomorphic functions with $u = i u_1^1 v_1^1 \neq 0$.

Now, the fundamental Cartan-Chern-Moser theory [CM] gives the following:

Let $M = \{r = 0\} \subset \mathbb{C}^2$, $(z_0, w_0) \in M$ such that (2.2) is satisfied and let $\widetilde{\mathcal{M}}$ be as in (2.3). Let $\widetilde{\pi} : \widetilde{\mathcal{Y}} \to \widetilde{\mathcal{M}}$ be the corresponding holomorphic projective structure bundle. Then besides the 3 holomorphic 1-forms in (2.9), there exist 5 more holomorphic 1-forms $\phi, \phi_1^1, \phi^1, \phi_1, \ \psi$, defined over $\widetilde{\mathcal{Y}}$, with holomorphic coordinates $z, w, p,\ u, u_1^1, u^1, v_1, t$, with $u, u_1^1 \neq 0$. These holomorphic 1-forms are $\mathbb{C}$ - linearly independent, and satisfy the following structure equations

$$
\begin{aligned}
d\omega &= i\omega^1 \wedge \omega_1 + \omega \wedge \phi, \\
d\omega^1 &= \omega^1 \wedge \phi_1^1 + \omega \wedge \phi^1, \\
d\omega_1 &= \phi_1^1 \wedge \omega_1 + \omega_1 \wedge \phi + \omega \wedge \phi_1, \\
d\phi &= i\omega^1 \wedge \phi_1 + i\phi^1 \wedge \omega_1 + \omega \wedge \psi, \\
d\phi_1^1 &= i\omega_1 \wedge \phi^1 - 2i\phi_1 \wedge \omega^1 - \tfrac{1}{2}\psi \wedge \omega, \\
d\phi^1 &= \phi \wedge \phi^1 + \phi^1 \wedge \phi_1^1 - \tfrac{1}{2}\psi \wedge \omega^1 + L^{11}\omega \wedge \omega_1, \\
d\phi_1 &= \phi_1^1 \wedge \phi_1 - \tfrac{1}{2}\psi \wedge \omega_1 + P_{11}\omega \wedge \omega^1, \\
d\psi &= \phi \wedge \psi + 2i\phi^1 \wedge \phi_1 + H_1\omega \wedge \omega^1 + K^1\omega \wedge \omega_1.
\end{aligned} \tag{2.10}
$$

All of these forms $\omega, \omega^1, \omega_1, \phi, \phi_1^1, \phi^1, \phi_1$ and ψ, as well as all of the curvature functions L^{11}, P_{11}, H_1 and K^1, have been calculated explicitly in [Theorem 3.1, HJY]. In particular, we have

(2.11)

$$
\begin{aligned}
L^{11} &= -\frac{i(u_1^1)^2}{6u^3}\frac{\partial^4 p_{11}}{\partial p^4}, \\
P_{11} &= \frac{i}{u(u_1^1)^2}\left[\frac{\partial^2 p_{11}}{\partial w^2} - \frac{1}{2}\frac{\partial p_{11}}{\partial w}\frac{\partial^2 p_{11}}{\partial p^2} + \frac{2}{3}\frac{\partial p_{11}}{\partial p}\frac{\partial^2 p_{11}}{\partial p\partial w} + \frac{p_{11}}{6}\frac{\partial^3 p_{11}}{\partial p^2\partial w}\right. \\
&\quad -\frac{1}{6}\frac{\partial p_{11}}{\partial p}\left(\frac{\partial^3 p_{11}}{\partial p^2\partial z} + p\frac{\partial^3 p_{11}}{\partial p^2\partial w}\right) - \frac{2}{3}\left(\frac{\partial^3 p_{11}}{\partial z\partial w\partial p} + p_{11}\frac{\partial^3 p_{11}}{\partial p^2\partial w} + p\frac{\partial^3 p_{11}}{\partial p\partial w^2}\right) \\
&\quad +\frac{1}{6}\left(\frac{\partial^4 p_{11}}{\partial p^2\partial z^2} + p_{11}\frac{\partial^4 p_{11}}{\partial p^3\partial z} + p\frac{\partial^4 p_{11}}{\partial p^2\partial z\partial w}\right) + \frac{1}{6}\frac{\partial^3 p_{11}}{\partial p^3}\left(\frac{\partial p_{11}}{\partial z} + p\frac{\partial p_{11}}{\partial w}\right) \\
&\quad \left. +\frac{p_{11}}{6}\left(\frac{\partial^4 p_{11}}{\partial z\partial p^3} + p_{11}\frac{\partial^4 p_{11}}{\partial p^4} + p\frac{\partial^4 p_{11}}{\partial p^3\partial w}\right) + \frac{p}{6}\left(\frac{\partial^4 p_{11}}{\partial z\partial p^2\partial w} + p_{11}\frac{\partial^4 p_{11}}{\partial p^3\partial w} + p\frac{\partial^4 p_{11}}{\partial p^2\partial w^2}\right)\right].
\end{aligned}
$$

On the CR structure bundle $\hat{Y}$ over $\hat{M} = S(\{(z, w, \overline{z}, \overline{w}) : (z, w) \in M\})$, there are $\mathbb{R}$-linearly independent 1-forms $\omega, \omega^1, \overline{\omega^1}, \phi_1^1, \phi = \phi_1^1 + \overline{\phi_1^1}, \phi^1, \overline{\phi^1}, \psi$ satisfying the structure equations

$$
\begin{aligned}
d\omega &= i\omega^1 \wedge \overline{\omega^1} + \omega \wedge \phi, \\
d\omega^1 &= \omega^1 \wedge \phi_1^1 + \omega \wedge \phi^1, \\
d\phi_1^1 &= i\overline{\omega^1} \wedge \phi^1 - 2i\overline{\phi^1} \wedge \omega^1 - \tfrac{1}{2}\psi \wedge \omega, \\
d\phi^1 &= \phi \wedge \phi^1 + \phi^1 \wedge \phi_1^1 - \tfrac{1}{2}\psi \wedge \omega^1 + \hat{L}^{11}\omega \wedge \overline{\omega^1}, \\
d\psi &= \phi \wedge \psi + 2i\phi^1 \wedge \overline{\phi_1} + (-\hat{H}_1\omega^1 - \overline{\hat{H}_1\omega^1}) \wedge \omega.
\end{aligned} \tag{2.12}
$$

Ii is known that the projective connection underlines the CR connection [C] [F]. Hence the structure equations (2.10), when restricted on $\hat{Y}$, reduce to (2.12). Consequently, $\hat{L}^{11} = L^{11}|_{\hat{Y}} = \overline{P_{11}|_{\hat{Y}}}$. $\hat{L}^{11}$, when pulled back to (Y, π, M), is the Cartan fundamental curvature function. Hence, $(z_0, w_0) \in M$ is an *umbilical point* if and only if $L^{11}|_{\hat{Y}} = 0$ along the fiber $\hat{\pi}^{-1}(z_0, w_0)$, where $\hat{\pi} : \hat{Y} \to \hat{M}$ is the natural projection. Notice that (z_0, w_0) is an umbilical point of M if and only if there is a biholomorphic change of coordinates under which (z_0, w_0) is mapped to the origin and $\hat{M}$ is defined by an equation of the form: $\mathrm{Im}(w) = |z|^2 + o(|z|^6)$. (See [CM]).

From (2.11), we notice that L^{11} vanishes at a point in the fiber $\hat{\pi}^{-1}(S(z_0, w_0, \overline{z_0}, \overline{w_0}))$ if and only if L^{11} vanishes along the whole fiber $\hat{\pi}^{-1}(S(z_0, w_0, \overline{z_0}, \overline{w_0}))$. Since $u \neq 0, u_1^1 \neq 0$ in (2.11), we obtain

Theorem 2.1. *Let $M = \{r = 0\} \subset \mathbb{C}^2$. Let r and $(z_0, w_0) \in M$ be as in (2.1). Assume that (2.2) is satisfied. Then $(z_0, w_0) \in M$ is an umbilical point if and only* *if*

$$\frac{\partial^4 p_{11}}{\partial p^4}(z_0, w_0, p_0) = 0$$

where $p_0 = -\frac{r_z}{r_w}(z_0, w_0, \overline{z_0}, \overline{w_0})$.

3. Umbilical points of ellipsoids in $\mathbb{C}^2$

Recall that a real ellipsoid $M \subset \mathbb{C}^n$ is the image of the unit sphere $\partial\mathbb{B}^n$ under a real-affine transformation of $\mathbb{R}^{2n} := \mathbb{C}^n$. It is known [We1] that after a holomorphic affine transformation, any real ellipsoid is given by an equation of the form: $\sum_{j=1}^n (A_j x_j^2 + B_j y_j^2) = 1$ where $A_j \geq B_j > 0$ and $z_j = x_j + iy_j$. The complex structure of ellipsoids was first studied by Webster in his famous paper [We1]. He showed that when $n \geq 2$, two ellipsoids are biholomorphically equivalent if and only if the set of ratios $(A_j - B_j)/(A_j + B_j)$ is the same for the two. Hence any ellipsoid M can be made into the form:

$$\sum_{j=1}^{n} \left((1 + a_j)x_j^2 + y_j^2 \right) = 1 \tag{3.1}$$

where $a_j \geq 0$. Notice that M is spherical if and only if $a_j = 0$ for all j. In particular, after a holomorphically linear change of coordinates, any ellipsoid M in $\mathbb{C}^2$ can be given by

$$(1 + a_1)x_1^2 + y_1^2 + (1 + a_2)x_2^2 + y_2^2 = 1, \quad a_1, a_2 \geq 0; \tag{3.2}$$

or equivalently,

$$a_1 z^2 + a_1 \overline{z}^2 + 2(2 + a_1)z\overline{z} + a_2 w^2 + a_2 \overline{w}^2 + 2(2 + a_2)w\overline{w} = 4. \tag{3.3}$$

We notice from (3.2) that M can be parameterized by three real parameters $\alpha, \beta \in [0, 2\pi]$, $c \in [0, 1]$ through the following equation:

$$z = \frac{c}{\sqrt{1 + a_1}} cos\ \alpha + i\ c\ sin\ \alpha, \quad w = \frac{\sqrt{1 - c^2}}{\sqrt{1 + a_2}} cos\ \beta + i\sqrt{1 - c^2} sin\ \beta \tag{3.4}$$

In fact, for any $c \in [0, 1]$, consider $w = x_2 + iy_2$ with $(1 + a_2)x_2^2 + y_2^2 = 1 - c^2$. Then $w = \frac{\sqrt{1-c^2}}{\sqrt{1+a_2}} cos\ \beta + i\sqrt{1 - c^2} sin\ \beta$ for $\beta \in [0, 2\pi]$. Since $(1 + a_1)x_1^2 + y_1^2 = c^2$, the formula for $z = x_1 + iy_1 = \frac{c}{\sqrt{1+a_1}} cos\ \alpha + i\ c\ sin\ \alpha$ follows.

Complexifying (3.3), we obtain the Segre family $\mathcal{M} \subset \mathbb{C}^2 \times \mathbb{C}^2$ of M, defined by the equation:

$$a_1 z^2 + a_1 \zeta^2 + 2(2 + a_1)z\zeta + a_2 w^2 + a_2 \eta^2 + 2(2 + a_2)w\eta = 4. \tag{3.5}$$

Choose the defining function of M to be $r := a_1 z^2 + a_1 \overline{z}^2 + 2(2+a_1)z\overline{z} + a_2 w^2 + a_2 \overline{w}^2 + 2(2+a_2)w\overline{w} - 4$. Then a point (z, w) satisfies (2.2) if and only if $a_2 w + (2+a_2)\overline{w} \neq 0$. By (3.4), this is equivalent to the condition that $c \neq 1$, or equivalently, $w \neq 0$. We assume

$$c \neq 1, \quad i.e., \quad w \neq 0. \tag{3.6}$$

Then making use of the implicit function theorem, we have a unique function $w = h(z, \overline{z}, \overline{w})$, which solves the the equation $r = 0$ near the point under study. Applying $\frac{\partial}{\partial z}$ and $\frac{\partial^2}{\partial z^2}$ to (3.5), we get $a_1 z + (2 + a_1)\zeta + a_2 w \frac{\partial h}{\partial z} + (2 + a_2)\eta \frac{\partial h}{\partial z} = 0$ and

$a_1 + a_2\left(\frac{\partial h}{\partial z}\right)^2 + a_2 w\frac{\partial^2 h}{\partial z^2} + (2+a_2)\eta\frac{\partial^2 h}{\partial z^2} = 0$. Since $p = \frac{\partial h}{\partial z}$ and $p_{11} = \frac{\partial^2 h}{\partial z^2}$ on $\mathcal{M}$, we obtain

$$a_1 z + (2+a_1)\zeta + a_2 wp + (2+a_2)\eta p = 0 \quad \text{and} \tag{3.7}$$

$$a_1 + a_2 p^2 + a_2 w p_{11} + (2+a_2)\eta p_{11} = 0. \tag{3.8}$$

At the point $(z, w, \overline{z}, \overline{w}) \in \mathcal{M}$, we then have

$$p = -\frac{a_1 z + (2+a_1)\overline{z}}{a_2 w + (2+a_2)\overline{w}}. \tag{3.9}$$

Now, we can use (3.5) (3.6) (3.7) and (3.8) to cancel out ξ, η as follows:

Multiplying $(2+a_1)^2$ to the equation (3.5) and making use of the equality: $(2+a_1)\zeta = -a_1 z - a_2 wp - (2+a_2)\eta p$ from (3.7), we have

$$\begin{aligned} &(2+a_1)^2 a_1 z^2 + a_1\Big[a_1 z + a_2 wp + (2+a_2)\eta p\Big]^2 \\ &+2(2+a_1)^2 z\Big(- a_1 z - a_2 wp - (2+a_2)\eta p\Big) \\ &+a_2(2+a_1)^2 w^2 + a_2(2+a_1)^2\eta^2 \\ &+2(2+a_1)^2(2+a_2)w\eta = 4(2+a_1)^2. \end{aligned} \tag{3.10}$$

Multiplying (3.10) by $(2+a_2)^2 p_{11}^2$ and making use of (3.8): $(2+a_2)\eta p_{11} = -a_1 - a_2 p^2 - a_2 w p_{11}$, we obtain the following

(3.11)

$$\begin{aligned} &a_1(2+a_1)^2(2+a_2)^2 z^2 p_{11}^2 + a_1(2+a_2)^2\Big(a_1 z p_{11} - a_1 p - a_2 p^3\Big)^2 \\ &-2(2+a_1)^2(2+a_2)^2 p_{11} z(a_1 z p_{11} - a_1 p - a_2 p^3) \\ &+a_2(2+a_1)^2(2+a_2)^2 p_{11}^2 w^2 + a_2(2+a_1)^2\Big(a_1 + a_2 p^2 + a_2 w p_{11}\Big)^2 \\ &-2(2+a_1)^2(2+a_2)^2 w p_{11}(a_1 + a_2 p^2 + a_2 w p_{11}) = 4(2+a_1)^2(2+a_2)^2 p_{11}^2. \end{aligned}$$

Write (3.11) as

$$\widetilde{A} p_{11}^2 + 2\widetilde{B} p_{11} + \widetilde{C} = 0, \text{ where} \tag{3.12}$$

(3.13)

$$\widetilde{A} = -4a_1(1+a_1)(2+a_2)^2 z^2 - 4a_2(1+a_2)(2+a_1)^2 w^2 - 4(2+a_1)^2(2+a_2)^2,$$

$$\widetilde{B} = 4(a_1 + a_2 p^2)\Big[(1+a_1)(2+a_2)^2 zp - (2+a_1)^2(1+a_2)w\Big], \tag{3.14}$$

$$\widetilde{C} = (a_1 + a_2 p^2)^2\Big[a_1(2+a_2)^2 p^2 + a_2(2+a_1)^2\Big]. \tag{3.15}$$

Assume that $\widetilde{A} \neq 0$ at the point $(z, w) \in M$ with $w \neq 0$. We can then solve p_{11} from (3.12):

$$p_{11} = \frac{-\widetilde{B} \pm \widetilde{H}}{\widetilde{A}} \tag{3.16}$$

where

$$
\begin{aligned}
\widetilde{H}^2 = \widetilde{B}^2 - \widetilde{A}\widetilde{C} = 4(a_1+a_2p^2)^2 \\
\Big\{4\Big[(1+a_1)(2+a_2)^2zp - (1+a_2)(2+a_1)^2w\Big]^2 \\
+\Big[a_1(2+a_2)^2p^2 + a_2(2+a_1)^2\Big]\cdot\Big[a_1(1+a_1)(2+a_2)^2z^2 \\
+a_2(1+a_2)(2+a_1)^2w^2 + (2+a_1)^2(2+a_2)^2\Big]\Big\}.
\end{aligned}
\tag{3.17}
$$

After taking out the common factor $2(a_1+a_2p^2)$, (3.16) can be simplified as

$$
p_{11} = \frac{-\hat{B}\pm\hat{H}}{\widetilde{A}}\cdot 2(a_1+a_2p^2)
\tag{3.18}
$$

where $2(a_1+a_2p^2)\hat{B} = \widetilde{B}$, and

(3.19)

$$
\begin{aligned}
\hat{H}^2 = 4\Big[(1+a_1)(2+a_2)^2zp - (1+a_2)(2+a_1)^2w\Big]^2 \\
+\Big[a_1(2+a_2)^2p^2 + a_2(2+a_1)^2\Big] \\
\cdot\Big[a_1(1+a_1)(2+a_2)^2z^2 + a_2(1+a_2)(2+a_1)^2w^2 + (2+a_1)^2(2+a_2)^2\Big].
\end{aligned}
$$

Write

$$
\hat{H}^2 = Ap^2 + Bp + C, \text{ where}
\tag{3.20}
$$

$$
\begin{aligned}
A = 4(1+a_1)^2(2+a_2)^4z^2 + a_1(2+a_2)^2\Big[a_1(1+a_1)(2+a_2)^2z^2 \\
+a_2(1+a_2)(2+a_1)^2w^2 + (2+a_1)^2(2+a_2)^2\Big],
\end{aligned}
\tag{3.21}
$$

$$
B = -8(1+a_1)(1+a_2)(2+a_1)^2(2+a_2)^2zw,
\tag{3.22}
$$

$$
\begin{aligned}
C = 4(1+a_2)^2(2+a_1)^4w^2 + a_2(2+a_1)^2\Big[a_1(1+a_1)(2+a_2)^2z^2 \\
+a_2(1+a_2)(2+a_1)^2w^2 + (2+a_1)^2(2+a_2)^2\Big].
\end{aligned}
\tag{3.23}
$$

Assume that $\hat{H}^2 = Ap^2+Bp+C \neq 0$ at the point $(z,w)\in M$ with p being given as before. Notice that $\widetilde{A}$ is independent of p and that the degree of $\hat{B}$ in p is 1. From the formula of p_{11} in (3.18), it follows that at $(z,w,\overline{z},\overline{w})$,

$$
\frac{\partial^4 p_{11}}{\partial p^4} = 0 \Leftrightarrow \frac{\partial^4}{\partial p^4}\Big((a_1+a_2p^2)\hat{H}\Big) = 0.
\tag{3.24}
$$

Assume that $\hat{H}(z^*,w^*,p^*) = 0$ with $(z^*,w^*)\in M$ and $p^* = p(z^*,w^*,\overline{z^*})$, where

$$
w^*,\ A(z^*,w^*), \widetilde{A}(z^*,w^*) \neq 0.
$$

Since $p_{11}(z,w,p)$ is a holomorphic function for $(z,w,p)\approx(z^*,w^*,p^*)$, we easily see from (3.18) that $J(z,w,p) := \hat{H}\cdot(a_1+a_2p^2)$ is also holomorphic for

$(z,w,p)\approx(z^*,w^*,p^*)$. In particular, $J(z^*,w^*,p)$ is holomorphic in p for $p\approx p^*$. Now, suppose that $2A(z^*,w^*)p^*+B(z^*,w^*)\neq 0$. Then $\hat{H}=\pm(p-p^*)^{1/2}h^*$ with $h^*\neq 0$ holomorphic for $p\approx p^*$, by (3.20). This clearly contradicts the fact that $J(z^*,w^*,p)$ is holomorphic in p for $p\approx p^*$. Hence, we conclude that $\hat{H}(z^*,w^*,p^*)=0$ can only occur at the point where

$$2A(z^*,w^*)p^*+B(z^*,w^*)=0. \tag{3.25}$$

Next, we have

$$\frac{\partial^4}{\partial p^4}\left((a_1+a_2p^2)\hat{H}\right)=12a_2\frac{\partial^2\hat{H}}{\partial p^2}+8a_2p\frac{\partial^3\hat{H}}{\partial p^3}+(a_1+a_2p^2)\frac{\partial^4\hat{H}}{\partial p^4}. \tag{3.26}$$

Since $\hat{H}^2=Ap^2+Bp+C$, we get $2\hat{H}\frac{\partial\hat{H}}{\partial p}=2Ap+B$. We continue to differentiate it to get $\left(\frac{\partial\hat{H}}{\partial p}\right)^2+\hat{H}\frac{\partial^2\hat{H}}{\partial p^2}=A$. Hence

$$\begin{aligned}\frac{\partial^2\hat{H}}{\partial p^2}&=\frac{A-(\frac{\partial\hat{H}}{\partial p})^2}{\hat{H}}=\frac{4A\hat{H}^2-(2Ap+B)^2}{4\hat{H}^3}\\&=\frac{4A(Ap^2+Bp+C)-(4A^2p^2+4ABp+B^2)}{4\hat{H}^3}=\frac{4AC-B^2}{4\hat{H}^3}.\end{aligned} \tag{3.27}$$

Continuing differentiation on $\left(\frac{\partial\hat{H}}{\partial p}\right)^2+\hat{H}\frac{\partial^2\hat{H}}{\partial p^2}=A$, we obtain $3\frac{\partial\hat{H}}{\partial p}\frac{\partial^2\hat{H}}{\partial p^2}+\hat{H}\frac{\partial^3\hat{H}}{\partial p^3}=0$ and thus

$$\begin{aligned}\frac{\partial^3\hat{H}}{\partial p^3}&=-\frac{3}{\hat{H}}\cdot\frac{\partial\hat{H}}{\partial p}\cdot\frac{\partial^2\hat{H}}{\partial p^2}\\&=-\frac{3}{\hat{H}}\cdot\frac{2Ap+B}{2\hat{H}}\cdot\frac{4AC-B^2}{4\hat{H}^3}=-\frac{3}{8\hat{H}^5}(2Ap+B)(4AC-B^2).\end{aligned} \tag{3.28}$$

Again from the equation $3\frac{\partial\hat{H}}{\partial p}\frac{\partial^2\hat{H}}{\partial p^2}+\hat{H}\frac{\partial^3\hat{H}}{\partial p^3}=0$, we get by differentiation

$$3\left(\frac{\partial^2\hat{H}}{\partial p^2}\right)^2+4\frac{\partial\hat{H}}{\partial p}\frac{\partial^3\hat{H}}{\partial p^3}+\hat{H}\frac{\partial^4\hat{H}}{\partial p^4}=0,\ \text{and thus}$$

$$\begin{aligned}\frac{\partial^4\hat{H}}{\partial p^4}&=\frac{1}{\hat{H}}\left[-3\left(\frac{\partial^2\hat{H}}{\partial p^2}\right)^2-4\frac{\partial\hat{H}}{\partial p}\frac{\partial^3\hat{H}}{\partial p^3}\right]\\&=\frac{3(4AC-B^2)}{16\hat{H}^7}\left(B^2-4AC+4(2Ap+B)^2\right).\end{aligned} \tag{3.29}$$

By Theorem 2.1, (3.24), (3.26), (3.27), (3.28) and (3.29), $(z,w)\in M$ is an umbilical point if and only if

$$\begin{aligned}&\frac{a_2(4AC-B^2)}{\hat{H}^3}-\frac{a_2p(2Ap+B)(4AC-B^2)}{\hat{H}^5}\\&+\left(a_1+a_2p^2\right)\frac{(4AC-B^2)[B^2-4AC+4(2Ap+B)^2]}{16\hat{H}^7}=0,\end{aligned}$$

which amounts to say that either $4AC-B^2=0$ or

$$a_2\hat{H}^4-a_2p(2Ap+B)\hat{H}^2+\frac{1}{16}[a_1+a_2p^2][B^2-4AC+4(2Ap+B)^2]=0. \tag{3.30}$$

Since $\hat{H}^2=Ap^2+Bp+C$, it follows from (3.30) that

$$4a_2(Bp+2C)^2+4a_1(2Ap+B)^2+(a_1+a_2p)(B^2-4AC)=0.$$

Hence, we have proved the following criterion on umbilical points.

Theorem 3.1. *Let $M \subset \mathbb{C}^2$ be as in (3.2). Let $(z,w) \in M$ be such that $w \neq 0$, $\widetilde{A}(z,w) \neq 0$ and $\hat{H}(z,w,p(z,w,\overline{z})) = Ap^2+Bp+C \neq 0$. Then (z,w) is an umbilical point if and only if either $4AC - B^2 = 0$ or*

$$4a_2(Bp+2C)^2 + 4a_1(2Ap+B)^2 + (a_1+a_2p)(B^2-4AC) = 0 \tag{3.31}$$

at (z,w,p). Here p is as in (3.9); A, B and C are as in (3.21), (3.22) and (3.23).

4. Proof of Theorem 1.2

Lemma 4.1. *Let M be as in (3.2). Assume that $a_1 > 0$. If $16a_1+16a_1a_2+3a_1a_2^2-4a_2^2 > 0$, then M is umbilical at $(\frac{c}{\sqrt{1+a_1}}, i\sqrt{1-c^2}) \in M$ for a certain $c \in (0,1)$.*

Proof: Consider the curve $\Gamma \subset M$ with the parameter $c \in [0,1]$, defined by:

$$z(c) = \frac{c}{\sqrt{1+a_1}}, \tag{4.1}$$

$$w(c) = i\sqrt{1-c^2}, \quad 0 \leq c < 1. \tag{4.2}$$

Then along Γ, from (3.9), we have

$$p(c) = -\frac{a_1 z + (2+a_1)\overline{z}}{aw + (2+a)\overline{w}} = -\frac{i(\sqrt{1+a_1})c}{\sqrt{1-c^2}}. \tag{4.3}$$

By (3.21),(3.22) and (3.23), we have

$$\begin{aligned} A(c) = 4(1+a_1)(2+a_2)^4c^2 + a_1(2+a_2)^2\Big[a_1(2+a_2)^2c^2 \\ -a_2(1+a_2)(2+a_1)^2(1-c^2) + (2+a_1)^2(2+a_2)^2\Big], \end{aligned} \tag{4.4}$$

$$B(c) = -8(1+a_1)(1+a_2)(2+a_1)^2(2+a_2)^2 ic\frac{\sqrt{1-c^2}}{\sqrt{1+a_1}}, \tag{4.5}$$

$$\begin{aligned} C(c) = -4(1+a_2)^2(2+a_1)^4(1-c^2) \ + a_2(2+a_1)^2\Big[a_1(2+a_2)^2c^2 \\ -a_2(1+a_2)(2+a_1)^2(1-c^2) + (2+a_1)^2(2+a_2)^2\Big]. \end{aligned} \tag{4.6}$$

By Theorem 3.1, it is enough to show that there is a certain $c \in (0,1)$ such that at the point $(z(c), w(c), p(c)) \in \widetilde{\mathcal{M}}$

$$\widetilde{A} \neq 0, \tag{4.7}$$

$$Ap^2 + Bp + C \neq 0, \quad \text{and} \tag{4.8}$$

$$4a_2(Bp+2C)^2 + 4a_1(2Ap+B)^2 + (a_1+a_2p^2)(B^2-4AC) = 0. \tag{4.9}$$

We first prove that (4.7) holds for any point in Γ. By (3.13), $\widetilde{A} = 0$ at $(z(c), w(c)) \in \Gamma$ if and only if

$$-4a_1(2+a_2)^2c^2 + 4a_2(1+a_2)(2+a_1)^2(1-c^2) - 4(2+a_1)^2(2+a_2)^2 = 0,$$

namely,

$$-4a_1(2+a_2)^2c^2 - 4(2+a_1)^2[4+3a_2+c^2a_2+c^2a_2^2] = 0.$$

But this is a contradiction because the left hand side is strictly negative for any $c \in [0,1]$.

We also notice that $A > 0$ along Γ, too.

Next, after restricted to Γ, (4.9) can be written as

$$\begin{aligned}\Big[4a_2B^2+16a_1A^2+a_2(B^2-4AC))\Big]p^2+(16a_2BC+16a_1AB)p\\ +\Big[16a_2C^2+4a_1B^2+a_1(B^2-4AC)\Big]=0.\end{aligned} \tag{4.10}$$

In order to solve the equation (4.9), by (4.3) and (4.10), it is enough to show that there exists a point $c \in (0,1)$ such that $K(c)=0$, where

$$\begin{aligned}K(c):\ &= \Big[4a_2B^2+16a_1A^2+a_2(B^2-4AC))\Big]\Big(a_1z+(2+a_1)\overline{z}\Big)^2\\ &-(16a_2BC+16a_1AB)\Big(a_1z+(2+a_1)\overline{z}\Big)\Big(a_2w+(2+a_2)\overline{w}\Big)\\ &+\Big[16a_2C^2+4a_1B^2+a_1(B^2-4AC)\Big]\Big(a_2w+(2+a_2)\overline{w}\Big)^2.\end{aligned} \tag{4.11}$$

By (4.11) (4.4) (4.5) and (4.6), $K(c)$ is a real-valued function defined on $[0,1]$. When $c=0$, we have $z=0$, $w=i$ and

$$\begin{aligned}A &= a_1(2+a_2)^2\Big[-a_2(1+a_2)(2+a_1)^2+(2+a_1)^2(2+a_2)^2\Big]\\ &= a_1(2+a_1)^2(2+a_2)^2(4+3a_2),\quad B=0,\end{aligned}$$

$$\begin{aligned}C &= -4(1+a_2)^2(2+a_1)^4+a_2(2+a_1)^2\Big[-a_2(1+a_2)(2+a_1)^2\\ &+(2+a_1)^2(2+a_2)^2\Big]=-(2+a_1)^4(2+a_2)^2.\end{aligned}$$

Hence

$$K(0)=-16C(4a_2C-a_1A)<0, \tag{4.12}$$

by noticing that $C<0$ and $A>0$.

When $c=1$, we have $z=\frac{1}{\sqrt{1+a_1}}$, $w=0$ and

$$\begin{aligned}A &= 4(1+a_1)(2+a_2)^4+a_1(2+a_2)^2\Big[a_1(2+a_2)^2+(2+a_1)^2(2+a_2)^2\Big]\\ &= (2+a_2)^4(1+a_1)(2+a_1)^2,\quad B=0,\end{aligned}$$

$$\begin{aligned}C &= a_2(2+a_1)^2\Big[a_1(2+a_2)^2+(2+a_1)^2(2+a_2)^2\Big]\\ &= a_2(2+a_1)^2(2+a_2)^2(1+a_1)(4+a_1)\end{aligned}$$

Hence

$$\begin{aligned}K(1)\ &= 4A(4a_1A-a_2C)4(1+a_1)\\ &= d^*[4a_1(2+a_2)^2-a_2^2(4+a_1)].\end{aligned} \tag{4.13}$$

Here $d^*>0$. Hence, when

$$4a_1(2+a_2)^2-a_2^2(4+a_1)=16a_1+16a_1a_2+3a_1a_2^2-4a_2^2>0,$$

$K(0)<0$ and $K(1)>0$. Thus, $K(c)=0$ for a certain $c\in(0,1)$. Namely, we showed that (4.9) holds for a certain c.

It remains to prove that (4.8) cannot hold for the above $c \in (0,1)$. Suppose that $\hat{H}(c)^2 = 0$. Since $\widetilde{A}(c) > 0$ and $A(c) > 0$, w conclude by (3.25), that $2Ap + B = 0$. Making use of (4.3), (4.4) and (4.5), we thus have

$$-8(1+a_1)(1+a_2)(2+a_1)^2(2+a_2)^2 \frac{ic\sqrt{1-c^2}}{\sqrt{1+a_1}} = \frac{2i(\sqrt{1+a_1})c}{\sqrt{1-c^2}} \cdot A(c) \tag{4.14}$$

This is a contradiction, for after dividing the fact i, the left hand side of (4.14) is negative, while its right hand side is strictly positive. The proof of Lemma 4.1 is complete. □

Proof of Theorem 1.2: If M is spherical, then every point is umbilical point. We assume that M is not spherical. Then $a_1 + a_2 > 0$. We notice that $(1+a_1)x_1^2 + y_1^2 + (1+a_2)x_2^2 + y_2^2 = 1$ is holomorphically equivalent to the ellipsoid defined by $(1+a_2)x_1^2 + y_1^2 + (1+a_1)x_2^2 + y_2^2 = 1$ through the map $(z,w) \to (w,z)$. Hence, we need only to prove Theorem 1.2 for the case when $a_1 \geq a_2$. Then the assumption in Lemma 4.1 holds automatically and thus we have an umbilical point of the form $(\frac{c}{\sqrt{1+a_1}}, i\sqrt{1-c^2})$ $(c \in (0,1))$. Notice that M has automorphisms sending (z,w) to $(\pm z, \pm w)$. We easily conclude that M possesses at least four umbilical points. □

5. Proof of Theorem 1.3

The ϵ-thickening Ω_ϵ of the unit circle $\{|z| = 1, w = 0\}$ is defined to be the set of points whose distance to the circle is less than ϵ. It is straightforward to verify that the boundary M_ϵ of Ω_ϵ is defined by the following equation, which is strictly plurisubharmonic when $0 < \epsilon < 1/4$:

$$|z|^2 - 2|z| + 1 + |w|^2 = \epsilon^2. \tag{5.1}$$

Here and in what follows, we assume $0 < \epsilon << 1$. Also, since Ω_ϵ is a Reinhardt domain, we need only to study the points $(z,w) \in M_\epsilon$ with $z = x_1 \geq 0$ and $w = x_2 \geq 0$. Also, we assume that $x_2 > 0$. Notice that when $\epsilon << 1$, $x_2 \approx 1$.

The complexification of (5.1) is given by

$$r := z\zeta - 2(z\zeta)^{1/2} + 1 + w\eta - \epsilon^2 = 0. \tag{5.2}$$

As we did in §3, we have

$$r_z = \zeta - (z\zeta)^{-1/2}\zeta + p\eta = 0, \text{ and} \tag{5.3}$$

$$r_{zz} = \frac{1}{2}(z\zeta)^{-3/2}\zeta^2 + p_{11}\eta = 0. \tag{5.4}$$

From (5.3), we have

$$z\zeta - (z\zeta)^{1/2} + pz\eta = 0. \tag{5.5}$$

Subtracting (5.2) from (5.5), we obtain

$$(z\zeta)^{1/2} = 1 - \epsilon^2 + (w - pz)\eta. \tag{5.6}$$

Returning to (5.4) and making use of (5.6), we get

$$1 - \epsilon^2 + (w - pz)\eta + 2\eta z^2 p_{11} = 0. \tag{5.7}$$

Here, we remark that near the point under study, $\eta \approx x_2 \neq 0$. Hence $\frac{1-\epsilon^2}{\eta} + (w - pz) + 2z^2 p_{11} = 0$ and

$$\frac{\partial^4 p_{11}}{\partial p^4} = 0 \iff \frac{\partial^4}{\partial p^4}\left(\frac{1}{\eta}\right) = 0. \tag{5.8}$$

Set $X = \frac{1}{\eta}$. Substituting (5.6) into (5.2), we get

$$\left[(1-\epsilon)^2 + (w-pz)\eta\right]^2 - 2\left[(1-\epsilon^2) + (w-pz)\eta\right] + 1 + w\eta - \epsilon^2 = 0, \text{ or}$$

$$(w-pz)^2\eta^2 + \left[2(1-\epsilon^2)(w-pz) - 2(w-pz) + w\right]\eta + (1-\epsilon^2)^2 - 2(1-\epsilon^2) + (1-\epsilon^2) = 0,$$

$$-\epsilon^2(1-\epsilon^2)X^2 + \left[-2\epsilon^2(w-pz) + w\right]X + (w-pz)^2 = 0.$$

Hence

$$X = \frac{-(-2\epsilon^2(w-pz)+w) \pm H}{-2\epsilon^2(1-\epsilon^2)} \tag{5.9}$$

where

$$H^2 := (2\epsilon^2(w-pz) - w)^2 + 4\epsilon^2(1-\epsilon^2)(w-pz)^2.$$

Hence

$$\frac{\partial^4 p_{11}}{\partial p^4} = 0 \iff \frac{\partial^4 H}{\partial p^4} = 0. \tag{5.10}$$

Write $H^2 = Ap^2 + Bp + C$ where

$$\begin{aligned} A &= 4\epsilon^2 z^2 + 4\epsilon^2(1-\epsilon^2)z^2 = 4\epsilon^2 z^2, \\ B &= -4\epsilon^2 z(2\epsilon^2 w) - 8\epsilon^2(1-\epsilon^2)wz = -4\epsilon^2 wz, \\ C &= \epsilon^2 w^2. \end{aligned} \tag{5.11}$$

By (3.29), we conclude that $\frac{\partial^4 p_{11}}{\partial p^4} = 0$ if and only if

$$\text{either } 4AC - B^2 = 0 \text{ or } B^2 - 4A + 4(2Ap+B)^2 = 0. \tag{5.12}$$

Since $4AC - B^2 = 4\epsilon^2(zw)^2(1-4\epsilon^2) \neq 0$, the first equality in (5.12) never occurs. The second equality in (5.12) is equivalent to $4AC - B^2 = 4(2Ap+B)^2$, namely,

$$2\epsilon zw\sqrt{1-4\epsilon^2} = \pm 2(2Ap+B). \tag{5.13}$$

At the point in M with $z = x_1 > 0$ and $w = x_2 > 0$, by (5.3), we find $x_1 - 1 + px_2 = 0$, or

$$p = \frac{1-x_1}{x_2}. \tag{5.14}$$

Hence we get from (5.11)

$$A = 4\epsilon^2 x_1^2, \; B = -4\epsilon^2 x_1 x_2 \text{ and } C = \epsilon^2 x_2^2. \tag{5.15}$$

Then (5.13) is equivalent to

$$2\epsilon x_1 x_2\sqrt{1-4\epsilon^2} = \pm 2\left(8\epsilon^2 x_1^2 \cdot \frac{1-x_1}{x_2} - 4\epsilon^2 x_1 x_2\right). \tag{5.16}$$

Since $x_1 \approx 1$, we get from (5.16):

$$x_2^2\sqrt{1-4\epsilon^2} = \pm\left(8\epsilon(x_1 - x_1^2) - 4\epsilon x_2^2\right). \tag{5.17}$$

Recall $x_2^2 = \epsilon^2 - (1-x_1)^2$. Let $T = 1 - x_1$. Then $x_1 - x_1^2 = T - T^2$ and $x_2^2 = \epsilon^2 - T^2$. Hence (5.17) is equivalent to

$$(\epsilon^2 - T^2)\sqrt{1-4\epsilon^2} = \pm 4\epsilon(2T - T^2 - \epsilon^2), \tag{5.18}$$

or

$$f(T) := (\sqrt{1-4\epsilon^2} \mp 4\epsilon)T^2 \pm 8\epsilon T + (-\epsilon^2\sqrt{1-4\epsilon^2} \mp \epsilon^2) = 0.$$

Notice that $-\epsilon < T < \epsilon$. From the fact that

$$f'(T) = 2(\sqrt{1-4\epsilon^2} \mp 4\epsilon)T \pm 8\epsilon = 0 \iff |T| \approx 4\epsilon$$

for $\epsilon << 1$, we conclude that the real-valued function $f(T)$ is monotonic for $T \in (-\epsilon, \epsilon)$. We further compute

$$f(-\epsilon) = (\sqrt{1-4\epsilon^2} \mp 4\epsilon)\epsilon^2 \mp 8\epsilon^2 + (-\epsilon^2\sqrt{1-4\epsilon^2} \mp 4\epsilon^3) \approx \mp 8\epsilon^2$$

and

$$f(\epsilon) = (\sqrt{1-4\epsilon^2} \mp 4\epsilon)\epsilon^2 \pm 8\epsilon^2 + (-\epsilon^2\sqrt{1-4\epsilon^2} \mp 4\epsilon^3) \approx \pm 8\epsilon^2$$

for $\epsilon << 1$. Then we see that (5.12) has two solutions in $(-\epsilon, \epsilon)$. A little more effort actually shows that these two solutions are different. Therefore, by Theorem 2.1, we conclude that M admits two distinct umbilical points with $z = x_1 > 0, w = x_2 > 0$. One can similarly verify that points in M with $w = 0$ are umbilical points. The statement Theorem 1.3 thus follows from the Reinhardt property of Ω_ϵ. $\square$

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X. Huang and S. Ji: School of Mathematical Sciences, Wuhan University, Wuhan 430072, P. R. China.

Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA
E-mail address: huangx@math.rutgers.edu

Department of Mathematics, University of Houston, Houston, TX 77204, USA
E-mail address: shanyuji@math.uh.edu