On Some Rigidity Problems in Cauchy-Riemann Analysis

Xiaojun Huang and Shanyu Ji

Abstract. This paper is a survey on some rigidity problems in Cauchy-Riemann Geometry. It includes the topics such as holomorphic mappings between balls, the CR version of the Bonnet-Type theorem, super-rigidity and CR transversality for holomorphic mappings between hyperquadrics, rigidity problems for holomorphic maps with symmetry, rigidity problems for holomorphic Segre maps.

1. Introduction

This paper has two parts. We first survey some recent work on various rigidity problems for CR mappings, CR submanifolds embedded into the spheres or hyperquadrics, and some rigidity problems for isolated complex singularities. We then present the computation of the Segre-isomorphisms as a first step to extend the investigation of rigidities for CR maps to holomorphic Segre maps.

We start with a classical result in the conformal geometry. In this setting, we consider open subsets U, V in $\mathbb{R}^n$, equipped with the flat metric ω. We say a smooth map f from U to V is a conformal map if $f^*(\omega) = e^u\omega$. The following result is classical:

Theorem 1.1. *(Liouville) Assume the above and let $n \geq 3$. Then f is conformal if and only if f is a Mobius transformation: A composition of the following type of transformations: (i) translations, (ii) rotations, (iii) scalings and inversions.*

In CR geometry, the replacement of $\mathbb{R}^n$ is the sphere— the boundary of the unit ball. The conformal transformations should be replaced by CR transformations (or the conjugate of CR transformations), which, roughly speaking, are just the boundary value of holomorphic mappings. (See [BER1] for most notations and definitions). Hence the following result of Poincaré, proved about 100 years ago, can be regarded as a complex version of the Liouville theorem:

Theorem 1.2. *(Poincaré 1907 [Po] , Tanaka [Ta], Chern-Moser [CM]) Let f be a non-constant holomorphic map from an open piece of the unit sphere $\partial\mathbb{B}^n$ into the unit sphere ($n \geq 2$). Then f is a linear fractional transformation and extends to a biholomorphic self-map of the unit ball.*

1991 *Mathematics Subject Classification.* Primary 32V40, 32H35; Secondary 32H02, 32V15.

Key words and phrases. Rigidity Problems, Cauchy-Riemann Geometry, holomorphic proper mappings.

The first author was supported in part by NSF-0500626.

One of our concerns here is to address the recent development along these lines of research. Unfortunately, we have to skip many important results in the related fields, where when a local map can be extended to a global object is studied. (See [BER1-2] [KZ] and the references therein).

The second classical result whose CR version we like to pursue here is the so-called Bonnet theorem. In this case, we consider two immersed surfaces M and M' in $\mathbb{R}^3$. Suppose they are parameterized by $U \subset \mathbb{R}^2$ through f and g, respectively. If M and M' are isometric to each other and if the second fundamental form is identical at the corresponding points: $l_{ij}(p) = \langle n(f(p)), f_{ij}(p)\rangle = \widetilde{l}_{ij}(p) = \langle \widetilde{n}(f(p)), g_{ij}(p)\rangle$. Then M and M' are the rigid motion of each other. Namely, there is a Euclidean transformation T such that $T(M) = M'$ (See [Spv]).

In CR geometry, immersed submanifolds are replaced by Levi non-degenerate CR submanifolds. The following result proved by S. Webster in 1977 can be regarded as a generalization of the complex variable version of the classical Bonnet theorem:

THEOREM 1.3. *(Webster [We4]) Let (M_1, p_1) and (M_2, p_2) be two germs of strongly pseudoconvex CR submanifolds of CR codimension 1 in $\partial\mathbb{B}^n$ with $n \geq 5$. If (M_1, p) and (M_2, p_2) are CR diffeomorphic to each other, then they are the rigidity motion of each other. Namely, there is a $T \in Aut(\partial\mathbb{B}^n)$ such that $T(p_1) = p_2$ and $T((M_1, p_1)) = (M_2, p_2)$.*

Acknowledgement: The authors thank D. Zaitsev and N. Mir for many helpful comments and suggestions to this paper

2. Holomorphic Mappings between Balls

We now discuss recent results along the lines of Poincaré's theorem. For simplicity, we restrict ourselves to the global setting, though all results to be stated in this section hold also in the local setting. By the Bochner theorem, any holomorphic (CR) function defined over $\partial\mathbb{B}^n$ extends holomorphically to $\mathbb{B}^n$. Hence, any non constant holomorphic map from $\partial\mathbb{B}^n$ to $\partial\mathbb{B}^n$ extends to a proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^n$. We use the notation $\text{Prop}(\mathbb{B}^n, \mathbb{B}^N)$ to denote the collection of all proper holomorphic maps from $\mathbb{B}^n$ into $\mathbb{B}^N$. $(N \geq n > 1)$. Now, the aforementioned theorem of Poincaré-Tanaka-Chern-Moser can be stated as follows:

THEOREM 2.1. *(Poincaré [Po]) Let $f \in Prop(\mathbb{B}^n, \mathbb{B}^n)$, that extends holomorphically across $\partial\mathbb{B}^n$. Then $f \in Aut(\mathbb{B}^n)$.*

PROOF. (Sketch) Need only to prove that f is a covering map. For that, it suffices to show that f is a local biholomorphic map. Indeed, if not, then E_f: $= \{z \in \mathbb{B}^n : J_f(z) = 0\}$ is a complex analytic variety of codimension 1. Since $n > 1$, E_f cannot be compact and thus must cut the boundary. Now, an application of the Hopf Lemma shows this is impossible. □

It had been wondered for years that if the boundary regularity assumption in Poincaré's theorem is necessary. That puzzle was answered by H. Alexander in [Alx1-2].

THEOREM 2.2. *(Alexander [Alx1-2]) Let $f \in Prop(\mathbb{B}^n, \mathbb{B}^n)$. Then $f \in Aut(\mathbb{B}^n)$.*

Notice that for an affine complex line L, $L\cap\mathbb{B}^n$ is a complex geodesic in terms of the hyperbolic Kobayashi metric of the ball and an automorphism of $\mathbb{B}^n$ maps an affine line to an affine line. The result of Alexander hence tells that a proper holomorphic self-map of $\mathbb{B}^n$ preserves the complex geodesics of $\mathbb{B}^n$ $(n>1)$. More generally, one says that a map from $\mathbb{B}^n$ into $\mathbb{B}^N$ is a linear map or a totally geodesic embedding if it maps a complex geodesic in $\mathbb{B}^n$ to a complex geodesic in $\mathbb{B}^N$. Using his result on the CR version of the Bonnet theorem (Theorem 1.2), Webster was the first one to look at the geometric structure of proper holomorphic maps between balls in complex spaces of different dimensions. He, in 1979, showed that a proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^{n+1}$ with $n>2$, which is three times differentiable up to the boundary, is a totally geodesic embedding. Subsequently, Cima-Suffridge ([CS1], 1983) reduced the boundary regularity in Webster's theorem to the C^2-regularity. Motivated by a conjecture posed by Cima-Suffridge, Faran in 1986 showed [Fa2] that any proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N<2n-1$, that is analytic up to the boundary, is also a totally geodesic embedding. Forstneric ([Fr1-3]) proved that any proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$ is rational, if the map is C^{N-n+1}-regular up to the boundary, which, in particular, reduces the regularity assumption in Faran's linearity theorem to the C^{N-n+1}-smoothness. It had been wondered for years if the super-rigidity holds for proper maps which is C^t-smooth with t independent of the codimension.

On the other hand, the discovery of inner functions had been used in the later 80's to construct proper holomorphic maps from $\mathbb{B}^n$ into $\mathbb{B}^{n+1}$, which cannot be C^2-smooth at any boundary point. (See Alexandroff, Harkim-Sibony [HS], Low, Forstneic[Fr1], Dor [Dor], Stensones, etc)

In 1999 [Hu1] and, subsequently, 2003 [Hu2], we considered two natural questions arising from the above mentioned work. In 1999, it was proved in [Hu1] that any proper holomorphic map which is only C^2-regular up to the boundary must be linear if $N<2n-1$, by applying a different method from the above mentioned work. It is not clear to us if this C^2-regularity is optimal or not for the super-rigidity to hold, the result in [Hu1] gives a first result in which the required regularity is independent of the codimension. Moreover, the basic approaches developed there seem to be quite useful for the study of many other related problems. For instance, it was used in 2005 by Hamada [Ham] to show that all rational proper holomorphic maps from $\mathbb{B}^n$ into $\mathbb{B}^{2n}$ $(n>3)$ are just the D'Angelo family up to the automorphisms.

In [HJ1], it was shown that any proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N=2n-1, n\geq 3$, which is C^2-smooth up to the boundary, is either linear or equivalent to the Whitney map $W: z=(z_1,\cdots,z_n)\rightarrow(z_1,\cdots,z_{n-1},z_nz)$ ([Theorem 1, Theorem 2.3; HJ1]). Since the Whitney map is not an immersion, together with the aforementioned work of Faran [Fa1], this shows that any proper holomorphic embedding from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N=2n-1$, which is twice continuously differentiable up to the boundary, must be a linear map.

THEOREM 2.3. *(Huang [Hu1], 1999): Let f be a proper holomorphic embedding from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $n\geq 3, N<2n-1$. Assume that f is twice differentiable up to the boundary. Then f is linear.*

THEOREM 2.4. *(Huang-Ji [HJ1], 2001): Let f be a proper holomorphic embedding from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N=2n-1>2$. Assume that f is twice differentiable up to the boundary. Then f is linear.*

The structure of the maps gets more complicated when $N > 2n-1$. Recall that two proper holomorphic maps f, g from $\mathbb{B}^n$ into $\mathbb{B}^N$ are called equivalent if there are $\sigma \in \mathrm{Aut}(\mathbb{B}^n)$ and $\tau \in \mathrm{Aut}(\mathbb{B}^N)$ such that $g = \tau \circ f \circ \sigma$. It is easy to verify that a map is linear if and only if it is equivalent to the standard big circle embedding $L(z) : z \to (z, 0)$. Faran in 1983 [Fa1] showed that there are four different inequivalent maps from $\mathbb{B}^2$ into $\mathbb{B}^3$, which are C^3-smooth up to the boundary. D'Angelo wrote down a family of holomorphic embeddings from $\mathbb{B}^n$ into $\mathbb{B}^{2n}$ which are mutually inequivalent.

Example 2.5. (D'Angelo [DA5][DA2]): Let $z = (z', z_n)$ and

$$F_t = (z', \cos(t) z_n, \sin(t) z_n z)$$

with $t \in (0, \pi/2)$. Then F_t is a proper holomorphic embeddings from $\mathbb{B}^n$ into $\mathbb{B}^{2n}$. Apparently, F_t cannot be linear; for it has degree two. Furthermore, F_t is equivalent to F_s if and only if $t = s$.

When $N \geq 2n$, the linearity breaks down even for embedding. However, by looking at the example of D'Angleo, one sees that when F is restricted to $z_n = const$, it is still linear. That motivated the first author in a recent paper [Hu2] to introduce the following definition:

Let F be a proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$. We say that F is κ-linear if for any point $p \in \mathbb{B}^n$, there is an affine complex subspace S_p^κ, that passes through p and is of dimension κ, such that for any affine complex line L contained in S_p^κ, $F(L \cap \mathbb{B}^n)$ is contained in an affine complex line in $\mathbb{C}^N$.

Theorem 2.6. *(Huang [Hu2], 2003) Let F be a proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$, which is C^3-smooth up to the boundary. Write $P(n, \kappa) = \frac{\kappa(2n-\kappa-1)}{2}$. If $1 \leq \kappa \leq n-1$ and $N - n < P(n, \kappa)$, then F is $(n-\kappa+1)$-linear.*

The following example shows that in Theorem 2.6, when $N - n \geq P(n, \kappa)$, one can not expect the $(n-k+1)$-linearity for the map in general.

Example 2.7. Let $W_{n,k} = (\psi_1, \cdots, \psi_k)$ where

$$\begin{aligned}
\psi_1 &= (z_1^2, \sqrt{2} z_1 z_2, \cdots, \sqrt{2} z_1 z_{k-1}, z_1 z_k, \cdots, z_1 z_n),\\
\psi_2 &= (z_2^2, \sqrt{2} z_2 z_3, \cdots, \sqrt{2} z_2 z_{k-1}, z_2 z_k, \cdots, z_2 z_n),\\
&\cdots\\
\psi_{k-1} &= (z_{k-1}^2, z_{k-1} z_k, \cdots, z_{k-1} z_n),\\
\psi_k &= (z_k, \cdots, z_n).
\end{aligned}$$

Then $W_{n,k}$ is a proper polynomial map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N = n + P(n, k)$. Notice that $W_{n,k}$ is not $(n-k+1)$-linear.

When $N < \frac{n(n+1)}{2}$, our result says that at each point in the ball, F has at least two independent directions along which the map is linear. When $N \geq \frac{n(n+1)}{2}$, F usually has no partial linearity. To see this, we just need to notice that the polynomial map H that sends $(z_1, \cdots, z_n)$ to

$$(z_1^2, \sqrt{2} z_1 z_2, \cdots, \sqrt{2} z_1 z_n, z_2^2, \sqrt{2} z_2 z_3, \cdots, \sqrt{2} z_2 z_n, \cdots, z_{n-1} z_{n-1}, \sqrt{2} z_{n-1} z_n, z_n^2)$$

is proper from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N = n + P(n, n-1) = \frac{n(n+1)}{2}$. We mention the interesting similarity between the minimal target dimension N for which the rigidity breaks down in the case we are considering here and the minimal target dimension in the classical Cartan-Janet theorem for which there is no more obstruction to locally isometrically embed an analytic Riemannian manifold of dimension n into $\mathbb{R}^N$.

The above mentioned partial linearity result has applications to some other related problems. For instance, it can be used to prove the following:

Theorem 2.8. *(Huang-Ji-Xu [HJX1], 2005) Let f be a proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N \le n(n+1)/2$. Assume that f is three times differentiable up to the boundary. Then f is a rational map.*

The following has been a well-known open question.

Problem 2.9. *Let $f \in Prop(\mathbb{B}^n, \mathbb{B}^N)$. Is there a fixed constant t such that whenever f is t-times differentiable up to the boundary, then f is rational.*

In a more recent paper, the partial linearity in [Hu2], together with the ideas used in [HJ1] and [Ha], has been used to get a new type of gap phenomenon for holomorphic maps:

Theorem 2.10. *(Huang-Ji-Xu [HJX2], 2005) Let F be a proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$, that is three times differentiable up to the boundary. Suppose that $4 \le n \le N \le 3n-4$. Then F is equivalent to*

$$F'_\theta := (F_\theta(z,w), 0, \cdots, 0) = (z, w\cos\theta, z_1 w \sin\theta, \cdots, z_{n-1} w \sin\theta, w^2 \sin\theta, 0, \cdots, 0)$$

for some θ with $(0 \le \theta \le \frac{\pi}{2})$.

An interesting feature of this result, which is somewhat surprising to us, is that there is no new map added when N runs from $2n$ to $3n-4$. When $N = 3n-3$, the generalized Whitney map $W_{n,2}$ defined in [Example 2.6] properly sends $\mathbb{B}^n$ into $\mathbb{B}^{3n-4}$. Notice that $W_{n,2}$ is not equivalent to F'_θ, for $W_{n,2}$ has geometric rank 2 $(n \ge 3)$. Indeed, [HJX2] contains more results which we discuss as follows.

Denote by $Prop_k(\mathbb{B}^n, \mathbb{B}^N)$ the collection of all proper holomorphic mappings from $\mathbb{B}^n$ into $\mathbb{B}^N$ that are C^k smooth up to the boundary $(k \ge 2)$. By the work in [Hu2], each map $F \in Prop_2(\mathbb{B}^n, \mathbb{B}^N)$ can be associated with an invariant integer $\kappa_0 \in \{0, 1, \cdots, n-1\}$, called its geometric rank (see [Definition 2.2, Hu2] for the precise definition of κ_0). An early result of the first author ([Hu1, Theorem 4.2]) states that F has geometric rank $\kappa_0 = 0$ if and only if F is equivalent to a linear fractional map. When $\kappa_0 \le n-2$, a map $F \in Prop_3(\mathbb{B}^n, \mathbb{B}^N)$ $(n \ge 3)$ has a very special geometric structure. Indeed, by the results of [Hu2] and [HJX1], such a map must be rational and $(n-\kappa_0)$-linear. At this point, we mention a theorem of Forstneric [Fr2] which states that $Prop_{N-n+1}(\mathbb{B}^n, \mathbb{B}^N) = Rat(\mathbb{B}^n, \mathbb{B}^N)$ for $N \ge n > 1$ (See also a very interesting paper of Mir [Mir] later on a more general situation).

At this point, we would like to give a discussion on a theorem of Forstneric in terms of a formulation of Ebenfelt and Zaitsev [EZ], then present a combining effort with results obtained in [HJX1]. (See already Corollary 2.1 and its proof in [Hu3] when $N = 2n-1$.) Without loss of generality, we need only consider the rationality problem for a CR map F from an open subset of the origin in the Heisenberg hypersurface $\mathbb{H}^n := \{(z,w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \mathrm{Im}(w) = \|z\|^2\}$ into $\mathbb{H}^N \subset \mathbb{C}^N$. $(N \ge n \ge 2)$. Let $L_j = \frac{\partial}{\partial z_j} + 2i\overline{z_j}\frac{\partial}{\partial w}$. For $\alpha = (\alpha_1, \cdots, \alpha_{n-1})$, we define L^α in a standard way. Write $F = (\widetilde{f}, g)$ and define $E_k(p) = \mathrm{span}_{\mathbb{C}}\{L^\alpha \widetilde{f}(p)\}_{\|\alpha\| \le k}$. Define $r_k(p) = \mathrm{rank} E_k(p)$. Then a result of Forstneric [Fr2] states that if F is C^m-smooth near 0 in $\mathbb{H}^n$ and $r_m(p) = N-1$ for a certain $p \approx 0$, then F is rational. Ebenfelt and Zaitsev [EZ] noticed that the same proof in [Fr2] can also be used to give the fact that if $r_{m-1}(p) \equiv r_m(p)$ for any p in a certain open subset of $\mathbb{H}^n$, then F is rational. Now, by the work of [HJX1] mentioned above, if the geometric rank of F is not $n-1$ in a neighborhood of 0 in M, then F is rational. On the other hand, by the work in [Hu2], if the geometric rank of F is $n-1$ at some point p close to the origin, then $r_2(p) = \frac{n(n-1)}{2} + (n-1)$. Hence, as a combination of the work mentioned above, one sees the following rationality result, which should be credited to Ebenfelt, Forstneric, Huang, Ji, Xu and Zaitsev:

Corollary 2.11. *Let F be a C^m-smooth ($m \geq 3$) CR map from an open piece of the unit sphere in $\mathbb{C}^n$ into the unit sphere in $\mathbb{C}^N$. If $\frac{n(n+1)}{2} + (m-2) \geq N$, then F is rational.*

In [HJX2], the following general normalization result has been obtained for maps without full rank:

Theorem 2.12. *Let F be a non-linear proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N \geq n \geq 3$. Assume that F is C^3-smooth up to the boundary and has geometric rank $\kappa_0 \leq n-2$. Then F is equivalent to a proper holomorphic map of the form*

$$H := (z_1, \cdots, z_{k^0}, H_1, \cdots, H_{N-k^0}),$$

where $k^0 = n - \kappa_0$ and $H_j = \sum_{l=k^0+1}^{n} z_l H_{j,l}$ with $H_{j,l}$ holomorphic over $\overline{\mathbb{B}^n}$. Moreover, when $\kappa_0 = 1$, $(H_1, \cdots, H_{N-n+1}) = z_n \cdot h$ with h a rational proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^{N-n+1}$. Both H and h are affine linear maps along each hyperplane defined by $z_n = constant$.

Theorem 2.12 has several immediate applications, which we discuss as follows:

Corollary 2.13. *Let $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ have geometric rank 1. Assume that $n \geq 3$. Then the degree of F is bounded by $\frac{N-1}{n-1}$.*

The degree estimate in Corollary 2.13 is optimal. Indeed, the Whitney map has degree 2. By letting h in Theorem 2.12 be the Whitney map, we get a proper polynomial map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N = 3n-2$, which is of geometric rank 1 and has degree 3. Inductively, we can construct a proper polynomial map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N = kn-(k-1)$, which has degree k and geometric rank 1. At this point, we mention a conjecture of D'Angelo which states that the degree of a rational proper map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $n \geq 3$ is bounded by $\frac{N-1}{n-1}$. Hence, Corollary 2.13 partially provides an affirmative solution to the aforementioned conjecture of D'Angelo.

Corollary 2.14. *Let $F \in Prop_3(\mathbb{B}^n, \mathbb{B}^N)$. Suppose $3 \leq n \leq N = 3n-3$ and F has geometric rank 1. Then F is equivalent to*

$$F'_\theta := (F_\theta(z,w), 0, \cdots, 0) = (z, w\cos\theta, z_1 w \sin\theta, \cdots, z_{n-1} w \sin\theta, w^2 \sin\theta, 0, \cdots, 0)$$

for some θ with $(0 \leq \theta \leq \frac{\pi}{2})$.

Notice that by [Lemma 3.1, Hu2], any non-linear map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N \leq 3n-4$ has geometric rank one. Hence, Theorem 2.10 is a special case of Corollary 2.14.

3. The CR version of the Bonnet-Type Theorem

Now, we let M be a submanifold of $\mathbb{C}^n$. For any $p \in M$, we can define $T_p^{(1,0)}M$ and $T_p^{(0,1)}M$. If the complex dimension $CR_M(p)$ of $T_p^{(1,0)}M$ is independent of p, we call M a CR submanifold. When the real dimension of M is $2CR_M(p)+1$, we call M a CR submanifold of hypersurface type. For two CR submanifolds M_1 and M_2 with $p_j \in M_j$, we say (M_1, p_1) is CR equivalent to (M_2, p) if there is a smooth dffeomorphism F from (M_1, p_1) to (M_2, p_2) such that $F_*(T^{(1,0)}M_1) = T^{(1,0)}M_2$. Assume that $M \subset \partial\mathbb{B}^n$ be a CR submanifold of hypersurface type. We say that M has CR codimension l if $l = n-1-CR_M$. Webster's theorem gives the rigidity result for CR codimension 1.

For higher codimension, we have the following:

Theorem 3.1. *(Ebenfelt-Huang-Zaitsev [EHZ1]) Let (M_j, p_j) be CR submanifold of $\partial\mathbb{B}^n$ of CR codimension k. If $k < (n-1)/2$ and (M_1, p_1) is CR equivalent to (M_2, p_2), then there is*

$$T \in Aut(\partial\mathbb{B}^n)$$

such that $T(p_1) = p_2$ and $T((M_1, p_1)) = (M_2, p_2)$.

CR submanifolds of the sphere of hypersurface type come naturally as the link of isolated complex singularity.

Let $(V, 0) \subset \mathbb{C}^n$ be a germ of complex analytic variety of codimension k with isolated singularity at 0. Let $0 < \epsilon << 1$. Then $M_\epsilon = V \cap \partial\mathbb{B}^n_\epsilon$ is called the link of V with radius ϵ. M_ϵ is a CR submanifold of $\partial\mathbb{B}^n_\epsilon$ of hypersurface type of CR codimension k. As an application of the above theorem, one has the following:

Theorem 3.2. *(Ebenfelt-Huang-Zaitsev [EHZ1]) Let $(V_j, 0)$ be germs of complex analytic variety with isolated singularity at 0 of codimension $k < (n-1)/2$. Let $M_{j,\epsilon} = V_j \cap \partial\mathbb{B}^n_\epsilon$ be the corresponding link. If $(M_{1,\epsilon}, p_1)$ is CR equivalent to $(M_{2,\epsilon}, p_2)$ for certain $p_j \in M_{j,\epsilon}$. Then there is unitary transformation T such that $T(V_1) = V_2$.*

Corollary 3.3. *Let $(V, 0)$ be the germ of a complex analytic variety with isolated singularity at 0 of codimension $k < (n-1)/2$. Let $M_\epsilon = V \cap \partial\mathbb{B}^n_\epsilon$ be the ϵ-link. If (M_{ϵ_1}, p_1) is CR equivalent to (M_ϵ, p_2) for certain $p_j \in M_{\epsilon_j}$. Then there is unitary transformation T such that $T(V) = \frac{\epsilon_2}{\epsilon_1}V$.*

It is not clear to us if $k < (n-1)/2$ in the above theorem is necessary or not. We certainly belive that k cannot be too big with respect to n as in the situation discussed in the previous section. However, we were not able to provide a counter-example for $k = (n-1)/2$.

4. Super-rigidity and CR transversality for holomorphic mappings between hyperquadrics

In this section, we discuss rigidity for holomorphic mappings from a piece of real hyperquadric with positive signature into a hyperquadric in a complex space of larger dimension. Different from the situation for Heisenberg hypersurfaces (hyperquadrics with 0-signature, that is holomorphically equivalent to the unit sphere), the maps always possess strong super-rigidity property no matter what the codimension is. This phenomenon has analogy in holomorphic maps between irreducible bounded symmetric domains of rank at least two, as studied by many authors: Mok, Tsai, Tu, etc... (See the book of Mok [Mok] for results and extended references on the matter.) To state recent results that the first author jointly obtained with S. Baouendi, to we define, for any $0 < \ell < n-1$, the generalized Siegel upper half space $\mathbb{S}^n_\ell$ as follows:

$$\mathbb{S}^n_\ell := \{(z,w) = (z_1, \cdots, z_{n-1}, w) \in \mathbb{C}^n : w = u + iv,\ v > -\sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n-1} |z_j|^2\}.$$

Its boundary is the standard hyperquadric

$$\mathbb{H}^n_\ell := \{(z,w) = (z_1, \cdots, z_{n-1}, w) \in \mathbb{C}^n : w = u + iv,\ v = -\sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n-1} |z_j|^2\},$$

where ℓ is called the signature of $\mathbb{H}^n_\ell$. For $0 < \ell < n-1$, it is well known that any CR function defined over a connected piece M of $\mathbb{H}^n_\ell$ extends to a holomorphic function in a neighborhood of M in $\mathbb{C}^n$. The main technical result in [BH] is the following Theorem 4.1, which has two parts. The first part is on the super-rigidity property for holomorphic maps and the second part is on the CR transversality for CR mappings along the lines of studies as in Baouendi-Rothschild [BR], etc.

Theorem 4.1. *(Theorem 1.6 of Baouendi-Huang [BH]) Let M be a small neighborhood of 0 in $\mathbb{H}^n_\ell$ with $0 < \ell < n-1$. Suppose that $F = (f_1, \cdots, f_{N-1}, g)$ is a holomorphic map from a neighborhood U of M in $\mathbb{C}^n$ into $\mathbb{C}^N$ with $F(M) \subset \mathbb{H}^N_\ell$, $N \geq n$, and $F(0) = 0$. Suppose either $\ell \leq (n-1)/2$ or F preserves sides in the sense that $F(U \cap \mathbb{S}^n_\ell) \subset \mathbb{S}^N_\ell$. Then the following hold:*

(I). If $\frac{\partial g}{\partial w}(0) \neq 0$, then F is linear fractional. Moreover, there exits $\tau \in Aut_0(\mathbb{H}^N_\ell)$ such that either

$$\tau \circ F(z_1, \cdots, z_{n-1}, w) = (z_1, \cdots, z_{n-1}, 0, \cdots, 0, w), \ \ or$$

$$\tau \circ F(z_1, \cdots, z_{n-1}, w) = (z_{\ell+1}, \cdots, z_{n-1}, z_1, \cdots, z_\ell, 0, \cdots, 0, -w),$$

and the latter can only happen when $\ell = (n-1)/2$.

(II). If $\frac{\partial g}{\partial w}(0) = 0$, then $F(U) \subset \mathbb{H}^N_\ell$. More precisely, there is a constant $\ell \times (N - \ell - 1)$ complex matrix V, with $V\,V^t = \mathrm{Id}_\ell$, such that

$$g \equiv 0, \qquad (f_1, \cdots, f_\ell) \cdot V \equiv (f_{\ell+1}, \cdots, f_{N-1}). \tag{4.1}$$

We remark that $\mathbb{H}^n_\ell$ is linearly equivalent to $\mathbb{H}^n_{n-1-\ell}$. Also, when $\ell > \frac{n-1}{2}$, the side preserving assumption in Theorem 4.1 is crucial for results to hold. Indeed, let $F = (z_3, z_2^3, z_1, z_2, z_2^3, -w)$. Then F embeds $\mathbb{H}^4_2$ into $\mathbb{H}^6_2$ with $g_w(0) \neq 0$. But F is not linear fractional.

When $N < 2n-1$, Theorem 4.1 (I) already follows from a recent work of Ebenfelt-Huang-Zaitsev [EHZ2].

The above theorem has applications to the study of super-rigidity problems for proper holomorphic mappings between certain classical domains in complex projective space of different dimensions. Write

$$\mathbb{B}^n_\ell := \{[z_0, \cdots, z_n] \in \mathbb{CP}^n :\ |z_0|^2 + \cdots + |z_\ell|^2 > |z_{\ell+1}|^2 + \cdots + |z_n|^2\}.$$

Its automorphism group is a non-compact semi-simple Lie-group of rank $(\ell+1)$:

$$\mathrm{Aut}(\mathbb{B}^n_\ell) = \{\sigma :\in \mathrm{Hol}(\mathbb{CP}^n, \mathbb{CP}^n) : \sigma([z_0, \cdots, z_n]) = [z_0, \cdots, z_n] \cdot A,$$
$$A \in GL(n+1, \mathbb{C}),\ AE_{\ell+1,n+1}A^t = E_{\ell+1,n+1}.\}$$

where $E_{k,m}$ denotes a diagonal matrix with its first k diagonal elements -1 and the rest $+1$. As a corollary of Theorem 4.1 (I), one has the following:

THEOREM 4.2. *(Theorem 1.1 of [BH]) Let p be a boundary point of $\mathbb{B}^n_\ell$ and let U_p be a small neigborhood of p in $\mathbb{CP}^n$ with $U_p \cap \mathbb{B}^n_\ell$ connected. Assume that F is a holomorphic map from $U_p \cap \mathbb{B}^n_\ell$ into $\mathbb{B}^N_\ell$ for $N \geq n$ and $0 < \ell < n-1$. If for any sequence $\{Z_j\} \subset U_p \cap \mathbb{B}^n_\ell$ with $\lim_{j\to\infty} Z_j \in \partial\mathbb{B}^n_\ell$, $\{F(Z_j)\}$ only has limit points in $\partial\mathbb{B}^N_\ell$. Then F extends to a totally geodesic embedding from $\mathbb{B}^n_\ell$ into $\mathbb{B}^N_\ell$ in the sense that there is a certain $\sigma \in Aut(\mathbb{B}^n_\ell)$ and $\tau \in Aut(\mathbb{B}^N_\ell)$ such that $\tau \circ F \circ \sigma([z_0, \cdots, z_n]) = [z_0, \cdots, z_n, 0, \cdots, 0]$.*

COROLLARY 4.3. *(Corollary 1.2 of Baouendi-Huang [BH]) Any proper holomorphic mapping F from $\mathbb{B}^n_\ell$ into $\mathbb{B}^N_\ell$ for $N \geq n$ and $0 < \ell < n-1$ is a totally geodesic embedding, in the sense that there is a certain $\sigma \in Aut(\mathbb{B}^n_\ell)$ and $\tau \in Aut(\mathbb{B}^N_\ell)$ such that $\tau \circ F \circ \sigma([z_0, \cdots, z_n]) = [z_0, \cdots, z_n, 0, \cdots, 0]$.*

There is no proper holomorphic map from $\mathbb{B}^n_\ell$ into $\mathbb{B}^N_{\ell'}$ for $\ell' < \ell$. One cannot expect, in general, that Corollary 4.3 holds for any $\ell' > \ell$. For instance, one has the following:

EXAMPLE 4.4. Let $F = [z_0^2, \sqrt{2}z_0z_1, z_1^2, z_2^2, \sqrt{2}z_3z_2, z_1^2]$. Then F maps properly $\mathbb{B}^3_1$ into $\mathbb{B}^5_2$. Notice that F is not a linear map.

However, one has the following: (In light of the above example, the target dimension $n+k$ can not be improved. Also, we will use 0_j to denote a zero component at the j^{th}-position in a vector)

THEOREM 4.5. *(Theorem 1.8 of Baouendi-Huang [BH]) Let M be a small neighborhood of 0 in $\mathbb{H}^n_\ell$ with $0 < \ell < n-1$. For $k \geq 0$, let $F = (f_1, \cdots, f_{n+k-1}, g)$ be a holomorphic map from a neighborhood U of M in $\mathbb{C}^n$ into $\mathbb{C}^{n+k}$ with $F(M) \subset \mathbb{H}^{n+k}_{\ell+k}$, and $F(0) = 0$. Suppose that F preserves sides in the sense that $F(U \cap \mathbb{S}^n_\ell) \subset \overline{\mathbb{S}^{n+k}_{\ell+k}}$. Then the following hold:*
(I). If $\frac{\partial g}{\partial w}(0) \neq 0$, then F is linear fractional. Moreover, there exits $\tau \in Aut_0(\mathbb{H}^{n+k}_{\ell+k})$ such that either

$$\tau \circ F(z_1, \cdots, z_{n-1}, w) = (z_1, \cdots, z_\ell, 0_{\ell+1}, \cdots, 0_{k+\ell}, z_{\ell+1}, \cdots, z_{n-1}, 0, \cdots, 0, w). \tag{4.2}$$

(II). If $\frac{\partial g}{\partial w}(0) = 0$, then $F(U) \subset \mathbb{H}^{n+k}_{\ell+k}$. More precisely, when $\ell + k \geq n - \ell - 1$, there is an $(n-\ell-1) \times (\ell+k)$ constant complex matrix V, with $V\,V^t = \mathrm{Id}_{n-\ell-1}$, such that

$$g \equiv 0, \qquad (f_1, \cdots, f_{\ell+k}) \equiv (f_{\ell+k+1}, \cdots, f_{n+k-1})V. \tag{4.3}$$

When $\ell + k \leq n - \ell - 1$, there is an $(\ell+k) \times (n-\ell-1)$ constant complex matrix V, with $V\,V^t = \mathrm{Id}_{\ell+k}$, such that

$$g \equiv 0, \qquad (f_1, \cdots, f_{\ell+k})V \equiv (f_{\ell+k+1}, \cdots, f_{n+k-1}). \tag{4.3$'$}$$

COROLLARY 4.6. *(Theorem 4.1 of [BH]) Any proper holomorphic mapping F from $\mathbb{B}^n_\ell$ into $\mathbb{B}^{n+k}_{\ell+k}$ for $N = n+k \geq n$ and $0 < \ell < n-1$ is a totally geodesic embedding, in the sense that there is a certain $\sigma \in Aut(\mathbb{B}^n_\ell)$ and $\tau \in Aut(\mathbb{B}^N_\ell)$ such that*

$$\tau \circ F \circ \sigma([z_0, \cdots, z_n]) = [z_0, \cdots, z_\ell, 0_{\ell+1}, \cdots, 0_{\ell+k}, z_{\ell+k+1}, \cdots, z_n, 0, \cdots, 0]. \tag{4.4}$$

The proof of Theorem 4.1 (II) and Theorem 4.5 (II) is based on the following, which is very useful in some other applicationst (For the definition of $\mathbb{H}^N_{\ell,\ell',n}$ see (2.2) of [BH]):

THEOREM 4.7. *(Lemma 4.1 of Baouendi-Huang [BH]) Let* $F=(f,\phi,g)$ *be a holomorphic map from a neighborhood* M *of* 0 *in* $\mathbb{H}^n_\ell$ *into* $\mathbb{H}^N_{\ell,\ell',n}$, $\ell'\geq\ell>0, N\geq n>1$, *with* $F(0)=0$. *Assume that either* $\ell'<n-1$ *or* $N-\ell'-1<n-1$. *For each* $p\in M$, *let* $F_p=(\widetilde{f_p},g_p)$ *be defined as in [(3.1), BH]. If* $(g_p)_w(0)=0$ *for all* p *sufficiently close to the origin, then* $g\equiv 0$ *and* $F(U)\in\mathbb{H}^N_{\ell,\ell',n}$, *where* U *is a small neighborhood of* 0 *in* $\mathbb{C}^n$. *More precisely, when* $\ell'\geq(N-\ell'-1)$, *there exists a constant* $(N-\ell'-1)\times\ell'$ *complex matrix* V *with* $VV^t=\mathrm{Id}_{N-\ell'-1}$ *such that*

(4.5)

$$g\equiv 0,\quad (f_1,\cdots,f_\ell,f_n,\cdots,f_{n+\ell'-\ell-1})\equiv\ (f_{\ell+1},\cdots,f_{n-1},f_{n+\ell'-\ell},\cdots,f_{N-1})\,V;$$

when $\ell'<(N-\ell'-1)$, *there exists a constant* $\ell'\times(N-\ell'-1)$ *complex matrix* V *with* $V\cdot V^t=\mathrm{Id}_{\ell'}$ *such that*

$(4.5)'$

$$g\equiv 0,\quad (f_1,\cdots,f_\ell,f_n,\cdots,f_{n+\ell'-\ell-1})\cdot V\equiv(f_{\ell+1},\cdots,f_{n-1},f_{n+\ell'-\ell},\cdots,f_{N-1}).$$

The last sentence of [Theorem 1.6 (ii), BH] is to give an equivalent expression of the mathematical content of [Theorem 1.6(ii), BH]). In that sentence, there is a typo which the authors of [BH] would like to correct by making use of this opportunity: Namely, $(f_1,\cdots,f_\ell)\equiv(f_{\ell+1},\cdots,f_{N-1})V$ should be correcetd to $(f_1,\cdots,f_\ell)V\equiv(f_{\ell+1},\cdots,f_{N-1})$. (Therefore, the size of the matrix V should be corrected as $\ell\times(N-\ell-1)$). Similar corections are needed in the last formula of [Theorem 1.8(ii), BH] in the case of $\ell+k<n-\ell-1$, and in the formula [(4.1), Lemma 4.1, BH] in the case of $\ell'<N-\ell'-1$. ([Theorem 1.6 (ii), BH], [Theorem 1.8 (ii), BH] and [Lemma 1.6, BH] (with these typos corrected) are copied here as Theorem 4.1(ii), Theorem 4.5 (ii) and Theorem 4.7, respectively.)

One can also study the rigidity for CR submanifolds embedded into $\mathbb{H}^n_\ell$. However, since $\mathbb{H}^n_\ell$ contains complex subspaces, we have to exclude those CR submanifolds embedded in the complex varieties in $\mathbb{H}^n_\ell$. For this, we say M is a CR transversal submanifold of $\mathbb{H}^n_\ell$ of hypersurface type if the complex normal direction of M is also along the complex normal direction of the hyperquadric. Then we have the following:

THEOREM 4.8. *(Ebenfelt-Huang-Zaitsev [EHZ2]) Let* M_j *be two CR transveral CR submanifolds of hypersurface type. Assume they are Levi non-degenerate with signature* ℓ, *and of CR codimension* k. *If* $2k<n-1$ *and they are CR equivalent, then they are rigid motion of each other.*

It is an open question if the codiemnsion restriction in the above result can be dropped or not.

5. Rigidity Problems for Holomorphic Maps with Symmetry

Here, we mention a well-known super-rigidity problem in differential geometry: Let G_1 and G_2 be two non-compact semi-simple Lie groups with $dim(G_2)\geq dim(G_1)$. Suppose $\Gamma_j\subset G_j$ are lattices (with certain density or co-compactness properties) and there is a injective homomorphism $\phi:\Gamma_1\to\Gamma_2$. Can one then extend ϕ to a global homomorphism from G_1 into G_2? This question has been quite well understood when the real ranks of the groups are at least 2 by the work of Mostow, Margulis and Mok-Siu-Yeung, etc. In the rank one case, there are essentially four cases to be considered: The automorphism groups of real, complex and quaternionic hyperbolic spaces, together with that of the hyperbolic Cayley plane. After the work of many people, an important remaining open case is when $G_1 = Aut(\mathbb{B}^n)$ and $G_2 = Aut(\mathbb{B}^N)$ with $1 < n < N$, Γ_1 co-compact in G_1 and $\phi(\Gamma_1)$ convex-cocompact. (Notice that when $n = 1$, certain kind of counter-examples have been constructed by Mostow). This problem, after applying the harmonic mapping theory of Siu, is reduced to the following :

Problem 5.1. *(Siu, Mok, etc) Let $\mathbb{B}^n \subset \mathbb{C}^n$ and $\mathbb{B}^N \subset \mathbb{C}^N$ be the unit balls, and let f be a proper holomorphic embedding from $\mathbb{B}^n$ into $\mathbb{B}^N$ $(n, N > 1)$. Suppose that there is a discrete subgroup $\Gamma \subset Aut(\mathbb{B}^N)$ such that Γ fixes $M = f(\mathbb{B}^n)$ and acts co-compactly over M (without fixed point). Is f then a linear embedding?*

This question has been answered in the affirmative by Cao-Mok [CaoM] in case($N \leq 2n - 1$).

Theorem 5.2. *(Cao-Mok [CM], 1990) Let Γ be a lattice of $Aut(\mathbb{B}^N)$ without fixed point. Let F be a proper holomorphic embedding from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N \leq 2n - 1$ such that Γ stablize and act co-compactly on the image of $F(\mathbb{B}^n)$. Then F is linear.*

Lastly, we mention two results on the bounded symmetric domains:

Theorem 5.3. *(Tsai [Ts], 1993) Let Ω_1 and Ω_2 be two bounded symmetric domains with the same rank at least 2. then any proper holomorphic map from Ω_1 into Ω_2 is a totally geodesic embedding with respect to their Bergman metric.*

Theorem 5.4. *(Tu [Tu], 2002) Let $D^I_{p,p-1}$ and $D^I_{p,p}$ be the standard bounded symmetric domains of the first type with $p - 1 \geq 2$. Then any proper holomorphic map from $D^I_{p,p-1}$ into $D^I_{p,p}$ is a totally geodesic embedding with respect to their Bergman metric.*

6. Segre family of real analytic hypersurfaces and rigidity problems for holomorphic Segre maps

Real analytic CR maps between real analytic CR manifolds in complex spaces naturally induce what we call the holomorphic Segre maps. However, not all holomorphic Segre maps are coming from CR maps. That motivated us to consider the more general rigidity problems for holomorphic Segre maps. This study seems still at a very early stage and we compute in this part of this paper the holomorphic Segre isomorphisms of the complexification of the Heisenberg hypersurface. We start with some basic notations.

Let M be a real analytic hypersurface in $D \subset \mathbb{C}^n$ with real analytic defining function $r \in C^\omega(D)$. Apparently, for any other local defining function r^* of M, $r^* = s^* r$ with $s^*|_M \neq 0$. Hence we can well define its complexification as the complex submanifold: $\mathcal{M} = \{(z, \xi) \in D \times Conj(D) \ : r(z, \xi) = 0\}$. $\mathcal{M}$ is a complex submanifold of complex codimension 1 in $\mathbb{C}^n \times \mathbb{C}^n$ near $M \times conj(M)$. Here for a set $E \subset \mathbb{C}^n$, $Conj(E) := \{\overline{z} \mid z \in E\}$. For each $\xi \sim Conj(M)$, we can define complex analytic varieties $Q_\xi := \{z \in \mathbb{C}^n : r(z, \xi) = 0\}$ and $\hat{Q}_z := \{\xi \in \mathbb{C}^n : r(z, \xi) = 0\}$. We call Q_ξ and $\hat{Q}_z$ the *Segre variety* of M with respect to ξ and z, respectively. Notice that $\mathcal{M}$ is holomorphically foliated by $\{Q_\xi \times \{\xi\}\}$ and also by $\{\{z\} \times \hat{Q}_z\}$.

Due to this reason, $\mathcal{M}$ is called the Segre family associated with M. A fundamental fact for Segre family is its invariant property for holomorphic maps. More precisely, if f is a local holomorphic map from (M,p) to $(\widetilde{M},\widetilde{p})$, then $f(Q_\xi) \subset \widetilde{Q}_{\overline{f}(\xi)}$ and $\overline{f}(\hat{Q}_z) \subset \hat{\widetilde{Q}}_{f(z)}$. Here, for instance, $\widetilde{Q}_{\overline{f}(\xi)}$ is the Segre variety of $\widetilde{M}$ with respect to $\overline{f}(\xi)$. In particular, when f is a holomorphic map from (M,p) to $(\widetilde{M},\widetilde{p})$, f induces a holomorphic map $\mathcal{F} := (f(z), \overline{f}(\xi))$ from $(\mathcal{M},(p,\overline{p}))$ to $(\widetilde{\mathcal{M}},(\widetilde{p},\overline{\widetilde{p}}))$.

Next, let $(\widetilde{M},\widetilde{p})$ be another real analytic hypersurface near $\widetilde{p} \in \mathbb{C}^N$. If there is a holomorphic map f from (M,p) to $(\widetilde{M},\widetilde{p})$, then we have a holomorphic map $(f(z),\overline{f}(\xi))$ from $(\mathcal{M},(p,\underline{p}))$ to $(\widetilde{\mathcal{M}},(\widetilde{p},\widetilde{p}))$. We say that Φ is a holomorphic Segre map from $(\mathcal{M},A)$ into $(\widetilde{\mathcal{M}},\widetilde{A})$ if Φ is a holomorphic map from $(\mathcal{M},A)$ into $(\widetilde{\mathcal{M}},\widetilde{A})$ such that Φ sends each $Q_\xi \times \{\xi\}$ of $\mathcal{M}$ near A into a certain $Q_{\widetilde{\xi}} \times \{\widetilde{\xi}\}$ of $\widetilde{\mathcal{M}}$ near $\widetilde{A}$ and sends each $\{z\} \times \hat{Q}_z$ into a certain $\{\widetilde{z}\} \times \hat{\widetilde{Q}}_{\widetilde{z}}$. A holomorphic Segre map Φ is called a holomorphic Segre isomorphism if it is also a local biholomorphism. If this is the case, we say $(\mathcal{M},A)$ is Segre-equivalent to $(\widetilde{\mathcal{M}},\widetilde{A})$. In what follows, we only consider the case when Φ is a holmorphic embedding. It can be seen that Φ can be made to take the form: $\Phi(z,\xi) = (\Phi_1(z),\Phi_2(\xi))|_{\mathcal{M}}$. (See Lemma 6.1 below). Apparently, when (M,p) is equivalent to $(\widetilde{M},\widetilde{p})$, then $(\mathcal{M},A)$ is Segre-equivalent to $(\widetilde{\mathcal{M}},\widetilde{A})$ with $A = (p,\overline{p})$ and $\widetilde{A} = (\widetilde{p},\overline{\widetilde{p}})$. We mention that even if M, $\widetilde{M}$ are strongly pseudoconvex, Faran has constructed examples showing that the converse of this statement fails (See [Fa3]).

For simplicity, assume that $p = 0 \in M$ and that M is defined by an equation of the form $r = 2\mathrm{Im}(z_n) + O(|z'| + |\mathrm{Re}(z_n)|)$. On $\mathcal{M}$, there are $(2n-1)$ independent holomorphic one forms

(6.1)
$$\theta^\alpha = dz^\alpha|_{\mathcal{M}},\ \theta_\alpha = d\xi_\alpha|_{\mathcal{M}},\ 1 \le \alpha < n-1, \theta = i d_z r|_{\mathcal{M}} = i r_\alpha dz^\alpha|_{\mathcal{M}} + i r_n dz^n|_{\mathcal{M}}.$$

$\{\theta,\theta^\alpha,\theta_\alpha\}$ is a co-frame for $\mathcal{M}$. In what follows, α,β have range from 1 to $(n-1)$ and the summation convention will be applied. Then $\{\theta,\theta_\alpha\}$ and $\{\theta,\theta^\alpha\}$ are two complete systems, whose corresponding foliations are precisely $\{Q_\xi \times \{\xi\}\}$ and $\{\{z\} \times \hat{Q}_z\}$, respectively.

Also, let $(\widetilde{M},\widetilde{p})$ be another real analytic hypersurface near $\widetilde{p} = 0$ in $\mathbb{C}^N$ with a defining function $\widetilde{r} = 2Im(\widetilde{z}_N) + O(|\widetilde{z}'| + Re(z_N))$. Define similarly the co-frame $\{\widetilde{\theta},\widetilde{\theta}^\alpha,\widetilde{\theta}_\alpha\}$ on $\widetilde{\mathcal{M}}$ near $(\widetilde{p},\widetilde{p})$.

Lemma 6.1. *Suppose Φ is a holomorphic Segre embedding from $(\mathcal{M},0)$ to $(\widetilde{\mathcal{M}},0)$. Then*

$$\begin{cases} \Phi^*(\widetilde{\theta}) = u\theta, \\ \Phi^*(\widetilde{\theta}^\alpha) = u^\alpha\theta + \sum_\beta u^\alpha_\beta \theta^\beta, \\ \Phi^*(\widetilde{\theta}_\alpha) = u_\alpha\theta + \sum_\beta v^\beta_\alpha \theta_\beta. (6.2) \end{cases} \tag{6.2}$$

where $u, u^\alpha, u^\alpha_\beta, v^\beta_\alpha$ are holomorphic near 0 and the holomorphic 1-forms are defined as in (6.1). Moreover $\Phi = (\Phi_1(z),\Phi_2(\xi))|_{\mathcal{M}}$.

Proof. Suppose that Φ is a holomorphic Segre embedding. Then

$$\{\Phi^*(\widetilde{\theta}),\Phi^*(\widetilde{\theta}^\alpha)\} \quad\text{and}\quad \{\Phi^*(\widetilde{\theta}),\Phi^*(\widetilde{\theta}_\alpha)\}$$

are complete differential systems over $\mathcal{M}$. Since Φ preserves the Segre varieties, the restriction of $\{\Phi^*(\widetilde{\theta}), \Phi^*(\widetilde{\theta}^\alpha)\}$ (respectively, $\{\Phi^*(\widetilde{\theta}), \Phi^*(\widetilde{\theta}_\alpha)\}$) to the leaves of $\{\theta, \theta^\alpha\}$ (respectively, $\{\theta, \theta_\alpha\}$) must be zero. We conclude the last two formulas in (6.2). The first formula is also obvious because the restriction of $\Phi^*(\widetilde{\theta})$ to leaves of both foliations through the Segre varieties is zero.

Write $\Phi = (\Phi_1, \Phi_2)$. Parametrize $\mathcal{M}$ by (z_α, ξ). Then Φ_2 now can be regarded as a holomorphic function in (z_α, ξ). Since for each fixed ξ, Φ maps $\{Q_\xi \times \{\xi\}\}$ into $\{\widetilde{Q}_{\widetilde{\xi}} \times \{\widetilde{\xi}\}\}$ for a certain $\widetilde{\xi}$, $\Phi_2(z_a, \xi)$ must be constant as z_a varies. That shows that Φ_2 can be expressed as the restriction of a holomorphic function in ξ to $\mathcal{M}$. Similarly, one can verify that Φ_1 can also be regarded as just a holomorphic map in z. □

Now, assume that $\mathcal{M}$ is defined near 0 by a real analytic function of the form: $r = z_n - \rho(z^\alpha, \xi_\alpha, \xi_n)$ with $\rho = O(|z^\alpha| + |\xi_\alpha| + |\xi_n|)$. Assume that M is Levi non-degenerate near 0. Then

$$\begin{cases} z^\alpha = z^\alpha \\ z_n = z_n \\ \rho_\alpha = \rho_\alpha(z^\alpha, \xi_\alpha, \xi_n) \end{cases}$$

can be used to uniquely solve for $(z^\alpha, \xi_\alpha, z_n)$ by the data $(z^\alpha, \rho_\alpha, z^n)$. (See [Ch], [CJ1], etc). Hence, we can use $(z^\alpha, z^n, \rho_\alpha)$ for the coordinates of $\mathcal{M}$ near 0. In the $(z^\alpha, z_n, \rho_\beta)$ coordinates, we define the holomorphic frame $\{\theta^\alpha, \theta_\alpha, \theta\}$ as follows:

$$\begin{cases} \theta = i(dz^n - \rho_\alpha dz^\alpha), \\ \theta^\alpha = dz^\alpha, \\ \theta_\alpha = d\rho_\alpha - \rho_{\alpha\beta} dz^\beta, \quad d\theta = i\theta^\alpha \wedge \theta_\alpha. \end{cases} \tag{6.3}$$

Here $\rho_\alpha = \frac{\partial \rho}{\partial z_\alpha}$, $\rho_{\alpha\beta} = \frac{\partial \rho}{\partial z_\alpha \partial z_\beta}$ are holomorphic functions in (z, ρ_α). Then $\{\theta, \theta_\alpha\}$($\{\theta, \theta^\alpha\}$) also generates the foliation $\{Q_\xi \times \{\xi\}\}$ ($\{\{z\} \times \hat{Q}_z\}$, respectively).

Next, let $(\widetilde{\mathcal{M}}, 0)$ be the complexification of another real analytic hypersurface $(\widetilde{M}, 0)$. We also choose the same type of the co-frame $(\widetilde{\theta}, \widetilde{\theta}^\alpha, \widetilde{\theta}_\alpha)$ on $(\widetilde{\mathcal{M}}, 0)$ as in (6.3). Suppose that Φ is a holomorphic Segre isomorphism from $(\mathcal{M}, 0)$ to $(\widetilde{\mathcal{M}}, 0)$. Then by Lemma 6.1, we have

$$\begin{cases} \Phi^*(\widetilde{\theta}) = u\theta, \\ \Phi^*(\widetilde{\theta}^\alpha) = u^\alpha_\beta \theta^\beta + u^\alpha \theta, \\ \Phi^*(\widetilde{\theta}_\alpha) = v^\beta_\alpha \theta_\beta + v_\alpha \theta. \end{cases}$$

with $u, u^\alpha_\beta, v^\alpha_\beta, u^\alpha, v_\beta$ holomorphic near the origin.

Since $d\widetilde{\theta} = i\widetilde{\theta}^\alpha \wedge \widetilde{\theta}_\alpha$, we get $du \wedge \theta + ud\theta = i(u^\alpha_\beta \theta^\beta + u^\alpha \theta) \wedge (v^\beta_\alpha \theta_\beta + v_\alpha \theta)$, from which we get the following

$$\begin{cases} \delta^l_k u = u^\alpha_l v^k_\alpha, \\ du = iu^\alpha_\beta v_\alpha \theta^\beta - iu^\alpha v^\beta_\alpha \theta_\beta + t\theta. \end{cases}$$

Next, we consider the $\mathbb{C}^* := \mathbb{C} - \{0\}$ fiber bundle $\mathcal{E}_0 = \mathcal{M} \times \mathbb{C}^*$ over $\mathcal{M}$, which can be identified with the $\mathbb{C}^*$-fiber bundle over $\mathcal{M}$, whose fiber $\pi^{-1}(P)$ over $P \in \mathcal{M}$ is precisely $\{u\theta|_P\}$ with $u \in \mathbb{C}^*$. Then $\omega = u\theta$ is a tautological global holomorphic 1-form on $\mathcal{E}_0$. Notice that

$$d\omega = ud\theta + du \wedge \theta = iu\theta^\alpha \wedge \theta_\alpha + \omega \wedge (-\frac{du}{u}).$$

Define

$$\begin{cases} \omega^\alpha = u^\alpha \theta + u^\alpha_\beta \theta^\beta, \\ \omega_\alpha = v_\alpha \theta + v^\beta_\alpha \theta_\beta, \end{cases}$$

where $u^\gamma_\beta v^\beta_\kappa = \delta^\gamma_\kappa u$. Then

$$d\omega = i\omega^\alpha \wedge \omega_\alpha + \omega \wedge \Big(- \frac{du}{u} - i\frac{u^\alpha}{u} v^\beta_\alpha \theta_\beta + iu^\alpha_\beta \frac{v_\alpha}{u}\theta^\beta \Big).$$

Let $\phi = -\frac{du}{u} - i\frac{u^\alpha}{u} v^\beta_\alpha \theta_\beta + iu^\alpha_\beta \frac{v_\alpha}{u}\theta^\beta + t\omega$. Then, the above motivates us to consider the following set of 1-forms:

$$\begin{cases} \omega = u\theta, \\ \omega^\alpha = u^\alpha \theta + u^\alpha_\beta \theta^\beta, \\ \omega_\alpha = v_\alpha \theta + v^\beta_\alpha \theta_\beta, \\ \phi = -\frac{du}{u} - i\frac{u^\alpha}{u} v^\beta_\alpha \theta_\beta + iu^\alpha_\beta \frac{v_\alpha}{u}\theta^\beta + t\theta \\ \delta^l_k u = u^l_\alpha v^\alpha_k. \end{cases} \tag{6.4}$$

A basic property for the above set of 1-forms is the relation:

$$d\omega = i\omega^\alpha \wedge \omega_\alpha + \omega \wedge \phi. \tag{6.5}$$

Choose a special co-frame:

$$\begin{cases} \omega^0 = u\theta, \\ \omega^{0\alpha} = u\theta^\alpha, \\ \omega^0_\alpha = \theta_\alpha, \\ \phi^0 = -\frac{du}{u}. \end{cases}$$

Then, we have

$$\begin{cases} \omega = \omega^0, \\ \omega^\alpha = \frac{u^\alpha}{u}\omega^0 + \frac{u^\alpha_\beta}{u}\omega^{0\beta} \\ \omega_\alpha = \frac{v_\alpha}{u}\omega^0 + v^\beta_\alpha \omega^0_\alpha, \\ \phi = \phi^0 - i\frac{u^\alpha}{u} v^\beta_\alpha \omega^0_\beta + iu^\alpha_\beta \frac{v_\alpha}{u^2}\omega^{0\alpha} + t\omega^0. \end{cases}$$

Hence the space of the set of 1-forms in (6.4) forms a G_1-structure bundle $\mathcal{Y}$ over $\mathcal{E}_0$, where G_1 consists of matrices of the following form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{u^\alpha}{u} & \frac{u^\alpha_\beta}{u} & 0 & 0 \\ \frac{v_\alpha}{u} & 0 & v^\beta_\alpha & 0 \\ t & i\frac{v^\alpha}{u^2}u^\beta_\alpha & -i\frac{u_\alpha}{u}v^\alpha_\beta & 1 \end{pmatrix}$$

with $u^l_\alpha v^\alpha_k = \delta^l_k u$. Or

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ u^\alpha & u^\alpha_\beta & 0 & 0 \\ v_\alpha & 0 & v^\beta_\alpha & 0 \\ t & iv^\alpha u^\beta_\alpha & -iu_\alpha v^\alpha_\beta & 1 \end{pmatrix}$$

with $u^l_\alpha v^\alpha_k = \delta^l_k$.

Now, the Segre family $(\mathcal{M}, 0)$ and $(\widetilde{\mathcal{M}}, 0)$ are equivalent if and only if there is a holomorphic map $\mathcal{F}$ from $(\mathcal{E}_0, P_0)$ to $(\widetilde{\mathcal{E}_0}, \widetilde{P_0})$ with $P_0 \in \pi^{-1}(0)$, $\widetilde{P_0} \in \widetilde{\pi}^{-1}(0)$, such that

$$F^* \begin{pmatrix} \widetilde{\omega}^0 \\ \widetilde{\omega}^{0\alpha} \\ \widetilde{\omega}^0_\alpha \\ \widetilde{\phi}^0 \end{pmatrix} (P) = \gamma_{\mathcal{F}}(P) \begin{pmatrix} \omega^0 \\ \omega^{0\alpha} \\ \omega^0_\alpha \\ \phi^0 \end{pmatrix} (P)$$

where $\gamma_{\mathcal{F}}(P)$ is a holomorphic map from a neighborhood of P_0 into G_1.

We consider the G_1-structure bundle $\mathcal{Y}$ over $\mathcal{E}_0$ and lift the above to globally defined holomorphic tautological 1-forms over $\mathcal{Y}$. A fundamental theorem of Chern-Moser and Chern ([CM], [Ch]) states that these forms can be indeed uniquely completed to make the G_1-equivalence problem into an e-equivalence problem over $\mathcal{Y}$. Also, $\mathcal{F}$ is uniquely determined by $\gamma_{\mathcal{F}}(P_0)$ and $\widetilde{P_0}$. (See [Hu4], [Ga]).

We discuss how to determine the holomorphic Segre isomorphisms through their jets at 0. Without loss of generality, we can assume that $\mathcal{M}, \widetilde{\mathcal{M}}$ are defined by the normalized equations as in Chern-Moser [CM]. Then $r = z_n - \xi_n = 2\pi z' \cdot \xi' + O(4)$. Write $F = (f(z), g(z), \phi(\xi), \psi(\xi))$ for a holomorphic Segre isomorphism from $(\mathcal{M}, 0)$ to $(\widetilde{\mathcal{M}}, 0)$. Hence, we have $g(z) - \psi(\xi) = 2\pi i f(z) \cdot \psi(\xi) + O(4)$. Here $z = (z', z_n)$, $\xi = (\xi', \xi_n)$. Write $F^*(\widetilde{\theta}) = k(z, \xi)\theta$, $P_0 = (0, 1) \in \pi^{-1}(0)$. Then the lifting of F to $\mathcal{F}$, a holomorphic map from $(\mathcal{E}_0, P_0)$ to $(\widetilde{\mathcal{E}_0}, \widetilde{P_0})$, takes the form: $\mathcal{F} = (F, u/k)$. Notice that $k(0) = g'_{z_n}(0)$. Hence, if $k(0)$ is fixed, then $\widetilde{P}_0 = \mathcal{F}(P_0)$ is fixed too. Next, in the matrix $\gamma_{\mathcal{F}}(P_0)$, $u^\alpha, u^\alpha_\beta, v^\alpha_\beta, v_\alpha$ are determined by the first jets of f, ϕ at 0. It is also clear that t is determined by $\frac{\partial^2 k}{\partial z_n^2}(0)$, which, through a direct computation, can be seen to be determined by $g''_{z_n z_n}(0)$ and $f'_{z_n}(0), \phi'_{\xi_n}(0)$. Therefore, we see that F is completely determined by the first jet of f, ϕ at 0, and $g'_{z_n}(0)$, $g''_{z_n z_n}(0)$.

Write $\mathcal{H}^n := \{(z', w, \xi', \eta) \in \mathbb{C}^{n-1} \times \mathbb{C}^1 \times \mathbb{C}^{n-1} \times \mathbb{C}^1 : w - \eta = 2iz' \cdot \xi\}$ for the complexification of the Heisenberg hypersurface $\mathbb{H}^n$. A natural question one would like to ask is the following:

Problem 6.2. *Let Φ be a holomorphic Segre embedding from $(\mathcal{H}^n, 0)$ into $(\mathcal{H}^N, 0)$ with $N \geq n \geq 2$. What type of rigidity can we get for Φ?*

We say $\Psi \in Aut(\mathcal{H}^n)$ if Ψ is a bimeromorphic map from $\mathcal{H}^n$ to itself, preserving the Segre varieties (both Q and $\hat{Q}$). For any $A_0 = (z'_0, w_0, \xi'_0, \eta_0) \in \mathcal{H}^n$, define $\sigma^0_{A_0}$ by sending $(z', w, \xi', \eta) \in \mathbb{C}^{n-1} \times \mathbb{C}^1 \times \mathbb{C}^{n-1} \times \mathbb{C}^1$ to $(z' + z'_0, w + w_0 + 2iz' \cdot \xi'_0, \xi' + \xi'_0, \eta + \eta_0 - 2i\xi' \cdot z'_0)$. Then $\sigma^0_{A_0} \in Aut(\mathcal{H}^n)$ with $\sigma^0_{A_0}(0) = A_0$. Hence, to understand $Aut(\mathcal{H}^n)$, it suffices to compute $Aut_0(\mathcal{H}^n)$, the set of local self-Segre isomorphisms of $(\mathcal{H}^n, 0)$.

Write $z' = (z_1, \cdots, z_{n-1})$, $\xi' = (\xi_1, \cdots, \xi_{n-1})$, $w, \eta \in \mathbb{C}$, and $z = (z', w)$ $\xi = (\xi', \eta)$ as before. Write $\mathcal{I}_0$ for the group generated by the following type of linear fraction maps: $F(z, \xi) = (S(z), T(\xi)) := (S_1(z), \cdots, S_{n-1}(z), S_n(z), T_1(\xi),$

$\cdots, T_{n-1}(\xi), T_n(\xi)) = (\widetilde{S}(z), S_n(z), \widetilde{T}(\xi), T_n(\xi))$ defined by

$$\widetilde{S}(z) = \frac{\lambda(z' + \vec{a}w)U}{1 - 2i\langle z', \vec{e}\rangle + e_n w}, \quad S_n(z) = \frac{\lambda w}{1 - 2i\langle z', \vec{e}\rangle + e_n w}, \tag{6.6}$$

$$\widetilde{T}(\xi) = \frac{(\xi' + \vec{e}\eta)V}{1 + 2i\langle \xi', \vec{a}\rangle + (e_n + 2i\langle \vec{e}, \vec{a}\rangle)\eta}, \tag{6.7}$$

$$T_n(\xi) = \frac{\lambda \eta}{1 + 2i\langle \xi', \vec{a}\rangle + (e_n + 2i\langle \vec{e}, \vec{a}\rangle)\eta}; \tag{6.8}$$

where U, V are non-singular $(n-1)\times(n-1)$ matrices of complex numbers with $U \cdot V^t = \mathrm{Id}$, $\vec{a} = (a_1, ..., a_{n-1})$, $\vec{e} = (e_1, ..., e_{n-1}) \in \mathbb{C}^{n-1}$, $\lambda \in \mathbb{C}^*$, $e_n \in \mathbb{C}$, $\langle \vec{x}, \vec{y}\rangle = \vec{x} \cdot \vec{y}^t$ for any $\vec{x}, \vec{y} \in \mathbb{C}^{n-1}$. Also, F is uniquely determined by the data $\lambda, \vec{a}, \vec{e}, e_n, A$.

A direct computation shows that $\mathcal{I}_0$ is a subgroup of $Aut_0(\mathcal{H}^n)$ under the action of compositions. From the above arguments, we have the following

Theorem 6.3. *Let Φ is a local holomorphic Segre self-isomorphism of $(\mathcal{H}_n, 0)$. Then Φ is the restriction of a certain element in $\mathcal{I}_0$. In particular, $Aut_0(\mathcal{H}^n) = \mathcal{I}_0$.*

Proof. Let $F = (f, g, \phi, \psi) \in Aut_0(\mathcal{H}^n)$. Assume that $g'_w(0) = \sigma$, $f'_{z'} = A$, $\phi'_{\xi'} = B$. Then it is easy to see that $\sigma = A \cdot B^t$. After composing F from the left by a map as in (6.6) (6.7) and (6.8) with $U = (\lambda A)^{-1}, V = B^{-1}, \lambda = \sigma^{-1}, \vec{a}, \vec{e}, e_n = 0$, we can assume that $A = B = \mathrm{Id}, \sigma = 1$. Composing F from the left with a map in (6.6)(6.7) and (6.8) with $\lambda = 1, U, V = \mathrm{id}, e_n = 0, \vec{a} = -f'_w(0), \vec{e} = -\phi'_w(0)$, we can further assume that $f_w = \phi_\eta = 0$. Finally, composing F from the left with $U = V = \mathrm{Id}, \lambda = 1, \vec{a}, \vec{e}, = 0, e_n = \frac{1}{2}g''_{ww}(0)$, we can also make $g''_{ww}(0) = 0$. Now, by the previous arguments on the unique determination of F, we conclude that F is inside the group $\mathcal{I}_0$. $\square$

Remark 6.4. Let $0 < \ell < n-1$. The Segre family $\mathcal{H}^n_\ell$ of $\mathbb{H}^n_\ell$ is apparently isomorphic to the Segre family of the Heisenberg group through the map

$$(z', w, \xi', \eta) \to (iz_1, \cdots, iz_\ell, z_{\ell+1}, \cdots, w, i\xi_1, \cdots, i\xi_\ell, \xi_{\ell+1}, \cdots, \eta).$$

Hence, we can use Theorem 6.3 to write down the holomorphic Segre self-isomorphisms of $\mathcal{H}^n_\ell$. Namely, $Aut_0(\mathcal{H}^n_\ell)$ consists of the elements of the following form:

$$\widetilde{S}(z) = \frac{\lambda(z' + \vec{a}w)U}{1 - 2i\langle z', \vec{e}\rangle_\ell + e_n w}, \quad S_n(z) = \frac{\lambda w}{1 - 2i\langle z', \vec{e}\rangle_\ell + e_n w},$$

$$\widetilde{T}(\xi) = \frac{(\xi' + \vec{e}\eta)V^t}{1 + 2i\langle \xi', \vec{a}\rangle_\ell + (e_n + 2i\langle \vec{e}, \vec{a}\rangle_\ell)\eta},$$

$$T_n(\xi) = \frac{\lambda \eta}{1 + 2i\langle \xi', \vec{a}\rangle_\ell + (e_n + 2i\langle \vec{e}, \vec{a}\rangle_\ell)\eta};$$

where U, V are non-singular $(n-1)\times(n-1)$ matrices of complex numbers with $U \cdot E_\ell \cdot V^t = E_\ell$ (E_ℓ is the diagonal matrix whose first ℓ diagonal elements are -1 and the other diagonal elements are 1,) $\vec{a} = (a_1, ..., a_{n-1})$, $\vec{e} = (e_1, ..., e_{n-1}) \in \mathbb{C}^{n-1}$, $\lambda \in \mathbb{C}^*$, $e_n \in \mathbb{C}$. Also, $\langle \vec{x}, \vec{y}\rangle_\ell = -\sum_{j=1}^{\ell} x_j y_j + \sum_{j=1+\ell}^{n-1} x_j y_j$ for $\vec{x}, \vec{y} \in \mathbb{C}^{n-1}$.

References

[Alx1] H. Alexander, *Holomorphic mappings from ball and polydisc*, Math. Ann. **209**(1974), 245–256.

[Alx2] H. Alexander, *Proper holomorphic maps in* $\mathbb{C}^n$, Ind. Univ. Math. Journal, **26** (1977), 137–146.

[BER1] S. Baouendi, P. Ebenfelt and L. Rothschild, *Real Submanifolds in Complex Space and Their Mappings, Princeton Mathematical Series*, **47**, Princeton University Press, Princeton, NJ, 1999.

[BER2] S. Baouendi, P. Ebenfelt and L. Rothschild, *Local geometric properties of real submanifolds in complex space*, Bull. Amer. Math. Soc. (N.S.) **37** (2000), no. 3, 309–33.

[BH] S. Baouendi and X. Huang, *Super-Rigidity for holomorphic mappings between hyperquadrtics with positive signature*, Jour. of Diff. Geom. Vol. 69, **2**(2005), 379–398.

[BR] M. S. Baouendi and L. P. Rothschild, *Geometric properties of mappings between hypersurfaces in complex spaces*, J. Differential Geom. **3**(1990), 473–499.

[Car] É. Cartan, *Sur les variétés pseudo-conformal des hypersurfaces de l'espace de deux variables complexes*, Ann. Mat. Pura Appl. (4) **11**(1932), 17–90.

[CD] D. Catlin and J.P. D'Angelo, *A stabilization theorem for Hermitian forms and applications to holomorphic mappings*, Math. Res. Lett. 3, **2**(1996), 149–166.

[CaoM] H. Cao and N. Mok, *Holomorphic immersions between compact hyperbolic space forms*, Invent. Math. 100 (1990), no. 1, 49–61.

[Ch] S. S. Chern, *On the projective structure of a real hypersurface in* $\mathbb{C}^{n+1}$, Math. Scand. **36**(1975), 74–82.

[CJ1] S.-S. Chern and S. Ji, *Projective geometry and Riemann's mapping theorem*, Math. Ann., **302**(1995), 581–600.

[CJ2] S.-S. Chern and S. Ji, *On the Riemann mapping theorem*, Ann. of Math. **144**(1996), 421-439.

[CM] S. S. Chern and J. K. Moser, *Real hypersurfaces in complex manifolds*, Acta Math. **133**(974), 219–271.

[CS1] J. Cima and T. J. Suffridge, *A reflection principle with applications to proper holomorphic mappings*, Math Ann. **265** (1983), 489–500.

[CS2] J.Cima and T. J. Suffridge, *Boundary behavior of rational proper maps*, Duke Math. J. **60**(1990), 135–138.

[CS3] J. Cima and T. Suffridge, *Proper mappings between balls in* $\mathbb{C}^n$*, Complex analysis (University Park, Pa., 1986)*, 66–82, Lecture Notes in Math., **1268**, Springer, Berlin-New York, 1987.

[CKS] J. Cima, S. Krantz, and T. J. Suffridge, *A reflection principle for proper holomorphic mappings of strictly pseudoconvex domains and applications*, Math. Z. **186**(1984), 1–8.

[Cor] K. Corlette, *Archimedean superrigidity and hyperbolic geometry*, Ann. of Math. (2) **135**(1992), no. 1, 165–182.

[DA1] J. D'Angelo, *Real hypersurfaces, orders of contact, and applications*, Ann. Math. **115**(1982), 615–637.

[DA2] J. P. D'Angelo, *Several Complex Variables and the Geometry of Real Hypersurfaces*, CRC Press, Boca Raton, 1993.

[DA3] J. P. D'Angelo, *Polynomial proper holomorphic mappings between balls*, Duke Math J. **57**(1988), 211–219.

[DA4] J. P. D'Angelo, *Proper holomorphic mappings between balls in different complex spaces*, Mich. Math. Jour. **38** (1991), 53–65.

[DA5] J. P. D'Angelo, *Proper holomorphic maps between balls of different dimensions*, Michigan Math. J. **35**(1988), no. 1, 83 – 90.

[Dor] A. Dor, *Proper holomorphic maps between balls in one co-dimension*, Ark. Math., **28**(1990),49–100.

[EHZ1] P. Ebenfelt, X. Huang and D. Zaitsev, *Rigidity of CR-immersions into spheres.* Comm. Anal. Geom. **12**(2004), no. 3, 631–670.

[EHZ2] P. Ebenfelt, X. Huang and D. Zaitsev, *The equivalence problem and rigidity for hypersurfaces embedded into hyperquadrics,* American Jour. of Math. Vol. **127**(2005), 169–191.

[EZ] P. Ebenfelt and D. Zaitsev, Private comminucation, Fall of 2003.

[Fa1] J. Faran, *Maps from the two ball to the three ball*, Invent Math. **68**(1982), 441–475.

[Fa2] J. Faran, *The linearity of proper holomorphic maps between balls in the low codimension case*, J. Differential Geom. **24** (1986), 15–17.

[Fa3] J. Faran, *Segre families and real hypersurfaces*, Invent. Math. **60**(1980), 135–172.

[Fr1] F. Forstneric, *A survey on proper holomorphic mappings*, Proceeding of Year in SCVs at Mittag-Leffler Institute, Math. Notes 38, Princeton, NJ: Princeton University Press , 1992.

[Fr2] F. Forstneric, *Extending proper holomorphic mappings of positive codimension*, Invent. Math. **95**(1989), 31–62.

[Fr3] F. Forstneric, *Complex tangents of real surfaces in complex surfaces*, Duke Math. J. **67** (1992), no. 2, 353–376.

[Ga] R. Gardner, *The method of equivalence and its applications*, CBMS-NSF, regional conference series in applied mathematics, 1989.

[Ham] : Hamada, *Hidetaka Rational proper holomorphic maps from $\mathbb{B}^n$ into $\mathbb{B}^{2n}$*, Math. Ann. **331**(2005), no.3, 693–711.

[Ha] M. Hakim, *Applications holomorphes propres continues de domaines strictment pseudoconvexes de $\mathbf{C^n}$ dans la boule unité $\mathbf{C^{n+1}}$*, Duke Math. J. **60** (1990), 115–133.

[HS] M. Hakim and N. Sibony, *Fonctions holomorphes bornes sur la boule unite de $\mathbb{C}^n$*, Invent. Math. **67**(1982), no. 2, 213–222.

[Hu1] X. Huang, *On a linearity problem of proper holomorphic mappings between balls in complex spaces of different dimensions*, J. of Diff. Geom. **51**(1999), 13–33.

[Hu2] X. Huang, *On a semi-linearity property for holomorphic maps*, Asian Jour. of Math. Vol 7 (No. 4)(2003), 463–492.

[Hu3] X. Huang, *On some problems in several complex variables and Cauchy-Riemann Geometry*, Proceedings of ICCM (edited by L. Yang and S. T. Yau), AMS/IP Stud. Adv. Math. **20**(2001), 383–396.

[Hu4] X. Huang, *Lectures on the Local Equivalence Problems for Real Submanifolds in Complex Manifolds*, Lecture Notes in Mathematics **1848** (C.I.M.E. Subseries), Springer-Verlag, 2004, 109–163.

[HJ1] X. Huang and S. Ji, *Mapping $\mathbb{B}^n$ into $\mathbb{B}^{2n-1}$*, Invent Math, **145**(2001), 219–250.

[HJ2] X. Huang and S. Ji, *Cartan-Chern-Moser theory on algebraic hypersurfaces and some applications*, Ann. Inst. Fourier (Grenoble) **52**(2002), no. 6, 1793–1831.

[HJX1] X. Huang, S. Ji, and D. Xu, *Several results for holomorphic mappings from $\mathbb{B}^n$ into $\mathbb{B}^N$*, Geometric analysis of PDE and several complex variables, 267–292, Contemp. Math., **368**, Amer. Math. Soc., Providence, RI, 2005.

[HJX2] X. Huang, S. Ji and D. Xu, *Proper holomorphic mappings from $\mathbb{B}^n$ into $\mathbb{B}^N$ with geometric rank one*, preprint, December, 2004.

[JX] S. Ji, and D. Xu, *Maps between $\mathbb{B}^n$ and $\mathbb{B}^N$ with geometric rank $k_0 \leq n-2$ and minimum N*, Asian J. Math. **8** (2004), no. 2, 233–257.

[KZ] W. Kaup and D. Zaitsev, *On local CR transformations of Levi-degenerate group orbits in compact Hermitian symmetric spaces*, Journal of the European Mathematical Society, 2005. (to appear)

[L] L. Lempert, *Imbedding strictly pseudoconvex domains into a ball*, Amer. J. Math. **104** (1982), no. 4,901–904.

[Mir] N. Mir, *Analytic regularity of CR maps into spheres.* Math. Res. Lett. 10 (no.4)(2003), 447–457.

[Mo] N. Mok, *Metric Rigidity Theorems on Hermitian Locally Symmetric Manifolds*, Series in Pure Mathematics, V. 6, World Scientific Publishing Co., 1989.

[MSY] N. Mok, Y. T. Siu and S. K. Yeung, *Geometric superrigidity*, Invent. Math. **113**(1993), 57–83.

[Moo] J. D. Moore, *On extendibility of isometric immersions of spheres*, Duke Math. J. **85**(1996), no. 3, 685–699.

[Mos] J. Moser, *Analytic surfaces in $\mathbb{C}^2$ and their local hull of holomorphy*, Annales Aca - demiæFennicae, Series A.I. Mathematica **10**(1985), 397–410.

[MW] J. Moser and S. Webster, *Normal forms for real surfaces in $\mathbb{C}^2$ near complex tangents and hyperbolic surface transformations*, Acta Math. **150**(1983), 255–296.

[Po] H. Poincaré, *Les fonctions analytiques de deux variables et la représentation conforme,* Rend. Circ. Mat. Palermo, (1907), 185–220.

[Sh] B. Shiffman, *Synthetic projective geometry and Poincaré's theorem on automorphisms of the ball*, Enseign. Math. (2) **41**(1995), no. 3-4, 201–215.

[Spv] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Volume (3), Publish or Persish, Inc., Wilmington, Delaware, 1979.

[Ste] B. Stensones, *Proper holomorphic mappings between the balls*, J. of Geom. Analysis, **6**(1996), 317–339.

[Ta] N. Tanaka, *On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables*, J. Math. Soc. Japan **14**(1962), 397–429.

[Ts] I. Tsai, *Rigidity of proper holomorphic maps between symmetric domains.* J. Differential Geom. **37**(1993), no. 1, 123–160.

[Tu] Z. Tu, *Rigidity of proper holomorphic mappings between nonequidimensional bounded symmetric domains,* Math. Z. **240**(2002), no. 1, 13–35.

[We1] S. Webster, *On the mapping problem for algebraic real hypersurfaces*, Invent. Math. **43**(1977), 53–68.

[We2] S. Webster, *Biholomorphic mappings and the Bergman kernels off diagonal*, Invent. Math. **51**(1979), 155–169.

[We3] S. Webster, *On mappings an (n+1)-ball in the complex space*, Pac. J. Math. **81**(1979), 267–272.

[We4] S. Webster, *The rigidity of CR hypersurfaces in a sphere*, Indiana Univ. Math. Jour. Vol **28** (1979), 405–416.

[Yue] C. Yue, *Dimension and rigidity of quasi-Fuchsian representations*, Ann. of Math. (2) **143** (1996), no. 2, 331–355.

Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA
E-mail address: huangx@math.rutgers.edu

Department of Mathematics, University of Houston, Houston, TX 77204, USA
E-mail address: shanyuji@math.uh.edu