Rational Holomorphic Maps from $\mathbb{B}^2$ into $\mathbb{B}^N$ with Degree 2

Zhihua Chen, Shanyu Ji and Dekang Xu

February 28, 2007

Dedicated to Professor Gong, Sheng in the Occasion of his 75th Birthday

1 Introduction

This paper continues the previous work in [HJX06] to study proper holomorphic mappings $F \in Rat(\mathbb{B}^2, \mathbb{B}^N)$ with degree 2. In [HJX06], it is proved that any such a map F is equivalent to a rational proper holomorphic map $(G, 0)$ where $G \in Rat(\mathbb{B}^2, \mathbb{B}^5)$. Also a normal form has been obtained for such a map ([Theorem 4.1, HJX06] or Lemma 2.3 below).

Here we write $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$ and $Prop(\mathbb{B}^n, \mathbb{B}^N)$ for the collection of all proper holomorphic mappings from $\mathbb{B}^n$ to $\mathbb{B}^N$ where $2 \leq n \leq N$. We say that $f, g \in Prop(\mathbb{B}^n, \mathbb{B}^N)$ are equivalent if there are automorphisms $\sigma \in Aut(\mathbb{B}^n)$ and $\tau \in Aut(\mathbb{B}^N)$ such that $f = \tau \circ g \circ \sigma$. We write $Rat(\mathbb{B}^n, \mathbb{B}^N)$ for the collection of all rational proper holomorphic mappings from $\mathbb{B}^n$ to $\mathbb{B}^N$.

Let us recall some known results on maps in $Rat(\mathbb{B}^2, \mathbb{B}^N)$ with degree 2. Faran [Fa82] proved that any F in $Rat(\mathbb{B}^2, \mathbb{B}^3)$ with degree two must be equivalent to either the Whitney map $(z, w) \mapsto (z, zw, w^2)$ or the map $(z, w) \mapsto (z^2, \sqrt{2}zw, w^2)$. D'Angelo [DA88] constructed the following continuous family of mutually inequivalent proper polynomial embeddings from $\mathbb{B}^n$ into $\mathbb{B}^{2n}$ of degree 2:

$$F_\theta(z, w) = (z, (\cos\theta)w, (\sin\theta)z_1 w, \cdots, (\sin\theta)z_{n-1}w, (\sin\theta)w^2), \ 0 < \theta \leq \pi/2, \qquad (1)$$

where $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$. In the same paper, he also gave a list of all mutually inequivalent monomial proper mappings from $\mathbb{B}^2$ to $\mathbb{B}^4$. Among the list, there are two mutually inequivalent continuous families of maps with degree 2: $\{F_\theta\}$ in (1) and the family $\{G_t\}$ defined by

$$G_t(z,w) = (z^2, \sqrt{1+\cos^2 t}zw, (\cos t)w^2, (\sin t)w),\ 0 \le t < \pi/2. \qquad (2)$$

M. S. B. Wono [Wo93] also constructed a family of monomial maps in $Rat(\mathbb{B}^2, \mathbb{B}^5)$ of degree 2:

$$H_{bc} = \left(\sqrt{1-b}z^2,\ \sqrt{1-c}w^2,\ \sqrt{2-b-c}zw,\ \sqrt{b}z,\ \sqrt{c}w\right),\quad \forall\ b,c \in [0,1].$$

In this paper, we shall prove the following result.

Theorem 1.1 *Any map F in $Rat(\mathbb{B}^2, \mathbb{B}^N)(N \ge 4)$ with degree 2 is equivalent to one of the following forms:*

(I) $(G_t, 0)$ where $G_t \in Rat(\mathbb{B}^2, \mathbb{B}^4)$ is defined by

$$G_t(z,w) = (z^2, \sqrt{1+\cos^2 t}\ zw, (\cos t)w^2, (\sin t)w),\quad 0 \le t < \pi/2.$$

(II A) $(F_\theta, 0)$ where $F_\theta \in Rat(\mathbb{B}^2, \mathbb{B}^4)$ is defined by

$$F_\theta(z,w) = (z, (\cos\theta)w, (\sin\theta)zw, (\sin\theta)w^2),\quad 0 < \theta \le \frac{\pi}{2}.$$

(IIB) $(H_{c_1}, 0)$ where $F_{c_1} \in Rat(\mathbb{B}^2, \mathbb{B}^4)$ so that $\rho_4^{-1} \circ H_{c_1} \circ \rho_2 = (f, \phi_1, \phi_2, g) \in Rat(\mathbb{H}^2, \mathbb{H}^4)$, where ρ_k is the Cayley transformations from $\mathbb{H}^k$ to $\mathbb{B}^k$, is of the form:

$$f = \frac{z + \frac{i}{2}zw}{1+e_2w^2},\ \phi_1 = \frac{z^2}{1+e_2w^2},\ \phi_2 = \frac{c_1 zw}{1+e_2w^2},\quad g = \frac{w}{1+e_2w^2},$$

where $-e_2 = \frac{1}{4} + c_1^2$ and $c_1 > 0$.

(IIC) $(F_{c_1,c_3,e_1,e_2}, 0)$ where $F_{c_1,c_3,e_1,e_2} \in Rat(\mathbb{B}^2, \mathbb{B}^5)$ so that $\rho_5^{-1} \circ F_{c_1,c_3,e_1,e_2} \circ \rho_2 = (f, \phi_1, \phi_2, \phi_3) \in Rat(\mathbb{H}^2, \mathbb{H}^5)$ is of the form:

$$\begin{aligned} f &= \frac{z + (\frac{i}{2} + ie_1)zw}{1+ie_1w+e_2w^2},\ \phi_1 = \frac{z^2}{1+ie_1w+e_2w^2}, \\ \phi_2 &= \frac{c_1 zw}{1+ie_1w+e_2w^2},\ \phi_3 = \frac{c_3 w^2}{1+ie_1w+e_2w^2},\ g = \frac{w+ie_1w^2}{1+ie_1w+e_2w^2}, \end{aligned}$$

where (c_1, c_3, e_1, e_2) is in a subset $\mathcal{K}_{IIC} \subset \mathbb{R}^4$ (i.e., $-e_1, -e_2 \ge 0$, $c_1 > 0$, $c_3 > 0$, $e_1e_2 = c_3^2$, $-e_1 - e_2 = \frac{1}{4} + c_1^2$ and it satisfies (34)).

Notice that F_0 is the linear map, $F_{\frac{\pi}{2}}$ and $G_{\frac{\pi}{2}}$ are equivalent to the map $(z,w) \mapsto (z, zw, w^2, 0)$, $G_0 = (z^2, \sqrt{2}zw, w^2, 0)$, $\{F_{0,c_3,e_1,e_2}\}$ with $c_3 > 0$ is the family $\{(G_t, 0)\}$, and $\{F_{c_1,0,e_1,e_2}\}$ with $c_1 \ge 0$ is the family $\{(F_\theta, 0)\}$.

It remains to study whether any two distinct maps in (I) (IIA)(IIB) or (IIC) above could be equivalent and to describe the domain $\mathcal{K}_{IIC}$ more explicitly.

2 Notation and preliminaries

•Maps between balls Write $\mathbb{H}^n := \{(z,w) \in \mathbf{C}^{n-1} \times \mathbf{C} :\ \mathrm{Im}(w) > |z|^2\}$ for the Siegel upper-half space. Since the Cayley transformation $\rho_n : \mathbb{H}^n \to \mathbb{B}^n$, $\rho_n(z,w) = (\frac{2z}{1-iw}, \frac{1+iw}{1-iw})$ is a biholomorphic mapping between $\mathbb{H}^n$ and $\mathbb{B}^n$, we can identify a map $F \in \mathrm{Prop}(\mathbb{B}^n, \mathbb{B}^N)$ or $\mathrm{Rat}(\mathbb{B}^n, \mathbb{B}^N)$ with $\rho_N^{-1} \circ F \circ \rho_n$ in the space $\mathrm{Prop}(\mathbb{H}^n, \mathbb{H}^N)$ or $\mathrm{Rat}(\mathbb{H}^n, \mathbb{H}^N)$, respectively.

It is known that any $F \in Rat(\mathbb{H}^n, \mathbb{H}^N)$ must be a smooth CR map from $\partial\mathbb{H}^n$ into $\partial\mathbb{H}^N$. Parameterize $\partial\mathbb{H}^n$ by $(z, \overline{z}, u)$ through the map $(z, \overline{z}, u) \to (z, u + i|z|^2)$. In what follows, we will assign the weight of z and u to be 1 and 2, respectively. For a non-negative integer m, a function $h(z, \overline{z}, u)$ defined over a small ball U of 0 in $\partial\mathbb{H}^n$ is said to be of quantity $o_{wt}(m)$ if $\frac{h(tz, t\overline{z}, t^2u)}{|t|^m} \to 0$ uniformly for (z,u) on any compact subset of U as $t (\in \mathbb{R}) \to 0$.

• Partial normalization of F Let $F = (f, \phi, g) = (\widetilde{f}, g) = (f_1, \cdots, f_{n-1}, \phi_1, \cdots, \phi_{N-n}, g)$ be a non-constant C^2-smooth CR map from $\partial\mathbb{H}^n$ into $\partial\mathbb{H}^N$ with $F(0) = 0$. For each $p \in \partial\mathbb{H}^2$, we write $\sigma_p^0 \in \mathrm{Aut}(\mathbb{H}^n)$ and $\tau_p^F \in \mathrm{Aut}(\mathbb{H}^N)$ for the maps

$$\sigma_p^0(z,w) = (z + z_0, w + w_0 + 2i\langle z, \overline{z_0}\rangle), \tag{3}$$

$$\tau_p^F(z^*, w^*) = (z^* - \widetilde{f}(z_0,w_0), w^* - \overline{g(z_0,w_0)} - 2i\langle z^*, \overline{\widetilde{f}(z_0,w_0)}\rangle). \tag{4}$$

F is equivalent to $F_p = \tau_p^F \circ F \circ \sigma_p^0 = (f_p, \phi_p, g_p)$. Notice that $F_0 = F$ and $F_p(0) = 0$.

Lemma 2.1 *([§2, Lemma 5.3, Hu99], [Lemma 2.0, Hu03]): Let F be a C^2-smooth CR map from $\partial\mathbb{H}^n$ into $\partial\mathbb{H}^N$, $2 \le n \le N$ with $F(0) = 0$. For each $p \in \partial\mathbb{H}^n$, there is an automorphism $\tau_p^{**} \in Aut_0(\mathbb{H}^N)$ such that $F_p^{**} := \tau_p^{**} \circ F_p$ satisfies the following normalization:*

$$f_p^{**} = z + \frac{i}{2} a_p^{**(1)}(z) w + o_{wt}(3),\ \ \phi_p^{**} = \phi_p^{**(2)}(z) + o_{wt}(2),\ \ g_p^{**} = w + o_{wt}(4), \tag{5}$$

$$\langle \overline{z}, a_p^{**(1)}(z)\rangle |z|^2 = |\phi_p^{**(2)}(z)|^2.$$

Let $\mathcal{A}(p) = -2i(\frac{\partial^2 (f_p)_l^{**}}{\partial z_j \partial w}|_0)_{1 \le j,l \le (n-1)}$. We call the rank of $\mathcal{A}(p)$, denoted by $Rk_F(p)$, the *geometric rank* of F at p. $Rk_F(p)$ is a lower semi-continuous function on p. We define the *geometric rank* of F to be $\kappa_0(F) = max_{p \in \partial\mathbb{H}^n} Rk_F(p)$. Notice $0 \le \kappa_0 \le n-1$. We define the geometric rank of $F \in \mathrm{Prop}_2(\mathbb{B}^n, \mathbb{B}^N)$ to be the one for the map $\rho_N^{-1} \circ F \circ \rho_n \in \mathrm{Prop}_2(\mathbb{H}^n, \mathbb{H}^N)$. It is proved that F is linear fractional if and only if the geometric rank of F is 0 ([Theorem 4.3, Hu99]). Hence, in all that follows, we assume that $\kappa_0(F) \ge 1$.

Denote by $\mathcal{S}_0 = \{(j,l) : 1 \le j \le \kappa_0, 1 \le l \le (n-1), j \le l\}$ and write $\mathcal{S} := \{(j,l) : (j,l) \in \mathcal{S}_0,\text{ or } j = \kappa_0 + 1, l \in \{\kappa_0+1, \cdots, \kappa_0 + N - n - \frac{(2n-\kappa_0-1)\kappa_0}{2}\}\}$. Then we further have the following normalization for F:

Lemma 2.2 *([Lemma 3.2, Hu03]): Let F be a C^2-smooth CR map from an open piece $M \subset \partial\mathbb{H}^n$ into $\partial\mathbb{H}^N$ with $F(0) = 0$ and $Rk_F(0) = \kappa_0$. Let $P(n,\kappa_0) = \frac{\kappa_0(2n-\kappa_0-1)}{2}$. Then $N \ge n + P(n,\kappa_0)$ and there are $\sigma \in Aut_0(\partial\mathbb{H}^n)$ and $\tau \in Aut_0(\partial\mathbb{H}^N)$ such that $F_p^{***} = \tau \circ F \circ \sigma := (f,\phi,g)$ satisfies the following normalization conditions:*

$$\begin{cases} f_j = z_j + \dfrac{i\mu_j}{2} z_j w + o_{wt}(3), \quad \dfrac{\partial^2 f_j}{\partial w^2}(0) = 0,\ j = 1\cdots,\kappa_0,\ \mu_j > 0, \\ f_j = z_j + o_{wt}(3), \quad j = \kappa_0+1,\cdots,n-1 \\ g = w + o_{wt}(4), \\ \phi_{jl} = \mu_{jl} z_j z_l + o_{wt}(2),\ \textit{where}\ \ (j,l) \in \mathcal{S}\ \textit{with}\ \ \mu_{jl} > 0\ \textit{for}\ \ (j,l) \in \mathcal{S}_0 \\ \qquad \textit{and}\ \mu_{jl} = 0\ \textit{otherwise}. \end{cases} \tag{6}$$

Moreover $\mu_{jl} = \sqrt{\mu_j + \mu_l}$ for $j,l \le \kappa_0$ $j \ne l$, $\mu_{jl} = \sqrt{\mu_j}$ if $j \le \kappa_0$ and $l > \kappa_0$ or if $j = l \le \kappa_0$.

• **Degree of a rational map** For a rational holomorphic map $H = \frac{(P_1,...,P_m)}{Q}$ over $\mathbb{C}^n$, where P_j, Q are holomorphic polynomials and $(P_1,...,P_m,Q) = 1$, we define

$$deg(H) = max\{deg(P_j),\ 1 \le j \le m,\ deg(Q)\}.$$

For a rational map H and a complex affine subspace S of dimension k, we say that H is linear fractional along S, if S is not contained in the singular set of H and for any linear parameterization $z_j = z_j^0 + \sum_{l=1}^k a_{jl} t_l$ with $j = 1, \cdots, n$, $H^*(t_1,\cdots,t_k) := H(z_1^0 + \sum_{l=1}^k a_{1l} t_l, \cdots, z_n^0 + \sum_{l=1}^k a_{jn} t_j)$ has degree 1 in $(t_1,\cdots,t_k)$.

• **Actions of the isotropic groups of the Heisenberg hypersurfaces** Let $\sigma \in Aut_0(\partial\mathbb{H}^2)$ and $\tau^* \in Aut_0(\partial\mathbb{H}^5)$ be defined by [(2.4.1), Hu03] and [(2.4.2), Hu03] respectively,

$$\sigma = \frac{(\lambda(z+aw)\cdot U,\ \lambda^2 w)}{q(z,w)}, \quad \tau^*(z^*,w^*) = \frac{(\lambda^*(z^* + a^* w^*)\cdot U^*, \lambda^{*2} w^*)}{q^*(z^*,w^*)}, \tag{7}$$

where $q(z,w) = 1 - 2i\langle\overline{a}, z\rangle + (r - i|a|^2)w$, $\lambda > 0$, $r \in \mathbb{R}$, $a \in \mathbb{C}$, $|U| = 1$, and $q^*(z^*,w^*) = 1 - 2i\langle\overline{a^*}, z^*\rangle + (r^* - i|a^*|^2)w^*$, $\lambda^* > 0$, $r^* \in \mathbb{R}$, $a^* = (a_1^*, a_2^*) \in \mathbb{C}^3$ and U^* is an 4×4 unitary matrix, such that [((2.5.1), (2.5.2), Hu03] holds:

$$\lambda^* = \lambda^{-1},\ a_1^* = -\lambda^{-1} a U,\ a_2^* = 0,\ r^* = -\lambda^{-2} r,\ U^* = \begin{pmatrix} U^{-1} & 0 \\ 0 & U_{22}^* \end{pmatrix}, \tag{8}$$

where $a^* = (a_1^*, a_2^*)$, U_{22}^* is an 3×3 unitary matrix. Define $F^* = \tau^*\circ F\circ\sigma$. By [Lemma 2.3(A), Hu03], we can write

$$\begin{aligned} &f(z,w) = z + \tfrac{i}{2}zAw + o_{wt}(3),\ \ f^*(z,w) = z + \tfrac{i}{2}zA^*w + o_{wt}(3),\\ &\phi(z,w) = \tfrac{1}{2}z(B^1, B^2, B^3)z + z\mathcal{B}w + \tfrac{1}{2}\tfrac{\partial^2\phi}{\partial w^2}(0)w^2 + o(|(z,w)|^2),\\ &\phi^*(z,w) = \tfrac{1}{2}z(B^{*1}, B^{*2}, B^{*3})z + z\mathcal{B}^*w + \tfrac{1}{2}\tfrac{\partial^2\phi^*}{\partial w^2}(0)w^2 + o(|(z,w)|^2), \end{aligned} \tag{9}$$

where $B^i = \frac{\partial^2\phi_i}{\partial z^2}(0)$, $B^{*i} = \frac{\partial^2\phi_i^*}{\partial z^2}(0)$ for $i = 1,2,3$ and $\mathcal{B} = (\frac{\partial^2\phi_1}{\partial z\partial w}, \frac{\partial^2\phi_2}{\partial z\partial w}, \frac{\partial^2\phi_3}{\partial z\partial w})$, $\mathcal{B}^* = (\frac{\partial^2\phi_1^*}{\partial z\partial w}, \frac{\partial^2\phi_2^*}{\partial z\partial w}, \frac{\partial^2\phi_3^*}{\partial z\partial w})$. Also, the same computation in [Hu03, Lemma 2.3 (A)] gives the following:

$$\begin{aligned} &\tfrac{\partial^2 g^*}{\partial z^2}(0) = 0,\ \tfrac{\partial^2 g^*}{\partial z\partial w}(0) = 0,\ \tfrac{\partial^2 g^*}{\partial w^2}(0) = 0,\ \tfrac{\partial^2 f^*}{\partial z^2}(0) = 0,\ \mathcal{A}^* = \lambda^2 U\mathcal{A}U^{-1},\\ &\tfrac{\partial^2 f^*}{\partial w^2}(0) = i\lambda^2 aU\mathcal{A}U^{-1} + \lambda^3\tfrac{\partial^2 f}{\partial w^2}(0)U^{-1},\\ &[B^{*1},\ B^{*2}, B^{*3}] = \lambda U[B^1,\ B^2,\ B^3]U^tU_{22}^*,\\ &\mathcal{B}^* = \lambda U[B^1,\ B^2,\ B^3]U^ta^tU_{22}^* + \lambda^2 U\mathcal{B}U_{22}^*,\\ &\tfrac{\partial^2\phi^*}{\partial w^2}(0) = \lambda aU[B^1,\ B^2,\ B^3]U^ta^tU_{22}^* + 2\lambda^2 aU\mathcal{B}U_{22}^* + \lambda^3\tfrac{\partial^2\phi}{\partial w^2}(0)U_{22}^*. \end{aligned} \tag{10}$$

- **A normal form for $F \in Rat(\mathbb{B}^2, \mathbb{B}^N)$ with degree 2**

Lemma 2.3 *([HXJ06, theorem 4.1]) Let $F \in Rat(\partial\mathbb{H}^2, \partial\mathbb{H}^N)$ have degree 2 with $N \geq 4$, $F(0) = 0$ and $Rk_F(0) = 1$. Then*

(1) F is equivalent to a new map $(F^{***}, 0)$ where $F^{***} = (f, \phi_1, \phi_2, \phi_3, g)$ in $Rat(\partial\mathbb{H}^2, \partial\mathbb{H}^5)$ defined by

$$f(z,w) = \frac{z - 2ibz^2 + (\frac{i}{2} + ie_1)zw}{1 + ie_1w + e_2w^2 - 2ibz},\ \ \phi_1(z,w) = \frac{z^2 + bzw}{1 + ie_1w + e_2w^2 - 2ibz}, \tag{11}$$

$$\phi_2(z,w) = \frac{c_2w^2 + c_1zw}{1 + ie_1w + e_2w^2 - 2ibz},\ \ \phi_3(z,w) = \frac{c_3w^2}{1 + ie_1w + e_2w^2 - 2ibz}, \tag{12}$$

$$g(z,w) = \frac{w + ie_1w^2 - 2ibzw}{1 + ie_1w + e_2w^2 - 2ibz}. \tag{13}$$

Here $b, -e_1, -e_2, c_1, c_2, c_3$ are real non-negative numbers satisfying

$$e_1e_2 = c_2^2 + c_3^2,\ \ -e_1 - e_2 = \tfrac{1}{4} + b^2 + c_1^2,\ \ -be_2 = c_1c_2,\ \ c_3 = 0\ if\ c_1 = 0. \tag{14}$$

(2) $c_1, c_2, c_3, e_1, e_2, b$ are uniquely determined by F [1]. Conversely, for any non-negative real numbers $c_1, c_2, c_3, e_1, e_2, b$ satisfying the relations in (14), the map F defined in (11) (12) (13) is an element in $Rat(\partial\mathbb{H}^2, \partial\mathbb{H}^5)$ of degree 2 with $F(0) = 0$ and $Rk_F(0) = 1$.

(3) If $e_2 = 0$, then F is equivalent to $(F_\theta, 0)$ with F_θ as in (1).

[1] In the sense that if $F^* = \tau\circ F\circ\sigma$ where both F and F^* satisfy the normalized condition in Lemma 2.3, $\tau \in Aut_0(\partial\mathbb{H}^5)$ and $\sigma \in Aut_0(\partial\mathbb{H}^2)$, then $F^* = F$.

Remarks (i) The new normalized map in Lemma 2.3(1) can be obtained by $F^{***} = \tau^* \circ F^{**} \circ \sigma$ where F^{**} is as in Lemma 2.2 and σ and τ^* are as in (7).
(ii) For the map F^{***} in Lemma 2.3(1), $b = \sqrt{-e_1 - e_2 - \frac{1}{4} - c_1^2}$ and $c_2 = \sqrt{e_1 e_2 - c_3^2}$ are determined by c_1, c_3, e_1 and e_2, which can be regarded as parameters. Then we denote $F^{***} = F_{c_1,c_3,e_1,e_2}$.
(iii) We denote by $\mathcal{K}$ a subset of $\mathbb{R}^4$ such that $(c_1, c_3, e_1, e_2) \in \mathcal{K}$ if and only if F_{c_1,c_3,e_1,e_2} is a map defined as above. We can identify a map F_{c_1,c_3,e_1,e_2} with the 4-tuple $(c_1, c_3, e_1, e_2) \in \mathcal{K}$. Sometimes we also denote $F_{c_1,c_3,e_1,e_2} \in \mathcal{K}$.
(iv) If $F \in Rat(\mathbb{H}^2, \mathbb{H}^5)$ with $F(0) = 0$ and rank 1 at 0, then $F^{***} \in \mathcal{K}$. Conversely, if $F \in \mathcal{K}$, then $F(0) = 0$ and F has rank 1 at 0.

To prove Theorem 1.1, the following results will be needed.

Lemma 2.4 *Let $F \in Rat(\mathbb{H}^2, \mathbb{H}^5)$ such that $F(0) = 0$, $deg(F) \leq 2$, the geometric rank at 0 $Rk_F(0) = 0$, and the associated map F^{**} satisfies*

$$\frac{\partial^2 f^{**}}{\partial w^2}(0) = 0, \ \frac{\partial^2 \phi^{**}}{\partial z^2}(0) = \frac{\partial^2 \phi^{**}}{\partial z \partial w}(0) = (0, 0, 0). \tag{15}$$

Then F must be a linear map.

Proof: By the hypothesis, F^{**} can be witten as

$$f = \frac{z + E_1 z^2 + E_2 zw}{1 + E_1 z + E_2 w + E_3 zw + E_4 w^2 + E_5 z^2} \tag{16}$$

$$\phi_1 = \frac{B_1 w^2}{1 + E_1 z + E_2 w + E_3 zw + E_4 w^2 + E_5 z^2} \tag{17}$$

$$\phi_2 = \frac{B_2 w^2}{1 + E_1 z + E_2 w + E_3 zw + E_4 w^2 + E_5 z^2} \tag{18}$$

$$\phi_3 = \frac{B_3 w^2}{1 + E_1 z + E_2 w + E_3 zw + E_4 w^2 + E_5 z^2} \tag{19}$$

$$g = \frac{w + E_1 zw + E_2 w^2}{1 + E_1 z + E_2 w + E_3 zw + E_4 w^2 + E_5 z^2} \tag{20}$$

Notice $w = u + i|z|^2$, $w^2 = u^2 - |z|^4 + 2iu|z|^2$, $|w|^4 = u^4 + 2u^2|z|^4 + |z|^8$. From $Im(g) = |\widetilde{f}|^2$ on $\partial\mathbb{H}^2$, we get

$$
\begin{aligned}
&\left(1+\overline{E_1}\overline{z}+\overline{E_2}(u-i|z|^2)+\overline{E_3}\overline{z}(u-i|z|^2)+\overline{E_4}(u^2-|z|^4-2iu|z|^2)+\overline{E_5}\overline{z}^2\right)\\
&\cdot\left(u+i|z|^2+E_1z(u+i|z|^2)+E_2(u^2-|z|^4+2iu|z|^2)\right)\\
&-\left(1+E_1z+E_2(u+i|z|^2)+E_3z(u+i|z|^2)+E_4(u^2-|z|^4+2iu|z|^2)+E_5z^2\right)\\
&\cdot\left(u-i|z|^2+\overline{E_1}\overline{z}(u-i|z|^2)+\overline{E_2}(u^2-|z|^4-2iu|z|^2)\right)\\
&=2i|z|^2\left[1+E_1z+E_2(u+i|z|^2)\right]\left[1+\overline{E_1}\overline{z}+\overline{E_2}(u-i|z|^2)\right]\\
&+2i(|B_1|^2+|B_2|^2+|B_3|^2)(u^4+2u^2|z|^4+|z|^8),\quad \forall z\in\mathbb{C},\forall u\in\mathbb{R}.
\end{aligned}
$$

Consider the uz^2 terms, we get $E_5=0$.
Consider the u^2z terms, we get $E_1\overline{E_2}-E_1\overline{E_2}-E_3=0$. Then $E_3=0$.
Let $z=0$ in the above equation. We get

$$
(1+\overline{E_2}u+\overline{E_4}u^2)(1+E_2u)-(1+E_2u+E_4u^2)(1+\overline{E_2}u)=2i(|B_1|^2+|B_2|^2)u^3.
$$

Then $\overline{E_4}-E_4=0$, i.e., E_4 is real, and

$$
E_4(E_2-\overline{E_2})=2i(|B_1|^2+|B_2|^2+|B_3|^2). \tag{21}
$$

Let $u=0$ in the above equation. We get

$$
\begin{aligned}
&\left(1+\overline{E_1}\overline{z}-i\overline{E_2}|z|^2-\overline{E_4}|z|^4\right)\left(i+iE_1z-E_2|z|^2\right)\\
&-\left(1+E_1z+iE_2|z|^2-E_4|z|^4\right)\left(-i-i\overline{E_1}\overline{z}-\overline{E_2}|z|^2\right)\\
&=2i\left[1+E_1z+E_2i|z|^2\right]\left[1+\overline{E_1}\overline{z}-\overline{E_2}i|z|^2\right]+2i(|B_1|^2+|B_2|^2+|B_3|^2)|z|^6.
\end{aligned}
$$

Consider the $z|z|^4$ terms, we get $\overline{E_4}E_1=0$. In case $E_4=0$, it implies $B_1=B_2=B_3=0$ by (21). Then it implies that F^{**} is linear and we are done. In case $E_1=0$, then the above equation becomes

$$\Big(1 - i\overline{E_2}|z|^2 - \overline{E_4}|z|^4\Big)\Big(i - E_2|z|^2\Big) - \Big(1 + iE_2|z|^2 - E_4|z|^4\Big)\Big(-i - \overline{E_2}|z|^2\Big)$$
$$= 2i\Big[1 + E_2 i|z|^2\Big]\Big[1 - \overline{E_2} i|z|^2\Big] + 2i(|B_1|^2 + |B_2|^2 + |B_3|^2)|z|^6.$$

Consider the $|z|^4$ terms, $i|E_2|^2 - i\overline{E_4} + i|E_2|^2 - iE_4 = 2i|E_2|^2$. Recall E_4 is real. It implies $E_4 = 0$. Hence $B_1 = B_2 = B_3 = 0$ so that F^{**} is linear. □

Lemma 2.5 *Let $F \in Rat(\partial\mathbb{H}^2, \partial\mathbb{H}^5)$ with $F(0) = 0$ and $deg(F) = 2$. Suppose that $p_m \in \partial\mathbb{H}^2$ is a sequence converging to 0, F_{p_m} is of rank 1 at 0 for any m and $F^{***}_{p_m}$ converges such that $\frac{\partial^2\phi^{***}_{1,m}}{\partial z\partial w}|_0$, $\frac{\partial^2\phi^{***}_{2,m}}{\partial w^2}|_0$, $\frac{\partial^2\phi^{***}_{2,m}}{\partial z\partial w}|_0$ and $\frac{\partial^2\phi^{***}_{3,m}}{\partial w^2}|_0$ are bounded* [2] *for all m. Then*

(i) F is of geometric rank 1 at 0: $Rk_F(0) = 1$, and hence F^{***} is well-defined.

(ii) $F^{***}_{p_m} \to F^{***}$.

(iii) If we write $F^{***}_{p_m} = \widetilde{G}_{2,m} \circ \tau_{p_m} \circ F \circ \sigma_{p_m} \circ \widetilde{G}_{1,m}$ where σ_{p_m} and $\tau_{p_m} := \tau^F_{p_m}$ are as in (3), $\widetilde{G}_{1,m}$ and $\widetilde{G}_{2,m}$ are as in (7), then $\widetilde{G}_{1,m}$ and $\widetilde{G}_{2,m}$ are convergent to some $\widetilde{G}_1 \in Aut_0(\partial\mathbb{H}^2)$ and $\widetilde{G}_2 \in Aut_0(\partial\mathbb{H}^5)$ respectively.

Proof: (i) Suppose that F has rank 0 at 0. We'll seek a contradiction.

Denote $F^{**} = (f^{**}, \phi^{**}, g^{**})$. We only need to prove the following claim:

$$\frac{\partial^2 f^{**}}{\partial w^2}(0) = 0, \ \frac{\partial^2 \phi^{**}}{\partial z^2}(0) = \frac{\partial^2 \phi^{**}}{\partial z\partial w}(0) = (0, 0, 0). \tag{22}$$

In fact, by Lemma 2.4, F must be linear but this is a contradiction with $deg(F) = 2$.

Since we have supposed that $Rk_F(0) = 0$, we have $\frac{\partial^2 f^{**}}{\partial z\partial w}(0) = 0$ so that F^{***} is not well defined.

Write $F^{**} = G_2 \circ F \circ G_1$, where $G_1 \in Aut_0(\partial\mathbb{H}^2)$ and $G_2 \in Aut_0(\partial\mathbb{H}^5)$. Since $Rk_F(p_m) = 1$ for any m, $(F_{p_m})^{***}$ is well-defined which is of the normal form as in Lemma 2.3(1). Write $q_m \in \partial\mathbb{H}^2$ so that $G_1(q_m) = p_m$. Consider

$$(\hat{f_m}, \hat{\phi_m}, \hat{g_m}) := \Big((F^{**})_{q_m}\Big)^{**} = \Big(H_2 \circ \tau^F_{q_m} \circ G_2\Big) \circ F \circ \Big(G_1 \circ \sigma_{q_m} \circ H_1\Big)$$

and

$$(\widetilde{f}_m, \widetilde{\phi}_m, \widetilde{g}_m) := (F_{p_m})^{***} = \Big(\widetilde{G}_2 \circ \tau^F_{p_m}\Big) \circ F \circ \Big(\sigma_{p_m} \circ \widetilde{G}_1\Big)$$

[2] This means that $b(p_m), c_1(p_m), c_2(p_m), c_3(p_m)$ are all bounded by the notation in Lemma 2.3(1).

where $H_1, \widetilde{G}_1 \in Aut_0(\partial\mathbb{H}^2)$, $H_2, \widetilde{G}_2 \in Aut_0(\partial\mathbb{H}^5)$, $\sigma_{q_m}(0) = q_m$, $\tau^F_{q_m}(G_2 \circ F(p_m)) = 0$, $\sigma_{p_m}(0) = p_m$, and $\tau^F_{p_m}(F(p_m)) = 0$ as in (3). Then

$$
\begin{aligned}
(F_{p_m})^{***} &= \widetilde{G}_2 \circ \tau^F_{p_m} \circ F \circ \sigma_{p_m} \circ \widetilde{G}_1 = \left(\widetilde{G}_2 \circ \tau^F_{p_m} \circ G_2^{-1} \circ (\tau^F_{q_m})^{-1} \circ H_2^{-1}\right) \\
&\circ \left(H_2 \circ \tau^F_{q_m} \circ G_2 \circ F \circ G_1 \circ \sigma_{q_m} \circ H_1\right) \circ \left(H_1^{-1} \circ \sigma_{q_m}^{-1} \circ G_1^{-1} \circ \sigma_{p_m} \circ \widetilde{G}_1\right) \\
&= \tau^m \circ \left((F^{**})_{q_m}\right)^{**} \circ \sigma_m,
\end{aligned}
$$

where $\sigma_m := H_1^{-1} \circ \sigma_{q_m}^{-1} \circ G_1^{-1} \circ \sigma_{p_m} \circ \widetilde{G}_1 \in Aut_0(\partial\mathbb{H}^2)$ and $\tau^m := \widetilde{G}_2 \circ \tau^F_{p_m} \circ G_2^{-1} \circ (\tau^F_{q_m})^{-1} \circ H_2^{-1} \in Aut_0(\partial\mathbb{H}^5)$.

Since $q_m \to 0$ as $m \to \infty$, we have $\left((F^{**})_{q_m}\right)^{**} \to F^{**}$ as $m \to \infty$. In order to prove Claim (22), it is enough to show that

$$
\frac{\partial^2 \hat{f}_m}{\partial w^2}|_0 \to 0, \ \frac{\partial^2 \hat{\phi}_m}{\partial z^2}|_0 \to (0,0,0), \ \frac{\partial^2 \hat{\phi}_m}{\partial z \partial w}|_0 \to (0,0,0), \ as \ m \to \infty. \tag{23}
$$

As in (7), we write

$$
\sigma_m(z,w) = \left(\frac{\lambda_m(z + a_m w)U_m}{1 - 2i\langle\overline{a_m}, z\rangle + (r_m - i|a_m|^2)w}, \frac{\lambda^2 w}{1 - 2i\langle\overline{a_m}, z\rangle + (r_m - i|a_m|^2)w}\right),
$$

$$
\tau^m(z^*, w^*) = \left(\frac{\lambda^*_m(z^* + a^*_m w^*)U^*_m}{1 - 2i\langle\overline{a^*_m}, z^*\rangle + (r^*_m - i|a^*_m|^2)w}, \frac{\lambda^{*2} w^*}{1 - 2i\langle\overline{a^*_m}, z^*\rangle + (r^*_m - i|a^*_m|^2)w^*}\right),
$$

where $\lambda_m > 0$, $a_m \in \mathbb{C}$, $U_m \in \mathbb{C}$ with $|U_m| = 1$, $\lambda^* = \lambda^{-1}$, $a^*_m = (a^*_{m,1}, a^*_{m,2}) \in \mathbb{C} \times \mathbb{C}^2$, $a^*_{m,1} = -\lambda_m^{-1} a_m U_m$, $a^*_{m,2} = 0$, $r^*_m = -\lambda_m^{-2} r_m$, $U^*_m = \begin{pmatrix} U_m^{-1} & 0 \\ 0 & U^*_{m,22} \end{pmatrix}$ is a 4×4 matrix, and $U^*_{m,22}$ is an unitary 3×3 unitary matrix.

By the formulas (10), the automorphisms σ_m and τ_m must satisfy the following relationship.

$$(i)\quad \frac{\partial^2 \hat{f}_m}{\partial z\partial w}|_0 = \lambda_m^2 \frac{\partial^2 \widetilde{f}_m}{\partial z\partial w}|_0,$$
$$(ii)\quad \frac{\partial^2 \hat{f}_m}{\partial w^2}|_0 = i\lambda_m^2 a_m \frac{\partial^2 \widetilde{f}_m}{\partial z\partial w}|_0 U_m^{-1} + \lambda_m^3 \frac{\partial^2 \widetilde{f}_m}{\partial w^2}|_0 U_m^{-1},$$
$$(iii)\quad \frac{\partial^2 \hat{\phi}_m}{\partial z^2}|_0 = \lambda_m U_m^2 \frac{\partial^2 \widetilde{\phi}_m}{\partial z^2}|_0 U_{22,m}^*,$$
$$(iv)\quad \frac{\partial^2 \hat{\phi}_m}{\partial z\partial w}|_0 = \lambda_m \frac{\partial^2 \widetilde{\phi}_m}{\partial z^2}|_0 a_m U_m^2 U_{22,m}^* + \lambda_m^2 U_m \frac{\partial^2 \widetilde{\phi}_m}{\partial z\partial w}|_0 U_{22,m}^*,$$
$$(v)\quad \frac{\partial^2 \hat{\phi}_m}{\partial w^2}|_0 = \lambda_m a_m^2 \frac{\partial^2 \widetilde{\phi}_m}{\partial z^2}|_0 U_m^2 U_{22,m}^* + 2\lambda_m^2 a_m U_m \frac{\partial^2 \widetilde{\phi}_m}{\partial z\partial w}|_0 U_{22,m}^* + \lambda_m^3 \frac{\partial^2 \widetilde{\phi}_m}{\partial w^2}|_0 U_{22,m}^*.$$

From (i), since F has rank 0 at 0, we see $\frac{\partial^2 \hat{f}_m}{\partial z\partial w}|_0 \to 0$. Recall that $\widetilde{F}_m$ has rank one at 0 and is of the form in Lemma 2.3(1). Then $\frac{\partial^2 \widetilde{f}_m}{\partial z\partial w}|_0 = \frac{i}{2}$ so that $\lambda_m \to 0$ as m goes to ∞.

From (ii), since $\frac{\partial \widetilde{f}_m}{\partial w^2}|_0 = 0$, we know that $\lambda_m^2 a_m$ is bounded.

From (iii), since $\lambda_m \to 0$ and $\frac{\partial^2 \widetilde{\phi}_m}{\partial z^2}|_0 = [1,0,0]$, we see $\frac{\partial^2 \hat{\phi}_m}{\partial z^2}|_0 \to \frac{\partial^2 \phi^{**}}{\partial z^2}|_0 = [0,0,0]$.

From (iv), the second term in the right hand side goes to zero for $\lambda_m \to 0$, and the first term in the right hand side is $\lambda_m \frac{\partial^2 \widetilde{\phi}_m}{\partial z^2}|_0 a_m U_m^2 U_{22,m}^* = \frac{\lambda_m^2 a_m}{\lambda_m}[1,0,0] U_m^2 U_{22,m}^*$. Recall from (ii) that $\lambda_m^2 a_m$ is bounded. On the other hand, $\frac{\partial^2 \hat{\phi}_m}{\partial z\partial w}|_0$ is bounded. All of these imply that $\lambda_m^2 a_m$ must go to zero. Then from (ii), $\frac{\partial^2 \hat{f}_m}{\partial w^2}|_0 \to \frac{\partial^2 f^{**}}{\partial w^2}|_0 = 0$.

From (v), the second and the third terms on the right hand side converge to zero because of λ_m and $a_m\lambda_m^2 \to 0$. The first term on the right hand side is bounded and can be written as $\frac{\lambda_m^2 a_m^2}{\lambda_m} \frac{\partial^2 \widetilde{\phi}_m}{\partial z^2}|_0 U_m^2 U_{22,m}^*$. This implies that $\lambda_m a_m \to 0$. Then from (iv), it proves $\frac{\partial^2 \hat{\phi}_m}{\partial z\partial w}|_0 \to \frac{\partial^2 \hat{\phi}}{\partial z\partial w} = [0,0,0]$. Our claim (23), as well as (22), is proved.

The part (ii) is already included in the above proof. For the part (iii), $\widetilde{G}_{1,m}$ is convergent because of the normalization procedure of F^{***} from F (cf. [Hu03]) and because of the part (i). □

3 A lemma for local computation

The only remaining way to further simplify F^{***} in Lemma 2.3 is to pass from F to F_p. This then gives us three new real parameters $p = (z_0, u_0 + i|z_0|^2)$ at our disposal. Here F_p is the same as defined in §2, which is equivalent to F.

Let F be as in Lemma 2.3 (1). By Lemma 2.3, F_p is equivalent to a map of the following form $F_p^{***} = (f_p^{***}, \phi_{1,p}^{***}, \phi_{2,p}^{***}, g_p^{***})$ for any $p \in \partial\mathbb{H}^2$ where $Rk_F(p) = 1$:

$$f_p^{***}(z,w) = \frac{z - 2ib(p)z^2 + (\frac{i}{2} + ie_1(p))zw}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z}, \tag{24}$$

$$\phi_{1,p}^{***}(z,w) = \frac{z^2 + b(p)zw}{1 + ie_1(p)w + e_2w^2 - 2ib(p)z}, \tag{25}$$

$$\phi_{2,p}^{***}(z,w) = \frac{c_2(p)w^2 + c_1(p)zw}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z}, \tag{26}$$

$$\phi_{3,p}^{***}(z,w) = \frac{c_3(p)w^2}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z}, \tag{27}$$

$$g_p^{***}(z,w) = \frac{w + ie_1(p)w^2 - 2ib(p)zw}{1 + ie_1(p)w + e_2w^2 - 2ib(p)z}. \tag{28}$$

Here $b(p), e_1(p), e_2(p), c_1(p), c_2(p), c_3(p)$ satisfy $e_2(p)e_1(p) = c_2^2(p) + c_3^2(p)$, $-e_2(p) = \frac{1}{4} + e_1(p) + b^2(p) + c_1^2(p)$, and $-b(p)e_2(p) = c_1(p)c_2(p)$, $c_3(p) = 0$ if $c_1(p) = 0$, with $c_1(p), c_2(p), b(p) \geq 0$, $e_2(p), e_1(p) \leq 0$.

Lemma 3.1 *Let $F = F_{c_1,c_3,e_1,e_2}$ and F_p^{***} be as above. Then for $p = (z_0, w_0) = (z_0, u_0 + i|z_0|^2) \in \partial\mathbb{H}^2$ near 0, we have real analytic functions*

$$b^2(p) = b^2 - 4b(2e_1 + c_1^2)\Im(z_0) + o(1), \quad c_1^2(p) = c_1^2 + 4c_1(bc_1 + 2c_2)\Im(z_0) + o(1),$$
$$e_2(p) + e_1(p) = e_2 + e_1 + 8b(e_1 + e_2)\Im(z_0) + o(1),$$
$$c_1^2(p) - e_1(p) - e_2(p) = c_1^2 - e_1 - e_2 + \Big(4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2)\Big)\Im(z_0) + o(1)$$

where we denote $o(k) = o(|(z_0, u_0)|^k)$.

The proof of Lemma 3.1 is long but tedious, and will be given in Section 5.

For any $F_{c_1,c_3,e_1,e_2} \in \mathcal{K}$, we define $\mathcal{W}(F_{c_1,c_3,e_1,e_2}) := \mathcal{W}(c_1, c_3, e_1, e_2) := c_1^2 - e_1 - e_2$.

4 Proof of Theorem 1.1

Proof of Theorem 1.1: For any given non-linear map $F \in Rat(\mathbb{H}^2, \mathbb{H}^N)$ with $deg(F) = 2$, by [Theorem 4.1, HJX06], we can assume that $F \in Rat(\mathbb{H}^2, \mathbb{H}^5)$.

Step 1. Define a limit map F_{c_1,c_3,e_1,e_2} Assume that $F(0) = 0$ has rank 1 at 0 and that $F = F^{***} \in \mathcal{K}$. Using notation as in Lemma 3.1, we consider $\ell = \inf \mathcal{W}(F_p^{***}) =$

$\inf\{c_1^2(p) - e_1(p) - e_2(p)\}$ where p runs through all points in $\partial\mathbb{H}^2$ with $Rk_F(p) = 1$. Take a sequence of points $p_m \in \partial\mathbb{H}^2$ such that

$$\lim_{m\to\infty} \mathcal{W}(F^{***}_{p_m}) = \ell. \tag{29}$$

Write $F^{***}_{p_m} = F_{c_1^{(m)},c_3^{(m)},e_1^{(m)},e_2^{(m)}} \in \mathcal{K}$. We claim that all $e_1(p_m), e_2(p_m), c_1(p_m),\ c_2(p_m)$, $c_3(p_m)$ and $b(p_m)$ are uniformly bounded for all m. In fact, since $c_1(p_m), -e_1(p_m), -e_2(p_m)$ are non-negative, $c_1(p_m), e_1(p_m)$ and $e_2(p_m)$ are uniformly bounded for all m. From $-e_1(p_m) - e_2(p_m) = \frac{1}{4} + b^2(p_m) + c_1^2(p_m)$, $b(p_m)$ is uniformly bounded for any m. Finally, from $e_1(p_m)e_2(p_m) = c_2^2(p_m) + c_3^2(p_m)$, $c_2(p_m)$ and $c_3(p_m)$ are uniformly bounded. Our claim is proved.

Since $c_1^{(m)}, c_3^{(m)}, e_1^{(m)}$ and $e_2^{(m)}$ are bounded for any m, by taking subsequences, we assume that $(c_1^{(m)}, c_3^{(m)}, e_1^{(m)}, e_2^{(m)}) \to (c_1, c_3, e_1, e_2) \in \mathcal{K}$ as $m \to \infty$, and hence $F_{c_1^{(m)},c_3^{(m)},e_1^{(m)},e_2^{(m)}}$ converges to a limit map $F_{c_1,c_3,e_1,e_2} \in \mathcal{K}$ as $m \to \infty$. We claim that

$$F_{c_1,c_3,e_1,e_2} \textit{ is equivalent to } F. \tag{30}$$

In fact, since $F^{***}_{p_m}$ is equivalent to F, we have $F^{***}_{p_m} = K_m \circ F \circ H_m$ where $H_m \in Aut(\mathbb{H}^2)$ and $K_m \in Aut(\mathbb{H}^5)$. Notice that the choices of such H_m and K_m are not unique. By taking subsequences, we assume $p_m := H_m(0) \to p_0 \in \overline{\partial\mathbb{H}^2}$ as $m \to \infty$.

We consider two possibilities. The first, suppose that $p_0 \neq \infty$. Then we can write

$$\begin{aligned}
&F^{***}_{p_m} = G_{2,m} \circ \tau^F_{p_m} \circ F \circ \sigma_{p_m} \circ G_{1,m} \\
&= G_{2,m} \circ \tau^F_{p_m} \circ (\tau^F_{p_0})^{-1} \circ \tau^F_{p_0} \circ F \circ \sigma_{p_0} \circ \sigma^{-1}_{p_0} \circ \sigma_{p_m} \circ G_{1,m} \\
&= \left(G_{2,m} \circ \tau^F_{p_m} \circ (\tau^F_{p_0})^{-1} \right) \circ F_{p_0} \circ \left(\sigma^{-1}_{p_0} \circ \sigma_{p_m} \circ G_{1,m} \right) \\
&= \left(G_{2,m} \circ \tau^F_{p_m} \circ (\tau^F_{p_0})^{-1} \circ (\tau^{F_{p_0}}_{q_m})^{-1} \right) \circ \left(\tau^{F_{p_0}}_{q_m} \circ F_{p_0} \circ \sigma_{q_m} \right) \circ \left(\sigma^{-1}_{q_m} \circ \sigma^{-1}_{p_0} \circ \sigma_{p_m} \circ G_{1,m} \right) \\
&= H_{2,m} \circ (F_{p_0})_{q_m} \circ H_{1,m} = (F_{p_0})^{***}_{q_m},
\end{aligned}$$

where $q_m = \sigma^{-1}_{p_0}(p_m)$, σ_{p_m}, σ_{q_m}, $\tau^F_{p_m}$, and $\tau^{F_{p_0}}_{q_m}$ are as in (3), $H_{1,m}, G_{1,m} \in Aut_0(\partial\mathbb{H}^2)$ and $H_{2,m}, G_{2,m} \in Aut_0(\partial\mathbb{H}^5)$. Since $q_m \to 0$ and $(F_{p_m})^{***}$ converges to F_{c_1,c_3,e_1,e_2}, we apply Lemma 2.5 to imply that F_{p_0} is of rank 1 at 0, and that $H_{1,m}$ and hence $H_{2,m}$ are convergent. Therefore $F_{p_0} = F_{c_1,c_3,e_1,e_2}$ and Claim (30) is proved.

In the second possibility: $p_0 = \infty$. We write

$$\begin{aligned}
&F^{***}_{p_m} = G_{2,m} \circ \tau^F_{p_m} \circ F \circ \sigma_{p_m} \circ G_{1,m} \\
&= \left(G_{2,m} \circ \tau^F_{p_m} \circ \tau^{-1}_\infty \right) \circ \tau_\infty \circ F \circ \sigma_\infty \circ \left(\sigma^{-1}_\infty \circ \sigma_{p_m} \circ G_{1,m} \right) = (\tau_\infty \circ F \circ \sigma_\infty)^{***}_{v_m}
\end{aligned}$$

where $\sigma_\infty \in Aut(\partial\mathbb{H}^2)$ and $\tau_\infty \in Aut(\partial\mathbb{H}^5)$ such that $\sigma_\infty(0) = \infty$, and $\tau_\infty \circ F \circ \sigma_\infty(0) = 0$, and $v_m = \sigma_\infty^{-1}(p_m)$. For example, we take $\sigma_\infty(z, w) = (z/w, -1/w)$. Since $v_m \to 0$, we apply Lemma 2.5 again to imply that the map $\tau_\infty \circ F \circ \sigma_\infty$ is of rank 1 at 0, and that $G_{1,m}$ and $G_{2,m}$ are convergent. Claim (30) is proved.

In the following sections, we always assume that $F = F_{c_1,c_3,e_1,e_2}$ as in (30), and we shall classify such F.

Step 2. Consequence from the critical point If $c_1 = 0$, we apply Lemma 3.1 to the function $\mathcal{W}(F_p^{***}) := (c_1^2 - e_1 - e_2)(p)$ to obtain

$$\mathcal{W}(F_p^{***}) = \mathcal{W}(F_0^{***}) - 8b(e_1 + e_2)\Im(z_0) + o(|p|),$$

for $p = (z_0, u_0 + i|z_0|^2)$ sufficiently closed to 0 in $\partial\mathbb{H}^2$. By the minimum property (29), it implies that the coefficient of $\Im(z_0)$ must be zero. Then we obtain $-8b(e_1 + e_2) = 0$. Since $-e_1 - e_2 = \frac{1}{4} + b^2 \neq 0$, it implies $b = 0$.

If $c_1 > 0$, we apply Lemma 3.1 to the function $\mathcal{W}(F_p^{***})$ to obtain

$$\mathcal{W}(F_p^{***}) = \mathcal{W}(F_0^{***}) + \Big[4c_1(c_1 b + 2c_2) - 8b(e_1 + e_2)\Big]\Im(z_0) + o(|p|),$$

for $p = (z_0, u_0 + i|z_0|^2)$. By the minimum property of $F = F_0^{***}$ (see(29)), it implies that $4c_1(c_1 b + 2c_2) - 8b(e_1 + e_2) = 0$. Since $-e_1 - e_2 = \frac{1}{4} + b^2 + c_1^2 \neq 0$ and $c_1, b, c_2, -e_1, -e_2 \geq 0$, it implies $b = c_2 = 0$.

To study F, we distinguish two cases: Case (I) $c_1 = b = 0$; Case (II) $c_1 \neq 0$ and $b = c_2 = 0$.

Step 3. Case (I) In Case (I): $c_1 = b = 0$. By Lemma 2.3, $c_3 = 0$. Hence F is of the form F_{e_2}

$$f(z, w) = \frac{z + (\frac{i}{2} + ie_1)zw}{1 + ie_1 w + e_2 w^2},\ \ \phi_1(z, w) = \frac{z^2}{1 + ie_1 w + e_2 w^2}, \tag{31}$$

$$\phi_2(z, w) = \frac{c_2 w^2}{1 + ie_1 w + e_2 w^2},\ \ \ \phi_3(z, w) = 0,\ \ \ g(z, w) = \frac{w + ie_1 w^2}{1 + ie_1 w + e_2 w^2} \tag{32}$$

where $e_1 e_2 = c_2^2,\ \ -e_1 - e_2 = \frac{1}{4}$. From these two equations, by noticing $e_1, e_2 \leq 0$ and $c_2 \geq 0$, we get $e_2(p) \in [-\frac{1}{4}, 0]$ and e_1 and c_2 are determined by e_2. Hence we can regard e_2 as the parameter for the family of maps in (31)(32). Therefore we obtain a family $\{F_{e_2}\}_{e_2 \in [-\frac{1}{4}, 0]}$.

For the family $\{F_{e_2}\}$, we consider one boundary point $e_2 = 0$. From $e_2 e_1 = c_2^2$, we know $c_2 = 0$. From $-e_2 = \frac{1}{4} + e_1$, we obtain $e_1 = -\frac{1}{4}$. By the same proof as in [§ 4, Step 2 and

Step 3, JX04], the map in (31)(32) is equivalent to $G_{\pi/2}$. We also consider another boundary point of $\{F_{e_2}\}$: $e_2 = -\frac{1}{4}$. From $-e_2 = \frac{1}{4} + e_1$, we have $e_1 = 0$. From $e_2 e_1 = c_2^2$, we know $c_2 = 0$. Using the same proof in [§ 6, the proof of Theorem 1.2, case (i) and (6.7), HJX05], such a map is equivalent to G_0.

Since the above family $\{F_{e_2}\}_{-\frac{1}{4}\le e_2<0}$ can be represented as real algebraic variety $\subseteq [-\frac{1}{4}, 0]$ and the family $\{G_t\}_{0\le t<\frac{\pi}{2}}$ in Theorem 1.1(I) is its connected subset with the same boundary points $\{-\frac{1}{4}\}$ and $\{0\}$, we identify $\{F_{e_2}\}_{-\frac{1}{4}\le e_2<0}$ with $\{G_t\}_{0\le t<\frac{\pi}{2}}$. Therefore, F is equivalent to $(G_t, 0)$ as in Theorem 1.1(I) in Case (I).

Step 4. Maps in Case (II) that can be embedded into $\mathbb{H}^4$ By F can be embedded into $\mathbb{H}^4$, we mean that $F(\mathbb{H}^2) \subset G(\mathbb{H}^4)$ for some automorphism $G \in Aut(\mathbb{H}^5)$.

Now consider Case (II): $c_1 > 0$ with $b = c_2 = 0$. Then F is of the form F_{c_1,c_3,e_1,e_2}:

$$f = \frac{z + (\frac{i}{2} + ie_1)zw}{1 + ie_1 w + e_2 w^2},\ \phi_1 = \frac{z^2}{1 + ie_1 w + e_2 w^2},$$
$$\phi_2 = \frac{c_1 zw}{1 + ie_1 w + e_2 w^2},\ \phi_3 = \frac{c_3 w^2}{1 + ie_1 w + e_2 w^2},\ g = \frac{w + ie_1 w^2}{1 + ie_1 w + e_2 w^2},$$

where $0 < c_1 < \infty$ and $0 \le c_3 \le \frac{1}{8} + \frac{c_1^2}{2}$ because e_1 and e_2 are non-negative real numbers determined by $e_1 e_2 = c_3^2$ and $-e_1 - e_2 = \frac{1}{4} + c_1^2$. We claim:

$$F \textit{ can be embedded into } \mathbb{H}^4 \iff c_3 = 0. \tag{33}$$

In fact, $F(\mathbb{H}^2)$ can be embedded into $\mathbb{H}^4$ if and only if for any point $(z, w) \in \partial\mathbb{H}^2$ sufficiently closed to $(0, 0)$, the tangent space $T^{(1,0)}_{F(z,w)}(\partial\mathbb{H}^5)$ is contained in a fixed hyperplane of $\mathbb{C}^5$. More precisely, the tangent space $T^{(1,0)}_{F(z,w)}(\partial\mathbb{H}^5)$ at the point $F(z, w)$ is spanned by the vectors $\vec{F_z} = (Lf, L\phi_1, L\phi_2, L\phi_3, Lg) = (1+\frac{i}{2}w+(\frac{e_1}{2}-e_2)w^2,\ 2z-2ie_1 zw,\ c_1 w-ie_1 c_1 w^2,\ 0,\ 0)+o(|(z, u)|^2)$ and $\vec{F_w} = (Tf, T\phi_1, T\phi_2, T\phi_3, Tg) = (\frac{i}{2}z + (e_1 - 2e_2)zw, -ie_1 z^2,\ c_1 z - 2ie_1 c_1 zw,\ 3c_3 w - 3ie_1 c_3 w^2,\ 1 - 3e_2 w^2) + o(|(z, w)|^2)$. The statement that F^{***} can be embedded into $\mathbb{H}^4$ is equivalent to the fact that there are constants $(A_1, A_2, ..., A_6) \ne (0, 0, ..., 0)$ such that

$$A_1(1 + \frac{i}{2}w + (\frac{e_1}{2} - e_2)w^2) + A_2(2z - 2ie_1 zw) + A_3(c_1 w - ie_1 c_1 w^2) = A_6 + o(|(z, w)|^2),$$
$$A_1(\frac{i}{2}z + (e_1 - 2e_2)zw) + A_2(-ie_1 z^2) + A_3(c_1 z - 2ie_1 c_1 zw) + A_4(3c_3 w - 3ie_1 c_3 w^2)$$
$$+A_5(1 - 3e_2 w^2) = A_6 + o(|(z, w)|^2), \quad \forall (z, w) \in \partial\mathbb{H}^2.$$

If $c_3 = 0$, we can take $A_4 = 1, A_1 = A_2 = A_3 = A_5 = A_6 = 0$ so that F^{***} can be embedded into $\mathbb{H}^4$. Conversely, suppose F can be embedded into $\mathbb{H}^4$ and $c_3 \ne 0$. We seek a contradiction. By considering the constant, z and u terms, we see $A_1 = A_5 = A_6$, $A_2 = 0$, $A_3 = -\frac{i}{2c_1}A_1$ and $A_4 = 0$ because $c_3 \neq 0$. By considering the zu terms, we get $A_1(e_1 - 2e_2) - 2ie_1c_1A_3 = 0$, i.e., $-2e_2A_1 = 0$. Recall $e_1e_2 = c_3^2 \neq 0$. This implies that $A_1 = 0$, i.e., $(A_1, ..., A_6) = 0$, which is a contradiction. Our claim (33) is proved.

Since $c_3 = e_1e_2$, by Claim (33), the case of $c_3 = 0$ can be divided into two subcases: Case (IIA) $c_3 = e_2 = 0$, and Case (IIB) $c_3 = e_1 = 0$.

Step 5. Case (IIA) In this subcase, F is of the form F_{c_1}

$$f = \frac{z + (\frac{i}{2} + ie_1)zw}{1 + ie_1w}, \ \phi_1 = \frac{z^2}{1 + ie_1w}, \ \phi_2 = \frac{c_1zw}{1 + ie_1w}, \ \phi_3 = 0, \ g = w,$$

where $-e_1 = \frac{1}{4} + c_1^2$ and $c_1 > 0$ can be regarded as a parameter. Since $e_2 = 0$, by Lemma 2.3(3), F is equivalent to F_θ as in (1). Therefore, F is equivalent to $(F_\theta, 0)$ as in Theorem 1.1(IIA) in Case (IIA).

By the way, for the family $\{F_{c_1}\}$, we consider one boundary point: when $c_1 = 0$, the map F_{c_1} is equivalent to $F_{\frac{\pi}{2}} \simeq G_{\frac{\pi}{2}}$. We also consider another boundary point: when $c_1 \to +\infty$, the map F_{c_1} tends to the linear map $F_0 = (z, 0, 0, w)$.

Step 6. Case (IIB) In this subcase, F is of the form

$$f = \frac{z + \frac{i}{2}zw}{1 + e_2w^2}, \ \phi_1 = \frac{z^2}{1 + e_2w^2}, \ \phi_2 = \frac{c_1zw}{1 + e_2w^2}, \ \phi_3 = 0, \ g = \frac{w}{1 + e_2w^2},$$

where $-e_2 = \frac{1}{4} + c_1^2$. Here $c_1 > 0$ can be regarded as a parameter.

Step 7. Case(IIC) Let us consider Case(II) in which $c_3 > 0$, i.e., F cannot be embedded into $\mathbb{H}^4$. From Step 4, such $F = F_{c_1,c_3,e_1,e_2}$ is of the form

$$f(z,w) = \frac{z + (\frac{i}{2} + ie_1)zw}{1 + ie_1w + e_2w^2}, \ \phi_1(z,w) = \frac{z^2}{1 + ie_1w + e_2w^2},$$
$$\phi_2(z,w) = \frac{c_1zw}{1 + ie_1w + e_2w^2}, \ \phi_3(z,w) = \frac{c_3w^2}{1 + ie_1w + e_2w^2}, \ g(z,w) = \frac{w + ie_1w^2}{1 + ie_1w + e_2w^2},$$

where $-e_1 \geq 0, -e_2 \geq 0, c_1 > 0, c_3 > 0$, $e_1e_2 = c_3^2$ and $-e_1 - e_2 = \frac{1}{4} + c_1^2$.

We say that $F = F_{c_1,c_3,e_1,e_2}$ is in Case (IIC) if $\mathcal{W}(F_p^{***}) = (c_1^2 - e_1 - e_2)(p)$ satisfies

$$\mathcal{W}(F_p^{***}) \geq \mathcal{W}(F_0^{***}), \quad \forall \, p \in U \subset \partial\mathbb{H}^2 \tag{34}$$

where U is some neighborhood of 0 in $\partial\mathbb{H}^2$. We denote $\mathcal{K}_{IIC}$ to be a subset of $\mathcal{K} \subset \mathbb{R}^4$ such that $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{IIC}$ if and only if F_{c_1,c_3,e_1,e_2} is in Case (IIC). Sometimes we may denote $F \in \mathcal{K}_{IIC}$.

Clearly, for any F_{c_1,c_3,e_1,e_2} of Case (IIC) with $c_3 > 0$ defined in Step 1, $F_{c_1,c_3,e_1,e_2} \in \mathcal{K}_{IIC}$. The proof of Theorem 1.1 is compete except of Lemma 3.1. □

5 Proof of Lemma 3.1

Proof of Lemma 3.1: Let $F = F_{c_1,c_3,e_1,e_2}$. We will first follow the procedure to normalize F_p to a map F_p^*, and then further normalize it to the map F_p^{**} satisfying the condition in Lemma 2.0. Write $p = (z_0, w_0)$. We obtain normalization F_p^{***}.

Step 1. Compute F_p We have

$$f(z,w) = [z - 2ibz^2 + (\frac{i}{2} + ie_1)zw]\Big[1 + (-ie_1w + 2ibz - e_2w^2) + (-ie_1w + 2ibz)^2\Big]$$
$$+o(3) = z + \frac{i}{2}zw - bz^2w + (\frac{e_1}{2} - e_2)zw^2 + o(3),$$

$$Lf(p) = 1 + \frac{i}{2}w_0 - 2bz_0w_0 + (\frac{e_1}{2} - e_2)w_0^2 - |z_0|^2 + o(2),$$

$$Tf(p) = \frac{i}{2}z_0 - bz_0^2 + (e_1 - 2e_2)z_0w_0 + o(2),$$

$$L^2f(p) = -2bw_0 - 2\overline{z}_0 + o(1),$$

$$TLf(p) = \frac{i}{2} - 2bz_0 + (e_1 - 2e_2)w_0 + o(1), \; T^2f_p = (e_1 - 2e_2)z_0 + o(1),$$

$$\phi_1(z,w) = (z^2 + bzw)\Big[1 + (-ie_1w + 2ibz)\Big] + o(3)$$
$$= z^2 + bzw + (-ie_1 + 2ib^2)z^2w + 2ibz^3 - ie_1bzw^2 + o(3),$$

$$L\phi_1(p) = 2z_0 + bw_0 + (-2ie_1 + 4ib^2)z_0w_0 + 6ibz_0^2 - ie_1bw_0^2 + 2ib|z_0|^2 + o(2),$$

$$T\phi_1(p) = bz_0 - ie_1z_0^2 - 2ibe_1z_0w_0 + 2ib^2z_0^2 + o(2),$$

$$L^2\phi_1(p) = 2 + (-2ie_1 + 4ib^2)w_0 + 12ibz_0 + 4ib\overline{z}_0 + o(1),$$

$$TL\phi_1(p) = b + (-2ie_1 + 4ib^2)z_0 - 2ie_1bw_0 + o(1),\ T^2\phi_1(p) = -2ie_1bz_0 + o(1),$$

$$\begin{aligned}\phi_2(z,w) &= (c_2w^2 + c_1zw)\Big[1 + (-ie_1w + 2ibz)\Big] + o(3)\\ &= c_2w^2 + c_1zw - ie_1c_2w^3 + (-ie_1c_1 + 2ibc_2)zw^2 + 2ibc_1z^2w + o(3),\end{aligned}$$

$$L\phi_2(p) = c_1w_0 + (-ie_1c_1 + 2ibc_2)w_0^2 + 4ibc_1z_0w_0 + 2i\overline{z}_0(c_1z_0 + 2c_2w_0) + o(2),$$

$$T\phi_2(p) = 2c_2w_0 + c_1z_0 - 3ie_1c_2w_0^2 + (-2ie_1c_1 + 4ibc_2)z_0w_0 + 2ibc_1z_0^2 + o(2),$$

$$L^2\phi_2(p) = 4ibc_1w_0 + 4ic_1\overline{z}_0 + o(1),$$

$$TL\phi_2(p) = c_1 + (-2ie_1c_1 + 4ibc_2)w_0 + 4ibc_1z_0 + 4ic_2\overline{z}_0 + o(1),$$

$$T^2\phi_2(p) = 2c_2 - 6ie_1c_2w_0 + (-2ie_1c_1 + 4ibc_2)z_0 + o(1),$$

$$\phi_3(z,w) = c_3 w^2\Big[1 + (-ie_1 w + 2ibz)\Big] + o(3) = c_3 w^2 - ie_1 c_3 w^3 + 2ibc_3 z w^2 + o(3),$$

$$L\phi_3(p) = 2ibc_3 w_0^2 + 4ic_3 \overline{z_0} w_0 + o(2),$$

$$T\phi_3(p) = 2c_3 w_0 - 3ie_1 c_3 w_0^2 + 4ibc_3 z_0 w_0 + o(2), \quad L^2\phi_3(p) = o(1),$$

$$TL\phi_3(p) = 4ibc_3 w_0 + 4ic_3\overline{z}_0 + o(1), \; T^2\phi_3(p) = 2c_3 - 6ie_1 c_3 w_0 + 4ibc_3 z_0 + o(1),$$

$$g(z,w) = (w + ie_1 w^2 - 2ibzw)\Big[1 + (-ie_1 w + 2ibz - e_2 w^2) + (-ie_1 w + 2ibz)^2\Big]$$
$$+ o(3) = w - e_2 w^3 + o(3),$$

$$Tg(p) = 1 - 3e_2 w_0^2 + o(2), \; T^2 g(p) = -6e_2 w_0 + o(1), \; \lambda(p) = Tg_p(0) = 1 + o(1).$$

Step 2. Compute F_p^{**}: As in [pp 467, (2.1.3), (2.1.4), Hu03], we get

$$F_p^* = F_p \begin{pmatrix} \overline{\frac{Lf(p)}{\lambda(p)}} & \overline{\frac{C_1^{(1)}(p)}{\sqrt{\lambda}}} & \overline{\frac{C_1^{(2)}(p)}{\sqrt{\lambda}}} & \overline{\frac{C_1^{(3)}(p)}{\sqrt{\lambda}}} & 0 \\ \overline{\frac{L\phi_1(p)}{\lambda(p)}} & \frac{C_2^{(1)}(p)}{\sqrt{\lambda}} & \frac{C_2^{(2)}(p)}{\sqrt{\lambda}} & \frac{C_2^{(3)}(p)}{\sqrt{\lambda}} & 0 \\ \overline{\frac{L\phi_2(p)}{\lambda(p)}} & \frac{C_3^{(1)}(p)}{\sqrt{\lambda}} & \frac{C_3^{(2)}(p)}{\sqrt{\lambda}} & \frac{C_3^{(3)}(p)}{\sqrt{\lambda}} & 0 \\ \overline{\frac{L\phi_3(p)}{\lambda(p)}} & \frac{C_4^{(1)}(p)}{\sqrt{\lambda}} & \frac{C_4^{(2)}(p)}{\sqrt{\lambda}} & \frac{C_4^{(3)}(p)}{\sqrt{\lambda}} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\lambda(p)} \end{pmatrix}$$

where

$$C_1^{(1)}(p) = -\frac{\overline{L\ \phi_1}}{\sqrt{|L\ f|^2 + |L\ \phi_1|^2}}(p) = -2\overline{z_0} - b\overline{w_0} + o(1),$$
$$C_2^{(1)}(p) = \frac{\overline{L\ f}}{\sqrt{|L\ f|^2 + |L\ \phi_1|^2}}(p) = 1 - \frac{i}{2}\overline{w_0} + o(1), \; C_3^{(1)}(p) = 0, \; C_4^{(1)}(p) = 0.$$

$$C_1^{(2)}(p) = -\frac{\overline{L\ \phi_2} L\ f}{\sqrt{|L\ f|^2 + |L\ \phi_1|^2}\sqrt{|L\ f|^2 + |L\ \phi_1|^2 + |L\ \phi_2|^2}}(p) = -c_1\overline{w_0} + o(1),$$
$$C_2^{(2)}(p) = -\frac{\overline{L\ \phi_2} L\ \phi_1}{\sqrt{|L\ f|^2 + |L\ \phi_1|^2}\sqrt{|L\ f|^2 + |L\ \phi_1|^2 + |L\ \phi_2|^2}}(p) = o(1),$$
$$C_3^{(2)}(p) = \frac{|L\ f|^2 + |L\ \phi_1|^2}{\sqrt{|L\ f|^2 + |L\ \phi_1|^2}\sqrt{|L\ f|^2 + |L\ \phi_1|^2 + |L\ \phi_2|^2}}(p) = 1 + o(1), \quad C_4^{(2)}(p) = 0,$$

$$C_1^{(3)}(p) = -\frac{\overline{L\ \phi_3} L\ f}{\sqrt{\lambda}\sqrt{|L\ f|^2 + |L\ \phi_1|^2 + |L\ \phi_2|^2}}(p) = o(1),$$
$$C_2^{(3)}(p) = -\frac{\overline{L\ \phi_3} L\ \phi_1}{\sqrt{\lambda}\sqrt{|L\ f|^2 + |L\ \phi_1|^2 + |L\ \phi_2|^2}}(p) = o(1),$$
$$C_3^{(3)}(p) = -\frac{\overline{L\ \phi_3} L\ \phi_2}{\sqrt{\lambda}\sqrt{|L\ f|^2 + |L\ \phi_1|^2 + |L\ \phi_2|^2}}(p) = o(1),$$
$$C_4^{(3)}(p) = \frac{|L\ f|^2 + |L\ \phi_1|^2 + |L\ \phi_2|^2}{\sqrt{\lambda}\sqrt{|L\ f|^2 + |L\ \phi_1|^2 + |L\ \phi_2|^2}}(p) = 1 + o(1).$$

Let F_p^{**} be as defined in Lemma 2.1(see [(2.1.8), Hu03]). By using the formula in [((2.1.6)-(2.1.8), Hu03] we have

$$\frac{\partial^2 f_p^{**}}{\partial z \partial w}|_0 = \frac{1}{\lambda(p)} LT\widetilde{f}(p) \cdot \overline{L\widetilde{f}(p)}^t - \frac{2i}{\lambda(p)^2}\left| T\widetilde{f}(p) \cdot \overline{L\widetilde{f}(p)}^t \right|^2$$
$$-\frac{1}{2\lambda(p)}\left(T^2 g(p) - 2iT^2\widetilde{f}(p) \cdot \overline{\widetilde{f}(p)}^t \right)$$
$$= LTf \cdot \overline{Lf} + LT\phi_1 \cdot \overline{L\phi_1} + LT\phi_2 \cdot \overline{L\phi_2} - \frac{1}{2}T^2 g + o(1) = \frac{i}{2} - 2bz_0 + 2b\overline{z_0} + o(1).$$

Here we used the formula $-e_1 - e_2 = \frac{1}{4} + b^2 + c_1^2$.

$$\frac{\partial^2 f_p^{**}}{\partial w^2}(0) = \frac{1}{\lambda(p)} T^2\widetilde{f}(p) \cdot \overline{L\widetilde{f}}^t - \frac{1}{\lambda(p)^2}\left(T\widetilde{f} \cdot \overline{L\widetilde{f}}^t \right)\left(T^2 g - 2iT^2\widetilde{f} \cdot \overline{\widetilde{f}}^t - 2i\|T\widetilde{f}\|^2 \right)(p)$$
$$= T^2 f \cdot \overline{Lf} + T^2\phi_2 \cdot \overline{L\phi_2} + o(1) = (e_1 - 2e_2)z_0 + 2c_1c_2u_0 + o(1),$$

$$\frac{\partial^2\phi^{**}_{p1}}{\partial z^2}(0) = \frac{1}{\sqrt{\lambda(p)}}L^2\widetilde{f}(p)\cdot\overline{C_1(p)}^t = L^2\phi_1\overline{C_2^{(1)}} + o(1)$$
$$= 2 + 12ibz_0 + 2i(2b^2 - e_1)u + 4ib\overline{z_0} + iu_0 + o(1),$$

$$\frac{\partial^2\phi^{**}_{p1}}{\partial z\partial w}(0) = \frac{1}{\sqrt{\lambda(p)}}TL\widetilde{f}(p)\cdot\overline{C_1(p)}^t - \frac{2i}{\lambda(p)^{3/2}}\left(T\widetilde{f}(p)\cdot\overline{C_1(p)}^t\right)\left(L\widetilde{f}(p)\cdot T\overline{\widetilde{f}}^t(p)\right)$$
$$= TLf\cdot\overline{C_1^{(1)}} + TL\phi_1\cdot\overline{C_2^{(1)}} + o(1) = b - iz_0 + 2i(2b^2 - e_1)z_0 - 2ibe_1u_0 + o(1),$$

$$\frac{\partial^2\phi^{**}_{p1}}{\partial w^2}(0) = \frac{1}{\sqrt{\lambda(p)}}T^2\widetilde{f}(p)\cdot\overline{C_1(p)}^t$$
$$-\frac{1}{\lambda(p)^{3/2}}\left(T\widetilde{f}(p)\cdot\overline{C_1(p)}^t\right)\left(T^2g(p) - 2iT^2\widetilde{f}(p)\cdot\overline{\widetilde{f}(p)}^t - 2i\|T\widetilde{f}\|^2\right)(p)$$
$$= T^2\phi_1\cdot\overline{C_2^{(1)}} + o(1) = -2ibe_1z_0 + o(1),$$

$$\frac{\partial^2\phi^{**}_{p2}}{\partial z^2}(0) = \frac{1}{\sqrt{\lambda(p)}}L^2\widetilde{f}(p)\cdot\overline{C_2(p)}^t = L^2\phi_2 + o(1) = 4ibc_1u_0 + 4ic_1\overline{z_0} + o(1),$$

$$\frac{\partial^2\phi^{**}_{p2}}{\partial z\partial w}(0) = \frac{1}{\sqrt{\lambda(p)}}TL\widetilde{f}(p)\cdot\overline{C_2(p)}^t - \frac{2i}{\lambda(p)^{3/2}}\left(T\widetilde{f}(p)\cdot\overline{C_2(p)}^t\right)\left(L\widetilde{f}(p)\cdot T\overline{\widetilde{f}}^t(p)\right)$$
$$= TLf\cdot\overline{C_1^{(2)}} + TL\phi_2 + o(1) = c_1 - \frac{i}{2}c_1u_0 + 2i(2bc_2 - c_1e_1)u_0 + 4ibc_1z_0 + 4ic_2\overline{z_0} + o(1),$$

$$\frac{\partial^2\phi^{**}_{p2}}{\partial w^2}(0) = \frac{1}{\sqrt{\lambda(p)}}T^2\widetilde{f}(p)\cdot\overline{C_2(p)}^t$$
$$-\frac{1}{\lambda(p)^{3/2}}\left(T\widetilde{f}(p)\cdot\overline{C_2(p)}^t\right)\left(T^2g(p) - 2iT^2\widetilde{f}(p)\cdot\overline{\widetilde{f}(p)}^t - 2i\|T\widetilde{f}\|^2\right)$$
$$= T^2\phi_2 + o(1) = 2c_2 - 6ic_2e_1u_0 + 2i(2bc_2 - c_1e_1)z_0 + o(1),$$

$$\frac{\partial^2 \phi^{**}_{p3}}{\partial z^2}(0) = \frac{1}{\sqrt{\lambda(p)}} L^2 \widetilde{f}(p) \cdot \overline{C_3(p)}^t = L^2\phi_3 + o(1) = o(1),$$

$$\frac{\partial^2 \phi^{**}_{p3}}{\partial z \partial w}(0) = \frac{1}{\sqrt{\lambda(p)}} TL\widetilde{f}(p) \cdot \overline{C_3(p)}^t - \frac{2i}{\lambda(p)^{3/2}} \left(T\widetilde{f}(p) \cdot \overline{C_3(p)}^t \right) \left(L\widetilde{f}(p) \cdot T\overline{\widetilde{f}}^t(p) \right)$$
$$= TL\phi_3 + o(1) = 4ibc_3u_0 + 4ic_3\overline{z_0} + o(1),$$

$$\frac{\partial^2 \phi^{**}_{p3}}{\partial w^2}(0) = \frac{1}{\sqrt{\lambda(p)}} T^2 \widetilde{f} \cdot \overline{C_3}^t$$
$$- \frac{1}{\lambda(p)^{3/2}} \left(T\widetilde{f}(p) \cdot \overline{C_3(p)}^t \right) \left(T^2 g(p) - 2iT^2\widetilde{f}(p) \cdot \overline{\widetilde{f}(p)}^t - 2i\|T\widetilde{f}\|^2 \right)$$
$$= T^2\phi_3 + o(1) = 2c_3 - 6ic_3e_1u_0 + 4ibc_3z_0 + o(1).$$

Step 3. Compute F^{***}_p: We next transform F^{**}_p into a normal form as in Lemma 2.2. For clarification, we do it in several steps.

Define $F^{**}_{pb} = \tau^* \circ F^{**}_p \circ \sigma$ so that $\frac{\partial^2 f^{**}_{pb}}{\partial z \partial w}(0) = 1$, where σ and τ^* are as in (7) with

$$\lambda = \frac{1}{\sqrt{-2i\frac{\partial^2 f^{**}_p}{\partial z \partial w}(0)}} = 1 - 2ibz_0 + 2ib\overline{z_0} + o(1),$$

$a = 0, r = 0, U^*_{22} = id, U = id$. Then by the formulas (10),

$$\frac{\partial^2 f^{**}_{pb}}{\partial w^2}(0) = \lambda^3 \frac{\partial^2 f^{**}_p}{\partial w^2}|_0 = \frac{\partial^2 f^{**}_p}{\partial w^2}|_0 + o(1) = (e_1 - 2e_2)z_0 + 2c_1c_2u_0 + o(1),$$

$$\frac{\partial^2 \phi^{**}_{pb1}}{\partial z^2}(0) = \lambda \frac{\partial^2 \phi^{**}_{p1}}{\partial z^2}|_0 = 2 + 8ibz_0 + 2iu_0(2b^2 - e_1) + 8bi\overline{z_0} + iu_0 + o(1),$$

$$\frac{\partial^2 \phi^{**}_{pb2}}{\partial z^2}(0)) = \lambda \frac{\partial^2 \phi^{**}_{p2}}{\partial z^2}|_0 = 4ibc_1u_0 + 4c_1i\overline{z_0} + o(1),$$

$$\frac{\partial^2 \phi^{**}_{pb3}}{\partial z^2}(0) = \lambda \frac{\partial^2 \phi^{**}_{p3}}{\partial z^2}|_0 = o(1),$$

$$\frac{\partial^2\phi^{**}_{pb1}}{\partial z\partial w}(0)=\lambda^2\frac{\partial^2\phi^{**}_{p1}}{\partial z\partial w}|_0=b-iz_0-2ie_1z_0-2ibe_1u_0+4ib^2\overline{z_0}+o(1),$$

$$\frac{\partial^2\phi^{**}_{pb2}}{\partial z\partial w}(0)=\lambda^2\frac{\partial^2\phi^{**}_{p2}}{\partial z\partial w}|_0=c_1-\frac{i}{2}c_1u_0+2i(2bc_2-c_1e_1)u_0+4ic_2\overline{z_0}+4ibc_1\overline{z_0}+o(1),$$

$$\frac{\partial^2\phi^{**}_{pb3}}{\partial z\partial w}(0)=\lambda^2\frac{\partial^2\phi^{**}_{p3}}{\partial z\partial w}|_0=4ibc_3u_0+4ic_3\overline{z_0}+o(1),$$

$$\frac{\partial^2\phi^{**}_{pb1}}{\partial w^2}(0)=\lambda^3\frac{\partial^2\phi^{**}_{p1}}{\partial w^2}|_0=-2ibe_1z_0+o(1),$$

$$\frac{\partial^2\phi^{**}_{pb2}}{\partial w^2}(0)=\lambda^3\frac{\partial^2\phi^{**}_{p2}}{\partial w^2}|_0=2c_2-2ic_1e_1z_0-6ic_2e_1u_0-8ibc_2z_0+12ibc_2\overline{z_0}+o(1),$$

$$\frac{\partial^2\phi^{**}_{pb3}}{\partial w^2}(0)=\lambda^3\frac{\partial^2\phi^{**}_{p3}}{\partial w^2}|_0=2c_3-6ic_3e_1u_0-8ibc_3z_0+12ibc_3\overline{z_0}+o(1).$$

Define $F^{**}_{pc}=\tau_2^*\circ F^{**}_{pb}\circ\sigma_2$ so that $\frac{\partial^2 f^{**}_{pc}}{\partial w^2}|_0=0$, where τ_2^* and σ_2 are as in (7) with $\lambda=1,r=0,U=id,U^*_{22}=id$,

$$a=i\frac{\partial^2 f^{**}_{pb}}{\partial w^2}(0)=i(e_1-2e_2)z_0+2ic_1c_2u_0+o(1).$$

Then by the formulas(10),

$$\frac{\partial^2\phi^{**}_{pc1}}{\partial z^2}(0)=\frac{\partial^2\phi^{**}_{pb1}}{\partial z^2}(0)=2+8ibz_0+2iu_0(2b^2-e_1)+8bi\overline{z_0}+iu_0+o(1),$$

$$\frac{\partial^2\phi^{**}_{pc2}}{\partial z^2}(0)=\frac{\partial^2\phi^{**}_{pb2}}{\partial z^2}(0)=4ibc_1u_0+4c_1i\overline{z_0}+o(1),$$

$$\frac{\partial^2\phi^{**}_{pc3}}{\partial z^2}(0)=\frac{\partial^2\phi^{**}_{pb3}}{\partial z^2}(0)=o(1),$$

$$\frac{\partial^2\phi^{**}_{pc1}}{\partial z\partial w}(0)=a\frac{\partial^2\phi^{**}_{pb1}}{\partial z^2}(0)+\frac{\partial^2\phi^{**}_{pb1}}{\partial z\partial w}(0)=b-iz_0-2ibe_1u_0+4ib^2\overline{z_0}-4ie_2z_0+4ic_1c_2u_0+o(1),$$

$$\frac{\partial^2\phi^{**}_{pc2}}{\partial z\partial w}(0) = a\frac{\partial^2\phi^{**}_{pb2}}{\partial z^2}(0)+\frac{\partial^2\phi^{**}_{pb2}}{\partial z\partial w}(0) = c_1-\frac{i}{2}c_1u_0+2iu_0(2bc_2-c_1e_1)+4ic_2\overline{z_0}+4ibc_1\overline{z_0}+o(1),$$

$$\frac{\partial^2\phi^{**}_{pc3}}{\partial z\partial w}(0) = a\frac{\partial^2\phi^{**}_{pb3}}{\partial z^2}(0)+\frac{\partial^2\phi^{**}_{pb3}}{\partial z\partial w}(0) = 4ibc_3u_0+4ic_3\overline{z_0}+o(1),$$

$$\frac{\partial^2\phi^{**}_{pc1}}{\partial w^2}(0) = a^2\frac{\partial^2\phi^{**}_{pb1}}{\partial z^2}(0)+2a\frac{\partial^2\phi^{**}_{pb1}}{\partial z\partial w}(0)+\frac{\partial^2\phi^{**}_{pb1}}{\partial w^2}(0) = -4ibe_2z_0+4ibc_1c_2u_0+o(1),$$

$$\begin{aligned}&\frac{\partial^2\phi^{**}_{pc2}}{\partial w^2}(0) = a^2\frac{\partial^2\phi^{**}_{pb2}}{\partial z^2}(0)+2a\frac{\partial^2\phi^{**}_{pb2}}{\partial z\partial w}(0)+\frac{\partial^2\phi^{**}_{pb2}}{\partial w^2}(0)\\ &= 2c_2-6ic_2e_1u_0-8ibc_2z_0+12ibc_2\overline{z_0}-4ic_1e_2z_0+4ic_1^2c_2u_0+o(1).\end{aligned}$$

$$\begin{aligned}&\frac{\partial^2\phi^{**}_{pc3}}{\partial w^2}(0) = a^2\frac{\partial^2\phi^{**}_{pb3}}{\partial z^2}(0)+2a\frac{\partial^2\phi^{**}_{pb3}}{\partial z\partial w}(0)+\frac{\partial^2\phi^{**}_{pb3}}{\partial w^2}(0)\\ &= 2c_3-6ic_3e_1u_0-8ibc_3z_0+12ibc_3\overline{z_0}+o(1).\end{aligned}$$

Define $F^{**}_{pd} = \tau_3^*\circ F^{**}_{pc}\circ\sigma_3$ so that $\frac{\partial^2\phi^{**}_{1pd}}{\partial z^2}|_0 = 2$ and $\frac{\partial^2\phi^{**}_{jpd}}{\partial z^2}|_0 = 0$ for $j=2$ and 3, where σ_3 and τ_3^* are as as in (7) with $\lambda=1, r=0, U=id, a=0, a^*=0, U^*_{22}=\overline{\tilde{U}}^t$, where the unitary matrix $\tilde{U}$ is defined by

$$\tilde{U} = \begin{pmatrix}\frac{u_{11}}{\mu_1} & \frac{u_{12}}{\mu_1} & \frac{u_{13}}{\mu_1}\\ \frac{u_{21}}{\mu_2} & \frac{u_{22}}{\mu_2} & \frac{u_{23}}{\mu_2}\\ \frac{u_{31}}{\mu_3} & \frac{u_{32}}{\mu_3} & \frac{u_{33}}{\mu_3}\end{pmatrix},$$

where

$$\begin{aligned}&u_{11} = \frac{\partial^2\phi^{**}_{pc1}}{\partial z^2}(0),\ u_{12} = \frac{\partial^2\phi^{**}_{pc2}}{\partial z^2}(0),\ u_{13} = \frac{\partial^2\phi^{**}_{pc3}}{\partial z^2}(0),\\ &\mu_1 = \sqrt{|\frac{\partial^2\phi^{**}_{pc1}}{\partial z^2}|_0^2+|\frac{\partial^2\phi^{**}_{pc2}}{\partial z^2}|_0^2+|\frac{\partial^2\phi^{**}_{pc3}}{\partial z^2}|_0^2} = 2+o(1),\end{aligned}$$

$$\begin{aligned}&u_{21} = -\overline{\frac{\partial^2\phi^{**}_{pc2}}{\partial z^2}}(0),\ u_{22} = \overline{\frac{\partial^2\phi^{**}_{pc1}}{\partial z^2}}(0),\ u_{23} = 0,\\ &\mu_2 = \sqrt{|\frac{\partial^2\phi^{**}_{pc1}}{\partial z^2}|_0^2+|\frac{\partial^2\phi^{**}_{pc2}}{\partial z^2}|_0^2} = 2+o(1),\end{aligned}$$

$$u_{31}=\overline{\frac{\partial^2\phi^{**}_{pc3}}{\partial z^2}}\left|\frac{\partial^2\phi^{**}_{pc1}}{\partial z^2}\right|^2=o(1),\ u_{32}=\overline{\frac{\partial^2\phi^{**}_{pc3}}{\partial z^2}}\frac{\partial^2\phi^{**}_{pc2}}{\partial z^2}\overline{\frac{\partial^2\phi^{**}_{pc1}}{\partial z^2}}=o(1),$$

$$u_{33}=-\overline{\frac{\partial^2\phi^{**}_{pc1}}{\partial z^2}}|_0\left(\left|\frac{\partial^2\phi^{**}_{pc1}}{\partial z^2}\right|_0^2+\left|\frac{\partial^2\phi^{**}_{pc2}}{\partial z^2}\right|_0^2\right)$$

$$=-8+32ib\overline{z_0}+8iu_0(2b^2-e_1)+32ibz_0+4iu_0+o(1),$$

$$\mu_3=\left|\frac{\partial^2\phi^{**}_{pc1}}{\partial z^2}\right|\sqrt{|\frac{\partial^2\phi^{**}_{pc1}}{\partial z^2}|^2+|\frac{\partial^2\phi^{**}_{pc2}}{\partial z^2}|^2+|\frac{\partial^2\phi^{**}_{pc3}}{\partial z^2}|^2}\sqrt{|\frac{\partial^2\phi^{**}_{pc1}}{\partial z^2}|^2+|\frac{\partial^2\phi^{**}_{pc2}}{\partial z^2}|^2}=8+o(1).$$

Then by the formulas (10)

$$\frac{\partial^2\phi^{**}_{pd1}}{\partial z\partial w}(0)=\frac{\partial^2\phi^{**}_{pc1}}{\partial z\partial w}(0)\frac{\overline{u_{11}}}{\mu_1}+\frac{\partial^2\phi^{**}_{pc2}}{\partial z\partial w}(0)\frac{\overline{u_{12}}}{\mu_1}+\frac{\partial^2\phi^{**}_{pc3}}{\partial z\partial w}(0)\frac{\overline{u_{13}}}{\mu_1}$$
$$=b-2ib^3u_0-biu_0e_1-4b^2iz_0-\frac{1}{2}biu_0$$
$$-iz_0-4ie_2z_0+4ic_1c_2u_0-2ibc_1^2u_0-2c_1^2iz_0+o(1),$$

$$\frac{\partial^2\phi^{**}_{pd2}}{\partial z\partial w}(0)=\frac{\partial^2\phi^{**}_{pc1}}{\partial z\partial w}(0)\frac{\overline{u_{21}}}{\mu_2}+\frac{\partial^2\phi^{**}_{pc2}}{\partial z\partial w}(0)\frac{\overline{u_{22}}}{\mu_2}+\frac{\partial^2\phi^{**}_{pc3}}{\partial z\partial w}(0)\frac{\overline{u_{23}}}{\mu_2}$$
$$=c_1+6bc_1i\overline{z_0}+4ibc_1z_0+4ic_2\overline{z_0}+4ibc_2u_0-3ic_1e_1u_0+o(1),$$

$$\frac{\partial^2\phi^{**}_{pd2}}{\partial w^2}(0)=\frac{\partial^2\phi^{**}_{pc1}}{\partial w^2}(0)\frac{\overline{u_{21}}}{\mu_2}+\frac{\partial^2\phi^{**}_{pc2}}{\partial w^2}(0)\frac{\overline{u_{22}}}{\mu_2}+\frac{\partial^2\phi^{**}_{pc3}}{\partial w^2}(0)\frac{\overline{u_{23}}}{\mu_2}$$
$$=2c_2+4ic_2b^2u_0-8ic_2e_1u_0+20ibc_2\overline{z_0}-4ic_1e_2z_0+4ic_1^2c_2u_0+c_2iu_0+o(1).$$

$$\frac{\partial^2\phi^{**}_{pd3}}{\partial z\partial w}(0)=\frac{\partial^2\phi^{**}_{pc1}}{\partial z\partial w}(0)\frac{\overline{u_{31}}}{\mu_2}+\frac{\partial^2\phi^{**}_{pc2}}{\partial z\partial w}(0)\frac{\overline{u_{32}}}{\mu_2}+\frac{\partial^2\phi^{**}_{pc3}}{\partial z\partial w}(0)\frac{\overline{u_{33}}}{\mu_2}$$
$$=-4ibc_3u_0-4ic_3\overline{z_0}+o(1),$$

$$\frac{\partial^2\phi^{**}_{pd3}}{\partial w^2}(0)=\frac{\partial^2\phi^{**}_{pc1}}{\partial w^2}(0)\frac{\overline{u_{31}}}{\mu_2}+\frac{\partial^2\phi^{**}_{pc2}}{\partial w^2}(0)\frac{\overline{u_{32}}}{\mu_2}+\frac{\partial^2\phi^{**}_{pc3}}{\partial w^2}(0)\frac{\overline{u_{33}}}{\mu_2}$$
$$=-2c_3-20ic_3b\overline{z_0}+(8ic_3e_1-4ic_3b^2-ic_3)u_0+o(1).$$

Step 4. Normalization such that $c_1(p)=\frac{\partial^2\phi^{**}_{pe2}}{\partial z\partial w}(0)\geq 0$ and $\frac{\partial^2\phi^{**}_{pe3}}{\partial z\partial w}(0)=0$ We define $F^{**}_{pe}=\tau_4^*\circ F^{**}_{pd}\circ\sigma_4$ so that $\frac{\partial^2\phi^{**}_{pe2}}{\partial z\partial w}(0)>0$ and $\frac{\partial^2\phi^{**}_{pe3}}{\partial z\partial w}(0)=0$, where σ_4 and τ_4^* are as in (7) with $U^{**}_{22}=\begin{pmatrix}1 & 0\\ 0 & \widetilde{U}\end{pmatrix}$, $\lambda=1$, $a=0$, $U=1$ and $r=0$, where

$$\widetilde{U}=\frac{\begin{pmatrix}\overline{A_2} & -A_3\\ \overline{A_3} & A_2\end{pmatrix}}{\sqrt{|A_2|^2+|A_3|^2}},$$

$$A_2=\begin{cases}\frac{\partial^2\phi^{**}_{pd2}}{\partial z\partial w}(0), & if\ \ \frac{\partial^2\phi^{**}_{pd2}}{\partial z\partial w}(0)\neq 0\\ 1, & if\ \ \frac{\partial^2\phi^{**}_{pd2}}{\partial z\partial w}(0)=0\end{cases},\quad A_3=\frac{\partial^2\phi^{**}_{pd3}}{\partial z\partial w}(0).$$

Notice that when $c_1>0$, $\frac{\partial^2\phi^{**}_{pd2}}{\partial z\partial w}(0)\neq 0$ holds as $|p|$ sufficiently small. While when $c_1=0$, from Lemma 2.3, we have $c_3=0$ so that $\phi_3\equiv 0$. Hence $\frac{\partial^2\phi^{**}_{pe3}}{\partial z\partial w}(0)=0$ is automatically true so that $A_3=0$. As a result, this step of normalization is not differentiable of p when $c_1(p)=0$.

We have

$$\frac{A_2}{\sqrt{|A_2|^2+|A_3|^2}}=|A_2|+o(1),\quad \frac{A_3}{\sqrt{|A_2|^2+|A_3|^2}}=-\frac{4ibc_3}{c_1}u_0-\frac{4ic_3}{c_1}\overline{z_0}+o(1),$$

By the formulas(10), we have

$$f^{**}_{pe}=f^{**}_{pd},\ \phi^{**}_{pe1}=\phi^{**}_{pd1},$$

$$\begin{aligned}\frac{\partial^2\phi^{**}_{pe2}}{\partial z\partial w}(0)&=\frac{\partial^2\phi^{**}_{pd2}}{\partial z\partial w}(0)\cdot\frac{\overline{A_2}}{\sqrt{|A_2|^2+|A_3|^2}}+\frac{\partial^2\phi^{**}_{pd3}}{\partial z\partial w}(0)\cdot\frac{\overline{A_3}}{\sqrt{|A_2|^2+|A_3|^2}}\\ &=\sqrt{\left|\frac{\partial^2\phi^{**}_{pd2}}{\partial z\partial w}(0)\right|^2+\left|\frac{\partial^2\phi^{**}_{pd3}}{\partial z\partial w}(0)\right|^2}=\left|\frac{\partial^2\phi^{**}_{pd2}}{\partial z\partial w}(0)\right|+o(1),\end{aligned}$$

and $\frac{\partial^2\phi^{**}_{pe3}}{\partial z\partial w}(0)=0$. Notice that although $\frac{\partial^2\phi^{**}_{pe2}}{\partial z\partial w}(0)$ may not be differentiable of p, $\left|\frac{\partial^2\phi^{**}_{pe2}}{\partial z\partial w}(0)\right|^2$ is real analytic.

Step 5. Normalization such that $b(p) \geq 0$, $c_1(p) \geq 0$, $e_1(p) \in \mathbb{R}$, and $c_3(p) \geq 0$
Define $F_p^{***} = \tau_5^* \circ F_{pe}^{**} \circ \sigma_5$ so that $e_1(p) \in \mathbb{R}$, $b(p) \geq 0$ and $c_3(p) \geq 0$, where σ_5 and τ_5^* are as in (7) with

$$r = -\Re(ie_1),\ U = e^{i\theta},\ U_{22}^{**} = \begin{pmatrix} e^{-2i\theta} & 0 & 0 \\ 0 & e^{i\beta_2} & 0 \\ 0 & 0 & e^{i\beta_3} \end{pmatrix}$$

where

$$e^{i\theta} = \begin{cases} \frac{\partial^2 \phi_{pe1}^{**}}{\partial z \partial w}(0) \Big/ \left| \frac{\partial^2 \phi_{pe1}^{**}}{\partial z \partial w}(0) \right| & if \quad \frac{\partial^2 \phi_{pe1}^{**}}{\partial z \partial w}(0) \neq 0, \\ 1 & if \quad \frac{\partial^2 \phi_{pe1}^{**}}{\partial z \partial w}(0) = 0, \end{cases} \tag{35}$$

$$e^{i\beta_2} = e^{-i\theta},$$

$$e^{i\beta_3} = \begin{cases} \overline{\frac{\partial^2 \phi_{pe3}^{**}}{\partial w^2}(0)} \Big/ \left| \frac{\partial^2 \phi_{pe3}^{**}}{\partial w^2}(0) \right| & if \quad \frac{\partial^2 \phi_{pe3}^{**}}{\partial w^2}(0) \neq 0, \\ 1 & if \quad \frac{\partial^2 \phi_{pe3}^{**}}{\partial w^2}(0) = 0. \end{cases}$$

Notice that $U = e^{i\theta}$ and U_{22}^* are not differentiable of p when $b(p) = 0$.

Then it turns out that

$$c_1^2(p) = \left| \frac{\partial^2 \phi_{p2}^{***}}{\partial z \partial w}(0) \right|^2 = \left| \frac{\partial^2 \phi_{pe2}^{**}}{\partial z \partial w}(0) \right|^2 = \left| \frac{\partial^2 \phi_{pd2}^{**}}{\partial z \partial w}(0) \right|^2 = c_1^2 + 4c_1(bc_1 + 2c_2)\Im(z_0) + o(1).$$

$$b^2(p) = \left| \frac{\partial^2 \phi_{p1}^{***}}{\partial z \partial w}(0) \right|^2 = \left| \frac{\partial^2 \phi_{pe1}^{**}}{\partial z \partial w}(0) \right|^2 = \left| \frac{\partial^2 \phi_{pd1}^{**}}{\partial z \partial w}(0) \right|^2 = b^2 - 4b(2e_1 + c_1^2)\Im(z_0) + o(1).$$

Here we used the formulas $-e_1 - e_2 = \frac{1}{4} + b^2 + c_1^2$ and $c_1 c_2 = -be_1$.

Since $e_2(p) + e_1(p) = -\frac{1}{4} - b^2(p) - c_1^2(p)$, we get $e_2(p) + e_1(p) = e_2 + e_1 + 8b(e_1 + e_2)\Im(z_0) + o(1)$. All of the formulas in Lemma 3.1(1) have been proved. Even $c_1(p), b(p)$ and U are not differentiable at $p_0 \in \partial\mathbb{H}^2$ when $b_1(p_0) = 0$, from the above, the function $c_1^2(p)$ and $b^2(p)$ are real analytic of p. □

References

[A77] H. Alexander, Proper holomorphic maps in $\mathbf{C}^n$, Indiana Univ. Math. Journal 26, 137-146 (1977).

[BER99] M. S. Baouendi, P. Ebenfelt and L. Rothschild, Real Submanifolds in Complex Spaces and Their Mappings, Princeton Univ. Mathematics Series 47, Princeton University, New Jersey, 1999.

[BR90] M. S. Baouendi, and L. P. Rothschild, Geometric properties of mappings between hypersurfaces in complex spaces, J. Differential Geom. 31, 473-499, 1990
[CS90] J.Cima and T. J. Suffridge, Boundary behavior of rational proper maps, Duke Math. J. 60, 135-138 (1990).
[DA93] J. P. D'Angelo, Several Complex Variables and the Geometry of Real Hypersurfaces, CRC Press, Boca Raton, 1993.
[DA88] J. P. D'Angelo, Proper holomorphic mappings between balls of different dimensions, *Mich. Math. J.* 35, 83-90 (1988).
[DC96] J. D'Angelo and D. Catlin, A stabilization theorem for Hermitian forms and applications to holomorphic mappings, Math Research Letters 3, 149-166 (1996).
[EHZ03] P. Ebenfelt, X. Huang and D. Zaitsev, The equivalence problem and rigidity for hypersurfaces embedded into hyperquadrics , preprint, 2003. (to appear in American Jour. of Math.)
[Fa82] J. Faran, Maps from the two ball to the three ball, Invent. Math. 68, 441-475 (1982).
[Fa86] J. Faran, On the linearity of proper maps between balls in the lower dimensional case, Jour. Diff. Geom. 24, 15-17 (1986).
[Fo89] F. Forstneric, Extending proper holomorphic mappings of positive codimension, Invent. Math., 95, 31-62 (1989).
[Ha04] H. Hamada, Rational proper holomorphic maps from $\mathbb{B}^n$ into $\mathbb{B}^{2n}$, Math. Ann.331 (2005), no. 3, 693–711.
[Hu99] X. Huang, On a linearity problem of proper holomorphic mappings between balls in complex spaces of different dimensions, Jour. of Diff. Geom. Vol (51) No. 1, 13-33 (1999).
[Hu01] X. Huang, On some problems in several complex variables and CR geometry. First International Congress of Chinese Mathematicians (Beijing, 1998), 383–396, AMS/IP Stud. Adv. Math., 20, Amer. Math. Soc., Providence, RI, 2001.
[Hu03] X. Huang, On a semi-rigidity property for holomorphic maps, Asian J. Math. Vol(7) No. 4(2003), 463-492.
[HJ01] X. Huang and S. Ji, Mapping $\mathbb{B}^n$ into $\mathbb{B}^{2n-1}$, Invent. Math. 145, 219-250(2001).
[HJ06] X. Huang and S. Ji, On some rigidity problems in Cauchy-Riemann geometry, to appear in: AMS/IP advanced study series.
[HJX05] X. Huang, S. Ji, and D. Xu, Several results for holomorphic mappings from $\mathbb{B}^n$ into $\mathbb{B}^N$, *Contemporay Math* vol. 368 (A special issue in honor of Professor F. Treves), (2005), 267-292.
[HJX06] X. Huang, S. Ji, and D. Xu, Proper holomorphic mappings from $\mathbb{B}^n$ into $\mathbb{B}^N$ with geometric rank one, to appear in: Math. Res. Lett., 2006.
[JX04] S. Ji and D. Xu, Rational maps between $\mathbb{B}^n$ and $\mathbb{B}^N$ with geometric rank $\kappa_0 \leq n-2$ and minimal target dimension, Asian J. Math. Vol(8) No. 2(2004), 233-258.

[W79] S. Webster, On mapping an (n+1)-ball in the complex space, Pac. J. Math. 81, 267-272 (1979).
[Wo93] Marcus S.-B. Wono, Dissertation, Proper holomorphic mappings in Several Complex Variables, University of Illinois at Urbana-Champaign, 1993.
[X06] D. Xu, Dissertation, University of Houston, 2006.

Zhihua Chen, zzzhcc@tongji.edu.cn, Department of Mathematics, Tongji University, Shanghai 200092, China

S. Ji, shanyuji@math.uh.edu, D. Xu, dekangxu@math.uh.edu, Department of Mathematics, University of Houston, Houston, TX 77204, USA