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Mapping $\mathbb{B}^n$ into $\mathbb{B}^{3n-3}$

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1. Introduction

Denote by $Rat(\mathbb{B}^n, \mathbb{B}^N)$ the collection of all proper holomorphic rational maps from the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ to the unit ball $\mathbb{B}^N \subset \mathbb{C}^N$, and denote by $Rat(\mathbb{H}_n, \mathbb{H}_N)$ the collection of all proper holomorphic rational maps from $\mathbb{H}_n$ to $\mathbb{H}_N$, where $\mathbb{H}_n = \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} \mid Im(w) > |z|^2\}$ is the Siegel upper half space. By the Cayley transform, we can identify $\mathbb{B}^n$ with $\mathbb{H}_n$, and identify $Rat(\mathbb{B}^n, \mathbb{B}^N)$ with $Rat(\mathbb{H}_n, \mathbb{H}_N)$. We say that $f, g \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ are *spherically equivalent* if there are $\sigma \in Aut(\mathbb{B}^n)$ and $\tau \in Aut(\mathbb{B}^N)$ such that $f = \tau \circ g \circ \sigma$. We use the convention as in [Le11] to extend the notion of spherical equivalence naturally to maps with different target dimensions. For instance, two maps $f \in Rat(\mathbb{B}^n, \mathbb{B}^{N_1})$ and $g \in Rat(\mathbb{B}^n, \mathbb{B}^{N_2})$ with $N_1 < N_2$ are said to be spherically equivalent if $(f, 0, \ldots, 0)$, with $(N_2 - N_1)$ 0-components added to f, is spherically equivalent to g. The study of rational and proper holomorphic maps has attracted much attention in the past several decades. Here, we refer the reader to [Fo92][DA93][Hu99][Hu03][DL09][FHJZ10] [Eb13] [LM07] [MMZ03] [Mir03] [YZ12] for discussions and many references therein for more related investigations on these matters.

The first gap theorem proved in [W79][Fa86][Hu99] is stated as follows: For $N \in (n, 2n-1)$ with $n \geq 2$, any map $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ is spherically equivalent to a map of the form $(G, 0)$ with $G \in Rat(\mathbb{B}^n, \mathbb{B}^n)$. When $N = n$, by a classical result of Alexander [A77], any map $F \in Rat(\mathbb{B}^n, \mathbb{B}^n)$ is an automorphism of $\mathbb{B}^n$. When $N = 2n-1$, it was proved in [HJ01] that any map $F \in Rat(\mathbb{B}^n, \mathbb{B}^{2n-1})$ is spherically equivalent to either the linear map or the Whitney map.

The second gap theorem was proved in [HJX06]: When $N \in (2n, 3n-3)$ and $n \geq 4$, any map $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ is spherically equivalent to a map of the form $(G, 0)$ with $G \in Rat(\mathbb{B}^n, \mathbb{B}^{2n})$. When $N = 2n$, it was proved by Hamada [Ha05] that any map $F \in Rat(\mathbb{B}^n, \mathbb{B}^{2n})$ must be spherically equivalent to a map from the D'Angelo family. In this paper, we consider the

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other limit case $N = 3n - 3$. We will give a complete classification up to the above mentioned spherical equivalence relation. This work is a continuation of the previous works by [Ha05] and [HJY14]. Our main theorem is stated as follows:

Theorem 1.1. *Let* $F \in Rat(\mathbb{B}^n, \mathbb{B}^{3n-3})$ *with* $n \geq 4$. *Then* F *is spherically equivalent to one of the following maps* G*:*

1. $G(z) = (z, 0, \ldots, 0)$.
2. $G(z) = (z_2, z_3, \ldots, z_n, z_1^2, z_1z_2, \ldots, z_1z_n, 0, \ldots, 0)$ *(the Whitney map).*
3. $G(z) = (\sqrt{t}z_1, z_2, \ldots, z_n, \sqrt{1-t}z_1^2, \sqrt{1-t}z_1z_2, \ldots, \sqrt{1-t}z_1z_n, 0, \ldots, 0)$, *where* $0 < t < 1$ *(the D'Angelo maps).*
4. $G(z) = (z_3, z_4, \ldots, z_n, z_1^2, z_2^2, \sqrt{2}z_1z_2, z_1z_3, \ldots, z_1z_n, z_2z_3, z_2z_4, \ldots, z_2z_n)$ *(the generalized Whitney map* $W_{n,2}$*).*

We remark that the maps in (2)-(3) are of geometric rank one (an invariant concept to be defined later); and the map in (4) is of geometric rank two. A special result of the above was proved in [JX04] that any map $F \in Rat(\mathbb{B}^4, \mathbb{B}^9)$ with $deg(F) = 2$ and geometric rank two must be spherically equivalent to the map $W_{4,2}$.

We outline the idea of the proof of Theorem 1.1 as follows. The paper is based on the previous two papers [HJY14] and [Le11]. First, for each map $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$, there is an associated invariant called its geometric rank κ_0. (For a precise definition, see the next section of this paper). By the inequality $N \geq n + \frac{(2n-\kappa_0-1)\kappa_0}{2}$ established in [Hu03], any map in $Rat(\mathbb{B}^n, \mathbb{B}^{3n-3})$ with $n \geq 4$ must have geometric rank $\kappa_0 \leq 2$. Since the rank one case was classified in [HJX06], it suffices to consider the rank two case. It was also proved in [JX04, Theorem 6.1], that for any $F \in Rat(\mathbb{B}^n, \mathbb{B}^{3n-3})$ with $\kappa_0 = 2$, we have $deg(F) \leq 4$. The main point of this paper is to show that we actually have $deg(F) \leq 2$. For this purpose, we make use of the techniques and formulas developed in a recent paper [HJY14]. We also need to develop several quite different approaches to deal with the degree estimate in our relatively large codimensional setting. Once $deg(F) \leq 2$ is proved, we apply Lebl's theorem [Le11] to complete our proof.

One of the main ingredients in our proof is to obtain the optimal degree estimate for maps in $Rat(\mathbb{B}^n, \mathbb{B}^{3n-3})$. Along these lines, there is a famous conjecture called the D'Angelo conjecture, which states as follows:

Conjecture 1.2. *For any* $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$, *the degree of* F *is bounded by* $2N - 3$ *when* $n = 2$*; and is bounded by* $\frac{N-1}{n-1}$ *when* $n \geq 3$.

When $N = 3n - 3$, $\frac{N-1}{n-1} < 3$. Our result hence provides a solution to the D'Angelo conjecture in this special setting.

The D'Angelo conjecture is a fundamental problem in the study of mappings between balls. It has a major impact for the classification of proper rational maps between balls, if answered affirmatively, as demonstrated in this paper. Forstnerič in [Fo89] first obtained a rough bound of the degree, which depends on the cubic power of the codimension. Meylan [Mey06] improved the Forstnerič bound to a quadratic bound. However, obtaining a linear bound on the codimension is substantially harder and more important. When $n = 2$, the D'Angelo conjecture for the monomial case was solved affirmatively in [DKR03] by introducing a new method. For the general n, the D'Angelo conjecture for the monomial case was completely solved by D'Angelo-Lebl-Peters [DLP07] and Lebl-Peters [LP12].

For more discussions on the gap conjecture and connections with other studies, we refer the reader to the papers of D'Angelo-Lebl [DL09], Huang-Ji-Yin [HJY09][HJY14], Ebenfelt [Eb13], and the references therein.

Remark: We use this opportunity to mention that in our paper [HJY14], the conjugate signs in some of equations were missed in its journal printed form due to the tex non-compatibility of ours with that of the publisher. (The readers may find our original tex-file submitted to Math. Ann. in arXiv:1201.6440 (v2) where there was no such an issue).

2. Preliminaries

2a. The associated maps F_p^{**}: Let $F = (f, \phi, g) = (\widetilde{f}, g) = (f_1, \ldots, f_{n-1}, \phi_1, \ldots, \phi_{N-n}, g)$ be a non-constant rational CR map from an open subset M of $\partial\mathbb{H}_n$ into $\partial\mathbb{H}_N$ with $F(0) = 0$. For each $p \in M$ close to 0, we write $\sigma_p^0 \in \mathrm{Aut}(\mathbb{H}_n)$ for the map sending (z, w) to $(z + z_0, w + w_0 + 2i\langle z, \overline{z_0}\rangle)$ and $\tau_p^F \in \mathrm{Aut}(\mathbb{H}_N)$ by defining

$$\tau_p^F(z^*, w^*) = (z^* - \widetilde{f}(z_0, w_0), w^* - \overline{g(z_0, w_0)} - 2i\langle z^*, \overline{\widetilde{f}(z_0, w_0)}\rangle).$$

Then F is equivalent to $F_p = \tau_p^F \circ F \circ \sigma_p^0 = (f_p, \phi_p, g_p)$. Notice that $F_0 = F$ and $F_p(0) = 0$. The following is fundamentally important for the understanding of the geometric properties of F. Let us denote $Prop(\mathbb{H}_n, \mathbb{H}_N)$:= {holomorphic proper maps from $\mathbb{H}_n$ into $\mathbb{H}_N$} and $Prop_k(\mathbb{H}_n, \mathbb{H}_N) := Prop(\mathbb{H}_n, \mathbb{H}_N) \cap C^k(\overline{\mathbb{H}_n})$.

Parameterize $\mathbb{H}_n$ by $(z, \overline{z}, u)$ through the map $(z, \overline{z}, u) \to (z, u + i|z|^2)$. In what follows, we will assign the weight of z and u to be 1 and 2, respectively. For a nonnegative integer m, a function $h(z, \overline{z}, u)$ defined over a small ball U of 0 in $\mathbb{H}_n$ is said to be of quantity $o_{wt}(m)$ if $h(tz, t\overline{z}, t^2u)/|t|^m \to 0$ uniformly for (z, u) on any compact subset of U as $t(\in \mathbb{R}) \to 0$. We use the notation $h^{(k)}$ to denote a polynomial h which has weighted degree k.

Lemma 2.1. *([Hu99]) Let $F \in Prop_2(\mathbb{H}_n, \mathbb{H}_N)$ with $2 \leq n \leq N$. For each $p \in \partial\mathbb{H}_n$, there is an automorphism $\tau_p^{**} \in Aut_0(\mathbb{H}_N)$ such that $F_p^{**} := \tau_p^{**} \circ F_p$ satisfies the following normalization:*

$$\begin{aligned} f_p^{**} &= z + \frac{i}{2}a_p^{**(1)}(z)w + o_{wt}(3), \\ \phi_p^{**} &= \phi_p^{**(2)}(z) + o_{wt}(2), \\ g_p^{**} &= w + o_{wt}(4), \\ \textit{with} \quad & \langle \overline{z}, a_p^{**(1)}(z)\rangle |z|^2 = |\phi_p^{**(2)}(z)|^2. \end{aligned}$$

2b. Geometric rank: Write $\mathcal{A}(p) := -2i(\frac{\partial^2 (f_p)_l^{**}}{\partial z_j \partial w}|_0)_{1\leq j,l\leq (n-1)}$. The rank of the $(n-1)\times(n-1)$ matrix $\mathcal{A}(p)$ is called the *geometric rank* of F at p. In what follows, we write $Rk_F(p)$ for the geometric rank of F at p, which depends only on p and F. $Rk_F(p)$ is a lower semi-continuous function on p, and is independent of the choice of $\tau_p^{**}(p)$. (See [Hu03] for more discussions on this matter). Define the *geometric rank* of F to be $\kappa_0(F) = max_{p\in\partial\mathbb{H}_n} Rk_F(p)$. Notice that it always holds that $0 \leq \kappa_0 \leq n-1$. Define the geometric rank of $F \in \mathrm{Prop}_2(\mathbb{B}^n, \mathbb{B}^N)$ to be the one for the map $\rho_N^{-1} \circ F \circ \rho_n \in \mathrm{Prop}_2(\mathbb{H}_n, \mathbb{H}_N)$. By [Hu03], $\kappa_0(F)$ depends only on the equivalence class of F and when $N < \frac{n(n+1)}{2}$, the geometric rank $\kappa_0(F)$ of F is precisely the κ_0 we mentioned in the introduction.

Under the condition that $1 \leq \kappa_0 \leq n-2$, the following theorem was proved in [Hu03].

Theorem 2.2. *([Hu03] and [HJX06]) Suppose that $F \in Prop_3(\mathbb{H}_n, \mathbb{H}_N)$ has geometric rank $1 \leq \kappa_0 \leq n-2$ with $F(0) = 0$. Then there are $\sigma \in Aut(\mathbb{H}_n)$ and $\tau \in Aut(\mathbb{H}_N)$ such that $\tau \circ F \circ \sigma$ takes the following form, which is still denoted by $F = (f, \phi, g)$ for convenience of notation:*

$$
\begin{cases}
f_l = \sum_{j=1}^{\kappa_0} z_j f^*_{lj}(z,w);\ l \le \kappa_0 \\
f_j = z_j,\ \mathit{for}\ \kappa_0+1 \le j \le n-1; \\
\phi_{lk} = \mu_{lk} z_l z_k + \sum_{j=1}^{\kappa_0} z_j \phi^*_{lkj}\ \mathit{for} \quad (l,k) \in \mathcal{S}_0, \\
\phi_{lk} = O_{wt}(3), \quad (l,k) \in \mathcal{S}_1, \\
g = w; \\
f^*_{lj}(z,w) = \delta^j_l + \frac{i\delta^j_l \mu_l}{2} w + b^{(1)}_{lj}(z) w + O_{wt}(4), \\
\phi^*_{lkj}(z,w) = O_{wt}(2), \quad (l,k) \in \mathcal{S}_0, \\
\phi_{lk} = \sum_{j=1}^{\kappa_0} z_j \phi^*_{lkj} = O_{wt}(3) \quad \mathit{for}\ (l,k) \in \mathcal{S}_1
\end{cases}
\tag{2.1}
$$

Here, for $1 \le \kappa_0 \le n-2$*, we write* $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$*, the index set for all components of* ϕ*, where* $\mathcal{S}_0 = \{(j,l) : 1 \le j \le \kappa_0, 1 \le l \le n-1, j \le l\}$, $\mathcal{S}_1 = \big\{(j,l) : j = \kappa_0+1, \kappa_0+1 \le l \le \kappa_0 + N - n - \frac{(2n-\kappa_0-1)\kappa_0}{2}\big\}$*, and*

$$
\mu_{jl} = \begin{cases} \sqrt{\mu_j + \mu_l} & \mathit{for}\ j < l \le \kappa_0; \\ \sqrt{\mu_j} & \mathit{if}\ j \le \kappa_0 < l\ \mathit{or}\ \mathit{if}\ j = l \le \kappa_0. \end{cases}
\tag{2.2}
$$

2c. A family of affine subspaces L_ϵ: Let us review some background materials on the semi-linearity property on $Rat(\mathbb{B}^n, \mathbb{B}^N)$ (cf. [H03] and [HJX06]). Let $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ with $1 \le \kappa_0 \le n-2$. Let E_0 be the proper complex variety consisting of points of indeterminacy and the non-immerse points of F. We define

$$
\begin{aligned}
\mathcal{V}_F := \{(Z, S_Z) \in (\mathbb{C}^n - E_0) \times G_{n,k^0}(\mathbb{C}), \\
F \text{ is linear fractional when restricted to } S_Z + Z\}.
\end{aligned}
$$

Here $Gr_{n,k^0}(\mathbb{C})$ is the Grassmannian manifold consisting of all $k^0 := n - \kappa_0$-dimensional complex subspaces in $\mathbb{C}^n$. Then $\mathcal{V}_F$ is a complex analytic variety such that the projection

$$
\pi : \mathcal{V}_F \to \mathbb{C}^n - E_0, \quad (Z, S_Z) \mapsto Z
\tag{2.3}
$$

is proper holomorphic. There is another proper complex variety $E_1 \subset \mathbb{C}^n - E_0$ such that for any $Z \in \mathbb{C}^n - E_0 \cup E_1$, π has a unique preimage in $\mathcal{V}_F$, i.e., for any $Z \in \mathbb{C}^n - E_0 \cup E_1$, there is a unique complex subspace S_Z of dimension k^0 such that F is linear fractional when restricted to $S_Z + Z$. Write $\mathcal{V}_F = \cup_j \mathcal{V}^{(j)}_F$ for the irreducible decomposition of $\mathcal{V}_F$. Then there is only one irreducible component, say $\mathcal{V}^{(1)}_F$, whose projection to $\mathbb{C}^n - E_0$ contains a sufficiently small domain inside $\mathbb{H}_n$ and has a small piece of $\partial\mathbb{H}_n$ as part of its boundary. If necessary, we can assume that $0 \notin E_1 \cap E$ and assume that π is biholomorphic near $(0, S_0) \in \mathcal{V}_F$ with $\pi^{-1}(0) = (0, S_0)$.

For the case of $\kappa_0 = 1$, we can assume that for any $\epsilon(\in \mathbb{C}) \approx 0$, there is a unique affine subspace L_ϵ of codimension one, on which the restriction of F is linear fractional, defined by equation:

$$z_1 = \sum_{j=2}^{n} a_j(\epsilon) z_j + \epsilon \tag{2.4}$$

where $a_j(\epsilon)$ are holomorphic functions of ϵ with $a_j(0) = 0$. We also denote $w := z_n$. It was shown [HJX06] that if we write $a_j(\epsilon) = \epsilon \hat{a}_j(\epsilon)$, then *all* $\hat{a}_j(\epsilon) = constant$. Then (2.4) can be written as

$$z_1 = \epsilon \left(\sum_{j=2}^{n} \hat{a}_j z_j + 1 \right). \tag{2.5}$$

Two different cases were considered in ([HJX06], p.521): (i) $a_j(\epsilon) = \epsilon \hat{a}_j$ with $\hat{a}_j = constant$ and $Im(\hat{a}_n) = -\sum_{j=2}^{n-1} |\frac{\hat{a}_j}{2}|^2$; (ii) $a_j(\epsilon) = \epsilon \hat{a}_j$ with $\hat{a}_j =$ *constant* but the above identity does not hold. It has been proved in ([HJX06], p. 521) that the first case cannot occur, and that in the second case, after some transformation, the hyperplanes L_ϵ are of the form $z_1 = constant(-iw + 1)$.

For the case $\kappa_0 = 2$, by a theorem of the second author in [Hu03], we can assume that for any $\epsilon = (\epsilon_1, \epsilon_2)(\in \mathbb{C}^2) \approx 0$, there is a unique affine subspace $L_\epsilon = L_{(\epsilon_1, \epsilon_2)}$ of codimension two defined by equations of the form:

$$\begin{cases} z_1 = \sum\limits_{i=3}^{n-1} a_i(\epsilon) z_i + a_n(\epsilon) w + \epsilon_1, \\ z_2 = \sum\limits_{i=3}^{n-1} b_i(\epsilon) z_i + b_n(\epsilon) w + \epsilon_2, \end{cases} \tag{2.6}$$

such that F is a linear map on L_ϵ, where $a_j(\epsilon), b_j(\epsilon)$ are holomorphic functions in ϵ near 0 with $a_j(0,0) = b_j(0,0) = 0$ for all j and for $j = n$.

2d. Basic notation: Let $F = (f, \phi, g) \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ be as in Theorem 2.2 with geometric rank $\kappa_0 = 2$. We have $N = \sharp(f) + \sharp(\phi) + \sharp(g)$ and $\sharp(\phi) = \sharp(\mathcal{S}_0) + \sharp(\mathcal{S}_1)$ where we denote by $\sharp(A)$ the number of elements in the set A, and $\sharp(f) = n - 1$, $\sharp(g) = 1$, $\sharp(\mathcal{S}_0) = \frac{(2n-1-\kappa_0)\kappa_0}{2} = n\kappa_0 - \frac{(\kappa_0+1)\kappa_0}{2} = 2n - 3$, $\sharp(\mathcal{S}_1) = N - n - \sharp(\mathcal{S}_0) = N - 3n + 3$.

We denote by $P^{(j,k)}(z,w)$ a polynomial in (z,w) with degree j in z and degree k in w, and denote by $P^{(j,k)}(z)$ the coefficient of w^k. For example, $P^{(1,1)}(z,w) = az_1w + bz_2w = P^{(1,1)}(z)w$ with $P^{(1,1)}(z) = az_1 + bz_2$. We also write

$$f^{(j_1I_1+j_2I_2+\cdots+j_nI_n)} = \frac{\partial^{j_1+j_2+\cdots+j_n} f}{\partial z_1^{j_1}\partial z_2^{j_2}\cdots\partial z_{n-1}^{j_{n-1}}\partial w^{j_n}}(0). \tag{2.7}$$

For any rational holomorphic map $H = \frac{(P_1,\ldots,P_m)}{Q}$ on $\mathbb{C}^n$, where $\{P_j, Q\}$ are relatively prime holomorphic polynomials, the *degree* of H is defined to be $deg(H) := \max\{\deg(P_j), deg(Q), 1 \le j \le m\}$.

- **The part ϕ:** Write $\phi = (\Phi_0, \Phi_1)$, $\Phi_0 = (\phi_{lk})_{(l,k)\in\mathcal{S}_0}$ and $\Phi_1 = (\phi_{lk})_{(l,k)\in\mathcal{S}_1}$. Here $\sharp(\Phi_0) = \sharp(\mathcal{S}_0)$ and $\sharp(\Phi_1) = \sharp(\mathcal{S}_1)$. Since $\kappa_0 = 2$, we can write

$$\Phi_0 = (\phi_{11}, \phi_{12}, \phi_{1j}, \phi_{22}, \phi_{2j})_{3\le j\le n-1} \quad \text{and} \quad \Phi_1 = (\phi_{33}, \phi_{3k})_{4\le k\le N-3n+3}.$$

- **The part $f_j^{(1,1)}(z)$:** Write $f^{(1,1)}(z) = (f_1^{(1,1)}(z), \ldots, f_{n-1}^{(1,1)}(z))$. By Theorem 2.2, we have $f_j^{(1,1)}(z) = \frac{i\mu_j}{2} z_j$, $\mu_j > 0$ for $1 \le j \le 2$, and $\mu_j = 0$ for $3 \le j \le n-1$.
- **The part $\Phi_0^{(2,0)}(z)$:** One important portion of Φ_0 is the z-quadric part (see Theorem 3.2): $\Phi_0^{(2,0)}(z) = \{\phi_{jl}^{(2)}(z) = \mu_{jl}z_jz_l\}_{(j,l)\in\mathcal{S}_0}$.
- **The part $\Phi_0^{(1,1)}(z)$:** Another portion of Φ_0 is $\Phi_0^{(1,1)}(z)w$ which is not described precisely in Theorem 2.2:

$$\Phi_0^{(1,1)}(z) = \sum_{j=1}^{\kappa_0} e_j z_j, \quad e_j \in \mathbb{C}^{\sharp(\mathcal{S}_0)}.$$

We mention that $\phi_{lk} = \mu_{lk}z_lz_k + \sum_{j=1}^{\kappa_0} z_j\phi_{lkj}^*$ where $\phi_{lkj}^* = O_{wt}(2)$ for $(l,k) \in \mathcal{S}_0$. These coefficients e_j are important parameters for F. Since $\kappa_0 = 2$, we have

$$\begin{cases} e_1 = (e_{1,11},\ e_{1,12},\ e_{1,1j},\ e_{1,22}, e_{1,2j})_{3\le j\le n-1}, \\ e_2 = (e_{2,11},\ e_{2,12},\ e_{2,1j},\ e_{2,22}, e_{2,2j})_{3\le j\le n-1}, \end{cases} \qquad \begin{cases} \phi_{11}^{(1,1)} = z_1e_{1,11} + z_2e_{2,11}, \\ \phi_{12}^{(1,1)} = z_1e_{1,12} + z_2e_{2,12}, \\ \phi_{1j}^{(1,1)} = z_1e_{1,1j} + z_2e_{2,1j}, \\ \phi_{22}^{(1,1)} = z_1e_{1,22} + z_2e_{2,22}, \\ \phi_{2j}^{(1,1)} = z_1e_{1,2j} + z_2e_{2,2j}. \end{cases}$$

We also introduce the Hermitian inner product $\xi = (\xi_1, \dots, \xi_{\kappa_0}) = \overline{(e_1, \dots, e_{\kappa_0})} \cdot \Phi_0^{(2,0)}$. When $\kappa_0 = 2$, we have

$$\begin{cases} \xi_1 = \Phi^{(2,0)} \cdot \overline{e_1} & = \phi_{11}^{(2,0)}\overline{e_{1,11}} + \phi_{12}^{(2,0)}\overline{e_{1,12}} + \phi_{22}^{(2,0)}\overline{e_{1,22}} \\ & \quad + \sum_{j=3}^{n-1} \big(\phi_{1j}^{(2,0)}\overline{e_{1,1j}} + \phi_{2j}^{(2,0)}\overline{e_{1,2j}}\big), \\ \xi_2 = \Phi^{(2,0)} \cdot \overline{e_2} & = \phi_{11}^{(2,0)}\overline{e_{2,11}} + \phi_{12}^{(2,0)}\overline{e_{2,12}} + \phi_{22}^{(2,0)}\overline{e_{2,22}} \\ & \quad + \sum_{j=3}^{n-1} \big(\phi_{1j}^{(2,0)}\overline{e_{2,1j}} + \phi_{2j}^{(2,0)}\overline{e_{2,2j}}\big). \end{cases} \tag{2.8}$$

2e. The components of $\Phi_1^{(3,0)}$: Recall the following result of [Corollary 3.4, [HJY14]]:

Lemma 2.3. *Let $\kappa_0 \geq 2$ and $(\kappa_0 + 1)n - \kappa_0 \leq N \leq (\kappa_0 + 2)n - \kappa_0(\kappa_0 + 1) + \kappa_0 - 2$. Then*

$$\Phi_1^{(3,0)}(z) = \Big(\frac{2}{\sqrt{\mu_j + \mu_l}}\big(\sqrt{\frac{\mu_j}{\mu_l}} z_j \xi_l - \sqrt{\frac{\mu_l}{\mu_j}} z_l \xi_j\big), \ \ 0'\Big)_{1 \leq j < l \leq \kappa_0}$$

$|\phi^{(3,0)}(z)|^2 = 4\big(\sum_{j \leq \kappa_0} \frac{1}{\mu_j} |\xi_j(z)|^2\big)|z|^2$. In particular, when $\kappa_0 = 2$, if the inequality $3n - 3 \leq N \leq 4n - 6$ holds, we have

$$\Phi_1^{(3,0)}(z) = \Big(\frac{2}{\sqrt{\mu_1 + \mu_2}}\big(\sqrt{\frac{\mu_1}{\mu_2}} z_1 \xi_2 - \sqrt{\frac{\mu_2}{\mu_1}} z_2 \xi_1\big), \ 0'\Big) \tag{2.9}$$

and $|\phi^{(3,0)}(z)|^2 = 4\big(\frac{1}{\mu_1}|\xi_1(z)|^2 + \frac{1}{\mu_2}|\xi_2(z)|^2\big)|z|^2$.

2f. Lebl's theorem: We recall a result of Lebl, which will be used in our paper.

Theorem 2.4. *[Le11] Let $F : \partial\mathbb{B}^n \to \partial\mathbb{B}^N, n \geq 2$, be a rational CR map of degree 2. Then F is is spherically equivalent to a map taking $(z_1, \dots, z_n)$ to a map of the following form:*

$$\begin{aligned} (\sqrt{t_1} z_1, \sqrt{t_2} z_2, \dots, \sqrt{t_n} z_n, \sqrt{1 - t_1} z_1^2, \sqrt{1 - t_2} z_2^2, \\ \dots, \sqrt{1 - t_n} z_n^2, \sqrt{2 - t_i - t_j} z_i z_j)_{i \neq j} \end{aligned} \tag{2.10}$$

where $0 \leq t_1 \leq \dots \leq t_n \leq 1$, $(t_1, t_2, \dots, t_n) \neq (1, 1, \dots, 1)$. Furthermore, maps in (2.10) are mutually spherically inequivalent for different parameters $(t_1, \dots, t_n)$.

When $N = 3n-3$ and $n \geq 4$, the map in Lebl's theorem is one of the maps stated in Theorem 1.1.

Indeed, suppose that the first h $t_i's$ are zero, the next $(k-h)$ $t_i's$ are in $(0,1)$ and the rest $t_i's$ are 1. Then the dimension of the image space of the map (2.10) is

$$(n-h)+k+\frac{n(n-1)}{2}-\frac{(n-k)(n-k-1)}{2}.$$

We next find all nonnegative integers h and k with $h \leq k \leq n$ such that

$$(n-h)+k+\frac{n(n-1)}{2}-\frac{(n-k)(n-k-1)}{2} \leq 3n-3.$$

Namely, h and k satisfy the following:

$$(k-2)n+3-\frac{k(k+1)}{2}+k-h \leq 0. \tag{2.11}$$

We claim that the following are all the possible solutions:

(1) $k=h=0$. In this case, (2.10) is the identity map with 0 components added to it.

(2) $k=1$ and $h=1$. In this case, then (2.10) is the Whitney map with 0 components added to it.

(3) $k=1$ and $h=0$. In this case, (2.10) is the D'Angelo map with 0-components added to it.

(4) $k=2$ and $h=2$. In this case, (2.10) is the generalized Whitney map in Theorem 1.1.

Indeed, when $k=2$, (2.11) takes the form $2-h \leq 0$. Since we also have $h \leq k = 2$, we obtain $h=2$. When $k=3$ or $k=4$, (2.11) takes the form $n+3-6+k-h \leq 0$ or $2n-7+k-h \leq 0$, both of which are impossible for $n \geq 4$. When $k \geq 5$, then $(k-2)n \geq \frac{k+1}{2}k$ and thus (2.11) can not hold neither.

When $N=3n-3$, by the above consideration, we see that the only map of geometric rank two in this setting is given by

$$(z_3, z_4, \ldots, z_n, z_1^2, z_2^2, \sqrt{2}z_1z_2, z_1z_3, \ldots, z_1z_n, z_2z_3, z_2z_4, \ldots, z_2z_n)$$

which is the generalized Whitney map $W_{n,2}$.

Hence, based on the Lebl theorem, to prove Theorem 1.1, we need only to prove the map in Theorem 1.1 has degree bounded by two. We will do this in the next two sections.

3. Lower order terms in the Taylor expansion of F

We start with the following

Proposition 3.1. *Let $F \in Rat(\mathbb{H}_n, \mathbb{H}_{3n-3})$ be as in Theorem 2.2 with geometric rank 2. Assume $\mu_1 \le \mu_2$. Then*

$$\begin{aligned} &f_j = z_j + \frac{i}{2}\mu_j z_j w + O(|(z,w)|^3) \quad \text{for } j = 1,2,\\ &f_k = z_k, \quad \text{for } 3 \le k \le n-1,\\ &\phi_{11} = \sqrt{\mu_1} z_1^2 + \sqrt{\mu_1} A z_1 w + O(|(z,w)|^3),\\ &\phi_{12} = \sqrt{\mu_1+\mu_2} z_1 z_2 + \frac{\mu_2 A}{\sqrt{\mu_1+\mu_2}} z_2 w + O(|(z,w)|^3),\\ &\phi_{22} = \sqrt{\mu_2} z_2^2 + O(|(z,w)|^3), \quad \phi_{1k} = \sqrt{\mu_1} z_1 z_k + O(|(z,w)|^3),\\ &\phi_{2k} = \sqrt{\mu_2} z_2 z_k + O(|(z,w)|^3), \quad g = w, \end{aligned} \tag{3.1}$$

where $A := \frac{e_{1,11}}{\sqrt{\mu_1}}$.

Proof. **Step I. The family L_ϵ:** Let L_ϵ be in (2.6) given by

$$\begin{cases} z_1 = \sum_{j=3}^{n-1} a_j(\epsilon) z_j + a_n(\epsilon) w + \epsilon_1,\\ z_2 = \sum_{j=3}^{n-1} b_j(\epsilon) z_j + b_n(\epsilon) w + \epsilon_2. \end{cases} \tag{3.2}$$

Consider the image $\hat{L}_\epsilon := \hat{\sigma}_{\vec{c}}(L_\epsilon)$ given by

$$\begin{cases} Z_1 = \sum_{j=3}^{n-1} A_j(\epsilon) Z_j + A_n(\epsilon) W + \rho_1(\epsilon),\\ Z_2 = \sum_{j=3}^{n-1} B_j(\epsilon) Z_j + B_n(\epsilon) W + \rho_2(\epsilon), \end{cases} \tag{3.3}$$

where the inverse of the the automorphism is given by

$$\begin{aligned} \hat{\sigma}_{\vec{c}}^{-1}(Z,W) &:= \frac{(Z_1, Z_2, Z_3 + c_3 W, \ldots, Z_{n-1} + c_{n-1} W, W)}{q_c}\\ &= (z_1, z_2, \ldots, z_{n-1}, z_n) \end{aligned} \tag{3.4}$$

and $q_{\vec{c}} := 1 - 2i\bar{\vec{c}} \cdot Z - i|\vec{c}|^2 W$, where $\vec{c} = (0, \dots, 0, c_3, \dots, c_{n-1})$. Substituting (3.4) into (3.2), we obtain

$$\begin{cases} Z_1 = \sum_{j=3}^{n-1} a_j(\epsilon)(Z_j + c_j W) + a_n(\epsilon)W + \epsilon_1 q_{\vec{c}}, \\ Z_2 = \sum_{j=3}^{n-1} b_j(\epsilon)(Z_j + c_j W) + b_n(\epsilon)W + \epsilon_2 q_{\vec{c}}. \end{cases}$$

Combining this with (3.3), we get

$$\begin{cases} \quad \sum_{j=3}^{n-1} A_j(\epsilon)Z_j + A_n(\epsilon)W + \rho_1(\epsilon) \\ = \sum_{j=3}^{n-1} a_j(\epsilon)(Z_j + c_j W) + a_n(\epsilon)W + \epsilon_1(1 - 2i\bar{\vec{c}} \cdot Z - i|\vec{c}|^2 W), \\ \quad \sum_{j=3}^{n-1} B_j(\epsilon)Z_j + B_n(\epsilon)W + \rho_2(\epsilon) \\ = \sum_{j=3}^{n-1} b_j(\epsilon)(Z_j + c_j W) + b_n(\epsilon)W + \epsilon_2(1 - 2i\bar{\vec{c}} \cdot Z - i|\vec{c}|^2 W). \end{cases}$$

By considering the Z_j, $3 \le j \le n-1$ terms, we obtain

$$A_j(\epsilon_1, \epsilon_2) = a_j(\epsilon_1, \epsilon_2) - \epsilon_1(2i\overline{c_j}), \quad B_j(\epsilon_1, \epsilon_2) = b_j(\epsilon_1, \epsilon_2) - \epsilon_2(2i\overline{c_j}).$$

Hence we can choose $\vec{c}$ such that $\frac{\partial a_j(\epsilon_1, \epsilon_2)}{\partial \epsilon_1}(0) = 0$.

Step II. Calculation of the linear parts of $a_n(\epsilon)$ and $b_n(\epsilon)$: For a map $F \in Rat(\mathbb{H}_n, \mathbb{H}_{3n-3})$ of geometric rank 2, we have $f_i, g \not\equiv 0$ and $\phi_{jl} \not\equiv 0$ for $(j, l) \in \mathcal{S}_0$. Notice that $n + \sharp\mathcal{S}_0 = n + n - 1 + n - 2 = 3n - 3$. Hence we have $\phi_{33} \equiv 0$. In particular, we obtain $\phi_{33}^{(2,1)} = 0$ and $e_{j,33} = 0$ for $j = 1, 2$. Following the notation of [HJY14, (4.20)], we have

$$\widetilde{\phi}_{33}^{(2,1)} := \phi_{33}^{(2,1)} - 2i \sum_{j=1}^{2} \frac{\xi_j}{\mu_j} e_{j,33} = 0.$$

In [HJY14, (4.46)], we also have

$$\widetilde{\phi}_{33}^{(2,1)}(z) = \frac{-2}{\sqrt{\mu_1 + \mu_2}} \left(\sqrt{\frac{\mu_1}{\mu_2}} z_1 f_2^{(1,2)}(z) - \sqrt{\frac{\mu_2}{\mu_1}} z_2 f_1^{(1,2)}(z) \right).$$

Thus $\mu_1 z_1 f_2^{(1,2)} = \mu_2 z_2 f_1^{(1,2)}$. From [HJY14, (4.3)], we know

$$\begin{aligned} &\frac{i}{2}\mu_1 a_n^{(1)}(\epsilon) + f_1^{(1,2)}(\epsilon, 0, \dots, 0) = 0, \\ &\frac{i}{2}\mu_2 b_n^{(1)}(\epsilon) + f_2^{(1,2)}(\epsilon, 0, \dots, 0) = 0. \end{aligned} \tag{3.5}$$

Hence $\epsilon_1 b_n^{(1)}(\epsilon) = \epsilon_2 a_n^{(1)}(\epsilon)$, from which we yield

$$\begin{aligned} &a_n^{(1)}(\epsilon) = \zeta\epsilon_1, \quad b_n^{(1)}(\epsilon) = \zeta\epsilon_2 \quad \text{for some } \zeta \in \mathbb{C}, \\ &f_1^{(1,2)}(z) = -\frac{i}{2}\mu_1\zeta z_1, \quad f_2^{(1,2)}(z) = -\frac{i}{2}\mu_2\zeta z_2. \end{aligned} \tag{3.6}$$

Step III. Proof for $e_{1,1j} = 0$: Let H be an affine linear function along L_ϵ, then we must have $\frac{\partial^2 H|_{L_\epsilon}}{\partial z_j \partial w} \equiv 0$. We can write

$$H|_{L_\epsilon} = H\left(\sum_{k=3}^{n-1} a_k z_k + a_n w + \epsilon_1,\ \sum_{k=3}^{n-1} b_k z_k + b_n w + \epsilon_2,\ z_3, \ldots, z_{n-1},\ w\right).$$

Then we calculate for $1 \leq j \leq n-1$

$$\begin{aligned} \frac{\partial^2 H|_{L_\epsilon}}{\partial z_j \partial w} = {} & \frac{\partial^2 H}{\partial z_1^2} a_j a_n + \frac{\partial^2 H}{\partial z_1 \partial z_2}(a_n b_j + a_j b_n) \\ & + \frac{\partial^2 H}{\partial z_1 \partial z_j} a_n + \frac{\partial^2 H}{\partial z_1 \partial w} a_j + \frac{\partial^2 H}{\partial z_2^2} b_j b_n \\ & + \frac{\partial^2 H}{\partial z_2 \partial z_j} b_n + \frac{\partial^2 H}{\partial z_2 \partial w} b_j + \frac{\partial^2 H}{\partial z_j \partial w} = 0, \quad \text{at } (\epsilon, 0). \end{aligned} \tag{3.7}$$

Choosing $H = f_1$ and collecting ϵ_1 and ϵ_2 terms in the above equation, we get

$$\frac{i}{2}\mu_1 a_j^{(1)} + f_1^{(I_1+I_j+I_n)}\epsilon_1 + f_1^{(I_2+I_j+I_n)}\epsilon_2 = 0, \quad \text{at } (\epsilon, 0). \tag{3.8}$$

In Step I, we have made $\frac{\partial a_j(\epsilon_1,\epsilon_2)}{\partial \epsilon_1}(0) = 0$. Hence we get $f_1^{(I_1+I_j+I_n)} = 0$. On the other hand, by [HJY14, (3.5)], we have $f_1^{(2,1)}(z) = -\xi_1$. Together with (2.8), we obtain $f_1^{(I_1+I_j+I_n)} = -\sqrt{\mu_1}\overline{e_{1,1j}}$, $f_1^{(I_2+I_j+I_n)} = 0$. Hence we get $e_{1,1j} = 0$ for $3 \leq j \leq n-1$.

Step IV. Calculating $e_{1,jk}$ and $e_{2,jk}$: Since $N = 3n-3$ and $\Phi_1 = \emptyset$, it implies $\Phi_1^{(3,0)}(z) \equiv 0$. From (2.9), we obtain

$$\mu_1 z_1 \xi_2 = \mu_2 z_2 \xi_1. \tag{3.9}$$

Observing that $\mu_1 z_1 \xi_2$ (resp. $\mu_2 z_2 \xi_1$) must be divided by z_2 (resp. z_1), and making use of (2.8), we obtain

$$e_{1,22} = e_{1,2j} = e_{2,11} = e_{2,1j} = 0.$$

Recall that $e_{1,1j} = 0$ for $j \geq 3$. Thus (3.9) takes the following form:

$$\mu_1 z_1 \left(\phi^{(2,0)}_{2,12} \overline{e_{2,12}} + \sum_{j=2}^{n-1} \phi^{(2,0)}_{2,2j} \overline{e_{2,2j}} \right) = \mu_2 z_2 \left(\phi^{(2,0)}_{1,11} \overline{e_{1,11}} + \phi^{(2,0)}_{1,12} \overline{e_{1,12}} \right).$$

Now a direct computation gives $e_{2,2j} = 0$ and

$$e_{2,12} = \frac{\mu_2}{\sqrt{\mu_1(\mu_1+\mu_2)}} e_{1,11}, \quad e_{2,22} = \frac{\sqrt{\mu_2(\mu_1+\mu_2)}}{\mu_1} e_{1,12}. \tag{3.10}$$

Step V. Calculation of Taylor series of F up to degree 3: Now we have obtained

$$\begin{aligned} & f^{(1,2)}_1(z) = -\frac{i}{2}\mu_1 \zeta z_1, \quad f^{(1,2)}_2(z) = -\frac{i}{2}\mu_2 \zeta z_2, \\ & f^{(1,1)}_1(z) = \frac{i}{2}\mu_1 z_1, \quad f^{(1,1)}_2(z) = \frac{i}{2}\mu_2 z_2, \\ & \phi^{(1,1)}(z) = \big(e_{1,11} z_1, e_{1,12} z_1 + e_{2,12} z_2, e_{2,22} z_2, 0 \ldots, 0\big). \end{aligned}$$

Substituting these relations into [HJY14, (4.10)], we obtain

$$\begin{aligned} & 2\mathrm{Re} \left\{ \overline{z_1} \cdot \left(-\frac{i}{2}\mu_1 \zeta z_1 \right) + \overline{z_2} \cdot \left(-\frac{i}{2}\mu_2 \zeta z_2 \right) \right\} + \left| \frac{i}{2}\mu_1 z_1 \right|^2 + \left| \frac{i}{2}\mu_2 z_2 \right|^2 \\ & \quad + \left| e_{1,11} z_1 \right|^2 + \left| e_{1,12} z_1 + e_{2,12} z_2 \right|^2 + \left| e_{2,22} z_2 \right|^2 = 0. \end{aligned}$$

Considering the coefficients of $|z_1|^2$, $|z_2|^2$ and $z_1 \overline{z_2}$, respectively, we get

$$2\mathrm{Re}\left\{ -\frac{i}{2}\zeta \right\} \mu_1 + \frac{\mu_1^2}{4} + \left| e_{1,11} \right|^2 + \left| e_{1,12} \right|^2 = 0, \tag{3.11}$$

$$2\mathrm{Re}\left\{ -\frac{i}{2}\zeta \right\} \mu_2 + \frac{\mu_2^2}{4} + \left| e_{2,22} \right|^2 + \left| e_{2,12} \right|^2 = 0, \tag{3.12}$$

$$e_{1,12} \overline{e_{2,12}} = 0. \tag{3.13}$$

By calculating $(3.11)\mu_2 - (3.12)\mu_1$, we get

$$\frac{\mu_1^2}{4}\mu_2 - \frac{\mu_2^2}{4}\mu_1 + \mu_2 \left| e_{1,11} \right|^2 - \mu_1 \left| e_{2,12} \right|^2 + \mu_2 \left| e_{1,12} \right|^2 - \mu_1 \left| e_{2,22} \right|^2 = 0.$$

Together with (3.10), we obtain

$$\frac{1}{4}\mu_1 \mu_2 (\mu_1 - \mu_2) + \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} \left| e_{1,11} \right|^2 - \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} \left| e_{2,22} \right|^2 = 0.$$

Namely, we have

$$\left|e_{1,11}\right|^2 = \left|e_{2,22}\right|^2 + \frac{1}{4}(\mu_1+\mu_2)(\mu_2-\mu_1).$$

By (3.10) and (3.13), either $e_{1,11}$ or $e_{2,22}$ is 0. Recall that $\mu_2 \geq \mu_1$, thus

$$e_{2,22} = 0, \quad \left|e_{1,11}\right|^2 = \frac{1}{4}(\mu_1+\mu_2)(\mu_2-\mu_1). \tag{3.14}$$

From all of the above, the proof of Proposition 3.1 is complete. □

4. Proof of Theorem 1.1

By Lebl's theorem, to prove our main theorem, we need only to show that the map F has degree two.

For a map $F \in Rat(\mathbb{H}_n, \mathbb{H}_{3n-3})$ with $n \geq 4$, by the inequality $N \geq n + \frac{(2n-\kappa_0-1)\kappa_0}{2}$ (cf. [Hu03]), we have that the geometric rank κ_0 of this map is less than or equal to 2. If $\kappa_0 = 0$, F is equivalent to $(z, 0, w)$. If $\kappa_0 = 1$, by [HJX06], Theorem 1.2, the map F is equivalent to a proper holomorphic map $F = (z_1, \ldots, z_{n-1}, z_n h)$ where $h \in Rat(\mathbb{B}^n, \mathbb{B}^{2n-2})$. By applying the first gap theorem [Hu99], h must be linear fractional. It suffices to prove Theorem 1.1 only for the case of $\kappa_0 = 2$.

If we are able to prove $deg(F) \leq 2$, then by applying Lebl's theorem (see Theorem 2.4) and by consideration of degree, it completes the proof of Theorem 1.1. In the rest of this section, we'll prove $deg(F) \leq 2$.

Step 1. The basic setting: In order to prove Theorem 1.1, we start with the equation

$$\frac{g(z,w)-\overline{g(z,w)}}{2i} = f(z,w)\cdot\overline{f(z,w)} + \phi(z,w)\cdot\overline{\phi(z,w)}, \quad \forall \mathrm{Im}(w) = |z|^2.$$

By complexification, we write

$$\begin{aligned}\frac{g(z,w)-\overline{g(\overline{\chi},\overline{\eta})}}{2i} &= \sum_{l=1}^{n-1} f_l(z,w)\overline{f_l(\overline{\chi},\overline{\eta})} \\ &\quad + \sum \phi_t(z,w)\overline{\phi_t(\overline{\chi},\overline{\eta})}, \quad \forall \frac{w-\eta}{2i} = z\cdot\chi.\end{aligned}$$

Applying $\mathcal{L}_j := \frac{\partial}{\partial z_j} + 2i\chi_j \frac{\partial}{\partial w}$ for $z = 0$ and $w = \eta = 0$ to the both sides of the above identity, we obtain

$$\frac{\mathcal{L}_j g(0,0)}{2i} = \sum_{l=1}^{n-1} \mathcal{L}_j f_l(0,0)\overline{f_l(\overline{\chi},0)} + \sum \mathcal{L}_j \phi_t(0,0)\overline{\phi_t(\overline{\chi},0)}$$

and

$$\frac{\mathcal{L}_j\mathcal{L}_k g(0,0)}{2i} = \sum_{l=1}^{n-1} \mathcal{L}_j\mathcal{L}_k f_l(0,0)\overline{f_l(\overline{\chi},0)} + \sum \mathcal{L}_j\mathcal{L}_k \phi_t(0,0)\overline{\phi_t(\overline{\chi},0)}.$$

We can write it in terms of matrix,

$$\begin{pmatrix} \chi_1 \\ \chi_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = B \begin{pmatrix} \overline{f_1(\overline{\chi},0)} \\ \overline{f_2(\overline{\chi},0)} \\ \overline{\phi(\overline{\chi},0)} \end{pmatrix} \tag{4.1}$$

where B is an $(2n-1) \times (2n-1)$ matrix:

$$B := \left.\begin{pmatrix}
\mathcal{L}_1 f_1 & \mathcal{L}_1 f_2 & \mathcal{L}_1 \phi_{11} & \mathcal{L}_1 \phi_{12} & \mathcal{L}_1 \phi_{22} & \mathcal{L}_1 \phi_{1j} & \mathcal{L}_1 \phi_{2j} \\
\mathcal{L}_2 f_1 & \mathcal{L}_2 f_2 & \mathcal{L}_2 \phi_{11} & \mathcal{L}_2 \phi_{12} & \mathcal{L}_2 \phi_{22} & \mathcal{L}_2 \phi_{1j} & \mathcal{L}_2 \phi_{2j} \\
\mathcal{L}_1\mathcal{L}_1 f_1 & \mathcal{L}_1\mathcal{L}_1 f_2 & \mathcal{L}_1\mathcal{L}_1 \phi_{11} & \mathcal{L}_1\mathcal{L}_1 \phi_{12} & \mathcal{L}_1\mathcal{L}_1 \phi_{22} & \mathcal{L}_1\mathcal{L}_1 \phi_{1j} & \mathcal{L}_1\mathcal{L}_1 \phi_{2j} \\
\mathcal{L}_1\mathcal{L}_2 f_1 & \mathcal{L}_1\mathcal{L}_2 f_2 & \mathcal{L}_1\mathcal{L}_2 \phi_{11} & \mathcal{L}_1\mathcal{L}_2 \phi_{12} & \mathcal{L}_1\mathcal{L}_2 \phi_{22} & \mathcal{L}_1\mathcal{L}_2 \phi_{1j} & \mathcal{L}_1\mathcal{L}_2 \phi_{2j} \\
\mathcal{L}_2\mathcal{L}_2 f_1 & \mathcal{L}_2\mathcal{L}_2 f_2 & \mathcal{L}_2\mathcal{L}_2 \phi_{11} & \mathcal{L}_2\mathcal{L}_2 \phi_{12} & \mathcal{L}_2\mathcal{L}_2 \phi_{22} & \mathcal{L}_2\mathcal{L}_2 \phi_{1j} & \mathcal{L}_2\mathcal{L}_2 \phi_{2j} \\
\mathcal{L}_1\mathcal{L}_k f_1 & \mathcal{L}_1\mathcal{L}_k f_2 & \mathcal{L}_1\mathcal{L}_k \phi_{11} & \mathcal{L}_1\mathcal{L}_k \phi_{12} & \mathcal{L}_1\mathcal{L}_k \phi_{22} & \mathcal{L}_1\mathcal{L}_k \phi_{1j} & \mathcal{L}_1\mathcal{L}_k \phi_{2j} \\
\mathcal{L}_2\mathcal{L}_k f_1 & \mathcal{L}_2\mathcal{L}_k f_2 & \mathcal{L}_2\mathcal{L}_k \phi_{11} & \mathcal{L}_2\mathcal{L}_k \phi_{12} & \mathcal{L}_2\mathcal{L}_k \phi_{22} & \mathcal{L}_2\mathcal{L}_k \phi_{1j} & \mathcal{L}_2\mathcal{L}_k \phi_{2j}
\end{pmatrix}\right|_{(0,0,\chi,0)}.$$

Step 2. The main idea to prove $deg(F) \leq 2$: Let $\widetilde{F} : \mathbb{C}^{n-1} \backslash \{1 - 2i\overline{A}z_1 = 0\} \to \mathbb{C}^{2n-1}$ be defined as follows:

$$\begin{aligned}
&\widetilde{f_1}(z) = z_1, \quad \widetilde{f_2}(z) = z_2, \quad \widetilde{\phi_{11}}(z) = \frac{\sqrt{\mu_1} z_1^2}{1 - 2i\overline{A}z_1}, \\
&\widetilde{\phi_{12}}(z) = \frac{\sqrt{\mu_1 + \mu_2} z_2 z_1}{1 - 2i\overline{A}z_1}, \quad \widetilde{\phi_{22}}(z) = \frac{\sqrt{\mu_2} z_2^2}{1 - 2i\overline{A}z_1}, \\
&\widetilde{\phi_{1j}}(z) = \frac{\sqrt{\mu_1} z_1 z_j}{1 - 2i\overline{A}z_1}, \quad \widetilde{\phi_{2j}}(z) = \frac{\sqrt{\mu_2} z_2 z_j}{1 - 2i\overline{A}z_1}.
\end{aligned} \tag{4.2}$$

If we can prove that

$$B|_{(0,0,\chi,0)} \text{ is non-singular} \tag{4.3}$$

and if the following holds:

$$\begin{pmatrix}\chi_1\\ \chi_2\\ 0\\ \vdots\\ 0\end{pmatrix} = B\begin{pmatrix}\overline{\widetilde{f}_1(\overline{\chi})}\\ \overline{\widetilde{f}_2(\overline{\chi})}\\ \overline{\widetilde{\phi}(\overline{\chi})}\end{pmatrix}, \tag{4.4}$$

we infer from (4.1) that

$$B\begin{pmatrix}\overline{f_1(\overline{\chi},0)} - \overline{\widetilde{f}_1(\overline{\chi})}\\ \overline{f_2(\overline{\chi},0)} - \overline{\widetilde{f}_2(\overline{\chi})}\\ \overline{\phi(\overline{\chi},0)} - \overline{\widetilde{\phi}(\overline{\chi})}\end{pmatrix} = 0.$$

Then by (4.3), it yields $F(z,0) = \widetilde{F}(z)$, and hence $deg(F(z,0)) \le 2$. Replacing F by F_p^{***} for any $p \in \partial\mathbb{H}_n$ near the origin, we can show $deg(F_p^{**}(z,0)) \le 2$ in a similar manner. By [HJ01, Section 5], we have that $deg(F) \le 2$. Then by Lebl's theorem (i.e., Theorem 2.4), the proof of Theorem 1.1 is complete.

In the rest of this section, we shall prove (4.3) and (4.4).

Step 3. Calculation of the partial derivatives of F up to degree 2:

•**Calculate $(\mathcal{L}_1 H)(0,0)$ for $H = f_i, \phi_{jk}$:** At the point $(0,0)$, we have

$$(\mathcal{L}_1 f_1)(0,0) = 1, \quad (\mathcal{L}_1 f_2)(0,0) = 0, \quad (\mathcal{L}_1 \phi_{jk})(0,0) = 0 \quad \text{for} \quad (j,k) \in \mathcal{S}_0.$$

Then

$$\sum_{j=1}^{2} (\mathcal{L}_1 f_j)(0,0)\cdot \overline{\widetilde{f}_j(\overline{\chi})} + \sum_t (\mathcal{L}_1 \phi_t)(0)\cdot \overline{\widetilde{\phi}_t(\overline{\chi})} = 1\cdot \chi_1 = \chi_1. \tag{4.5}$$

• **Calculate $(\mathcal{L}_2 H)(0,0)$ for $H = f_i, \phi_{jk}$:** At the point $(0,0)$, we have

$$(\mathcal{L}_2 f_2)(0,0) = 1, \quad (\mathcal{L}_2 f_1)(0,0) = 0, \quad (\mathcal{L}_2 \phi_{jk})(0,0) = 0 \quad \text{for } (j,k) \in \mathcal{S}_0.$$

Corresponding to (4.5), we have the following:

$$\sum_{j=1}^{2} (\mathcal{L}_2 f_j)(0,0)\cdot \overline{\widetilde{f}_j(\overline{\chi})} + \sum_t (\mathcal{L}_2 \phi_t)(0,0)\cdot \overline{\widetilde{\phi}_t(\overline{\chi})} = 1\cdot \chi_2 = \chi_2. \tag{4.6}$$

- **Calculate** $(\mathcal{L}_1^2 H)(0,0)$ **for** $H = f_i, \phi_{jk}$**:** A direct computation shows that $\mathcal{L}_1^2 = \frac{\partial^2}{\partial z_1^2} + 4i\chi_1 \frac{\partial^2}{\partial z_1 \partial w} + (2i\chi_1)^2 \frac{\partial^2}{\partial w^2}$. At the point $(0,0)$, we have

$$\begin{aligned}
&(\mathcal{L}_1^2 f_1)(0,0) = 4i\chi_1 \cdot \frac{i}{2}\mu_1 = -2\mu_1\chi_1,\\
&(\mathcal{L}_1^2 \phi_{11})(0,0) = 2\sqrt{\mu_1} + 4i\chi_1\sqrt{\mu_1}A,\\
&(\mathcal{L}_1^2 f_2)(0,0) = 0, \quad (\mathcal{L}_1^2 \phi_{jk})(0,0) = 0 \quad \text{for } (j,k) \neq (1,1).
\end{aligned}$$

Then we get

$$\begin{aligned}
\text{(4.7)} \qquad &\sum_{j=1}^{2} (\mathcal{L}_1^2 f_j)(0,0) \cdot \overline{\widetilde{f}_j(\overline{\chi})} + \sum_t (\mathcal{L}_1^2 \phi_t)(0,0) \cdot \overline{\widetilde{\phi}_t(\overline{\chi})}\\
&= (-2\mu_1\chi_1) \cdot \chi_1 + \big(2\sqrt{\mu_1} + 4i\chi_1\sqrt{\mu_1}A\big) \cdot \frac{\sqrt{\mu_1}\chi_1^2}{1 + 2iA\chi_1} = 0.
\end{aligned}$$

- **Calculate** $(\mathcal{L}_1\mathcal{L}_2 H)(0,0)$ **for** $H = f_i, \phi_{jk}$**:** A direct computation shows that $\mathcal{L}_1\mathcal{L}_2 = \frac{\partial^2}{\partial z_1 \partial z_2} + 2i\chi_2 \frac{\partial^2}{\partial z_1 \partial w} + 2i\chi_1 \frac{\partial^2}{\partial z_2 \partial w} - 4\chi_1\chi_2 \frac{\partial^2}{\partial w^2}$. At the point $(0,0)$, we have

$$\begin{aligned}
&(\mathcal{L}_1\mathcal{L}_2 f_1)(0,0) = 2i\chi_2 \cdot \frac{i}{2}\mu_1 = -\mu_1\chi_2,\\
&(\mathcal{L}_1\mathcal{L}_2 f_2)(0,0) = 2i\chi_1 \cdot \frac{i}{2}\mu_2 = -\mu_2\chi_1,\\
&(\mathcal{L}_1\mathcal{L}_2 \phi_{11})(0,0) = 2i\chi_2 \cdot \sqrt{\mu_1}A = 2i\sqrt{\mu_1}\chi_2 A,\\
&(\mathcal{L}_1\mathcal{L}_2 \phi_{12})(0,0) = \sqrt{\mu_1 + \mu_2} + 2i\chi_1 \cdot \frac{\mu_2 A}{\sqrt{\mu_1 + \mu_2}},\\
&(\mathcal{L}_2 \phi_{jk})(0,0) = 0 \quad \text{for } (j,k) \neq (1,1), (1,2).
\end{aligned}$$

We get

$$\begin{aligned}
\text{(4.8)} \qquad &\sum_{j=1}^{2} (\mathcal{L}_1\mathcal{L}_2 f_j)(0,0) \cdot \overline{\widetilde{f}_j(\overline{\chi})} + \sum_t (\mathcal{L}_1\mathcal{L}_2 \phi_t)(0,0) \cdot \overline{\widetilde{\phi}_t(\overline{\chi})}\\
&= -\mu_1\chi_2 \cdot \chi_1 - \mu_2\chi_1 \cdot \chi_2 + 2i\sqrt{\mu_1}\chi_2 A \cdot \frac{\sqrt{\mu_1}\chi_1^2}{1 + 2iA\chi_1}\\
&\quad + \left(\sqrt{\mu_1 + \mu_2} + 2i\chi_1 \cdot \frac{\mu_2 A}{\sqrt{\mu_1 + \mu_2}}\right) \cdot \frac{\sqrt{\mu_1 + \mu_2}\chi_1\chi_2}{1 + 2iA\chi_1}\\
&= -(\mu_1 + \mu_2)\chi_1\chi_2 + \chi_1\chi_2\big(\mu_1 + \mu_2 + 2i(\mu_1 + \mu_2)A\chi_1\big)\frac{1}{1 + 2iA\chi_1}\\
&= 0.
\end{aligned}$$

• **Calculate** $(\mathcal{L}_2^2 H)(0,0)$ **for** $H = f_i, \phi_{jk}$**:** A direct computation shows that $\mathcal{L}_2^2 = \frac{\partial^2}{\partial z_2^2} + 4i\chi_2 \frac{\partial^2}{\partial z_2 \partial w} + (2i\chi_2)^2 \frac{\partial^2}{\partial w^2}$. At the point $(0,0)$, we have

$$\begin{aligned}
&(\mathcal{L}_2^2 f_2)(0,0) = 4i\chi_2 \cdot \frac{i}{2}\mu_2 = -2\mu_2\chi_2,\\
&(\mathcal{L}_2^2 \phi_{12})(0,0) = 4i\chi_2 \frac{\mu_2 A}{\sqrt{\mu_1+\mu_2}} = \frac{4i\mu_2\chi_2 A}{\sqrt{\mu_1+\mu_2}},\\
&(\mathcal{L}_2^2 \phi_{22})(0,0) = 2\sqrt{\mu_2},\\
&(\mathcal{L}_2^2 f_1)(0,0) = 0, \quad (\mathcal{L}_2^2 \phi_{jk})(0,0) = 0 \quad \text{for } (j,k) \neq (1,2),(2,2).
\end{aligned}$$

We get

$$\begin{aligned}
(4.9) \qquad &\sum_{j=1}^{2} (\mathcal{L}_2^2 f_j)(0,0) \cdot \overline{\widetilde{f}_j(\overline{\chi})} + \sum_t (\mathcal{L}_2^2 \phi_t)(0,0) \cdot \overline{\widetilde{\phi}_t(\overline{\chi})}\\
&= -2\mu_2\chi_2 \cdot \chi_2 + \frac{4i\mu_2\chi_2 A}{\sqrt{\mu_1+\mu_2}} \cdot \frac{\sqrt{\mu_1+\mu_2}\chi_1\chi_2}{1+2iA\chi_1} + 2\sqrt{\mu_2} \cdot \frac{\sqrt{\mu_2}\chi_2^2}{1+2iA\chi_1}\\
&= 0.
\end{aligned}$$

• **Calculate** $(\mathcal{L}_1\mathcal{L}_k H)(0,0)$ **for** $H = f_i, \phi_{jl}$**:** A direct computation shows that

$$\mathcal{L}_1\mathcal{L}_k = \frac{\partial^2}{\partial z_1 \partial z_k} + 2i\chi_k \frac{\partial^2}{\partial z_1 \partial w} + 2i\chi_1 \frac{\partial^2}{\partial z_k \partial w} + 2i\chi_1 \cdot 2i\chi_k \frac{\partial^2}{\partial w^2}.$$

At the point $(0,0)$, we have

$$\begin{aligned}
&(\mathcal{L}_1\mathcal{L}_k f_1)(0,0) = 2i\chi_k \cdot \frac{i}{2}\mu_1 = -\mu_1\chi_k,\\
&(\mathcal{L}_1\mathcal{L}_k \phi_{11})(0,0) = 2i\chi_k \cdot \sqrt{\mu_1} A = 2i\sqrt{\mu_1}\chi_k A,\\
&(\mathcal{L}_1\mathcal{L}_k \phi_{1k})(0,0) = \sqrt{\mu_1},\\
&(\mathcal{L}_1\mathcal{L}_k f_2)(0,0) = 0, \quad (\mathcal{L}_1\mathcal{L}_k \phi_{jk})(0,0) = 0 \quad \text{for } (j,k) \neq (1,1),(1,k).
\end{aligned}$$

We get

$$\begin{aligned}
(4.10) \qquad &\sum_{j=1}^{2} (\mathcal{L}_1\mathcal{L}_k f_j)(0,0) \cdot \overline{\widetilde{f}_j(\overline{\chi})} + \sum_t (\mathcal{L}_1\mathcal{L}_k \phi_t)(0,0) \cdot \overline{\widetilde{\phi}_t(\overline{\chi})}\\
&= -\mu_1\chi_k \cdot \chi_1 + 2i\sqrt{\mu_1}\chi_k A \cdot \frac{\sqrt{\mu_1}\chi_1^2}{1+2iA\chi_1} + \sqrt{\mu_1} \cdot \frac{\sqrt{\mu_1}\chi_1\chi_k}{1+2iA\chi_1} = 0.
\end{aligned}$$

• **Calculate** $(\mathcal{L}_2\mathcal{L}_k H)(0,0)$ **for** $H = f_i, \phi_{jl}$**:** A direct computation shows that

$$\mathcal{L}_2\mathcal{L}_k = \frac{\partial^2}{\partial z_2 \partial z_k} + 2i\chi_k \frac{\partial^2}{\partial z_2 \partial w} + 2i\chi_2 \frac{\partial^2}{\partial z_k \partial w} + 2i\chi_2 \cdot 2i\chi_k \frac{\partial^2}{\partial w^2}.$$

At the point $(0,0)$, we have

$$\begin{aligned}
&(\mathcal{L}_2\mathcal{L}_k f_2)(0,0) = 2i\chi_k \cdot \frac{i}{2}\mu_2 = -\mu_2\chi_k,\\
&(\mathcal{L}_2\mathcal{L}_k \phi_{12})(0,0) = 2i\chi_k \cdot \frac{\mu_2 A}{\sqrt{\mu_1+\mu_2}} = \frac{2i\mu_2\chi_k A}{\sqrt{\mu_1+\mu_2}},\\
&(\mathcal{L}_2\mathcal{L}_k \phi_{2k})(0,0) = \sqrt{\mu_2},\\
&(\mathcal{L}_2\mathcal{L}_k f_1)(0,0) = 0, \quad (\mathcal{L}_2\mathcal{L}_k \phi_{jk})(0,0) = 0 \quad \text{for } (j,k) \neq (1,2),(2,k).
\end{aligned}$$

By a similar computation as that of (4.10), we get

$$\sum_{j=1}^{2} (\mathcal{L}_2\mathcal{L}_k f_j)(0,0) \cdot \overline{\widetilde{f}_j(\overline{\chi})} + \sum_t (\mathcal{L}_2\mathcal{L}_k \phi_t)(0,0) \cdot \overline{\widetilde{\phi}_t(\overline{\chi})} = 0. \tag{4.11}$$

By all of the above, (4.4) is proved. Also, we see

$$\begin{aligned}
B = {}& \mathrm{diag}\big(1, 1, 2\sqrt{\mu_1}, \sqrt{\mu_1+\mu_2}, 2\sqrt{\mu_2}, \sqrt{\mu_1}, \ldots, \sqrt{\mu_1}, \sqrt{\mu_2}, \ldots, \sqrt{\mu_2}\big)\\
&+ O(|\chi|).
\end{aligned}$$

Hence (4.3) is proved. The proof of Theorem 1.1 is complete. ☐

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