Spherical CR Submanifolds of a Sphere

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PREFACE

This lecture notes is an extended version of my lecture series given at the workshop in the Department of Mathematics, Seoul National University, in Seoul in November, 2009.

It surveys the theory of proper holomorphic mappings between balls. This theory was originated from Poincaré's work in 1807: any non-constant holomorphic map $f : U \to V$ satisfying $f(U \cap \partial\mathbb{B}^2) \subset V\partial\mathbb{B}^2$ is a map in $Aut(\partial\mathbb{B}^2)$, where U, V are open subsets of $\mathbb{C}^2$. Over time many mathematicians made contribution to this theory.

In Chapter 1, we introduce some background information.

In Chapter 2 we introduce the first gap theorem, which was initiated from 1979 by Webster, and is an accumulative result by many mathematicians over 20 years.

In Chapter 3, we illustrate a lots of examples of proper holomorphic mappings between balls, from which a general conjecture about gap phenomenon is formulated. All constructed examples seem to be polynomial maps, nevertheless, not every proper rational map between balls can be equivalent to polynomial maps. A criterion, which tells when a proper rational map can be equivalent to a polynomial one, is introduced. To illustrate the method that used to study the classification problem, we first show a new proof for Faran's theorem on classification of maps from $\mathbb{B}^2$ to $\mathbb{B}^3$, and then outline how to find complete classification for proper holomorphic rational maps from $\mathbb{B}^2$ to $\mathbb{B}^N$ with degree 2.

In Chapter 4, we start with a result on maps from $\mathbb{B}^n$ to $\mathbb{B}^{2n-1}$. We list five main facts in the ingredient of the proof, and discuss its generalization for higher codimensional case. As a result, by using analytic approach, we shall demonstrate applications of these generalizations, including the rationality problems, and the proof of the second gap theorem.

In Chapter 5, besides the analytic approach, we also introduce a geometric approach: the Cartan's moving frame theory in differential geometry, as well as its applications.

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Chapter 1

Real Hypersurfaces

1.1 Domains and Their Boundaries

Geometry and analysis on domains in $\mathbb{C}^n$ and on their boundaries are closely related. We start with several theorems concerning domains in $\mathbb{C}^n$ and their boundaries.

Theorem 1.1.1 *[Fe74][B43] Let $D_1, D_2 \subset \mathbb{C}^n$ be smooth strongly pseudoconvex domains with C^∞ boundaries. Then the following statements are equivalent:*
(i) There exists a biholomorphic map $f : D_1 \to D_2$.
(ii) There is a C^∞ CR isomorphism $F : \partial D_1 \to \partial D_2$.

Theorem 1.1.2 *(i) [CJ96] If Ω is a bounded simply connected domain in $\mathbb{C}^{n+1}$ with connected smooth spherical real analytic boundary, then Ω is globally biholomorphic to the unit ball $\mathbb{B}^{n+1}$.*
(ii) [HJ98] The "simply connected" condition can be dropped if the boundary is defined by a real polynomial.

One could pass problems in domains into the ones in boundaries. Conversely, one could pass problems in boundaries into the ones in domains.

Siegel upper-half space and Heisenberg hypersurfaces For a domain $D \subset \mathbb{C}^n$, its boundary ∂D is a real hypersurface in $\mathbb{C}^n$.

[Example 1.1 A]

1. Let
$$\mathbb{B}^n = \{z = (z_1, ..., z_n) \in \mathbb{C}^n \mid |z|^2 = |z_1|^2 + ... + |z_n|^2 < 1\}$$

be the *unit ball*. Its boundary

$$\partial\mathbb{B}^n = \{z = (z_1, ..., z_n) \in \mathbb{C}^n \mid |z|^2 = 1\}$$

is the *unit sphere*.

2. Let

$$\mathbb{H}^n := \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \ \mathrm{Im}(w) > |z|^2\}$$

be the *Siegel upper-half space*. Its boundary

$$\partial\mathbb{H}^n := \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \ \mathrm{Im}(w) = |z|^2\}$$

is the *Heisenberg hypersurface*. When $n = 1$, $\mathbb{H}^1$ is the upper-half plane $\{w \in \mathbb{C} \mid Im(w) > 0\}$ and $\partial\mathbb{H}^1 = \{w \in \mathbb{C} \mid Im(w) = 0\}$ is the x-axis. Among all (non-degenerate) boundaries of domains in $\mathbb{C}^n$ with $n \geq 2$, the most simplest one is the $\partial\mathbb{H}^n$.

[Example 1.1 B]

1. More generally, we can define

$$\mathbb{H}^n_\ell := \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} \mid Im(w) > |z|^2_\ell\}$$

where $|z|^2_\ell := -\sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n-1} |z_j|^2$. Its boundary

$$\partial\mathbb{H}^n_\ell = \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} \mid Im(w) = |z|^2_\ell, \quad z \in \mathbb{C}^{n-1}\}, \tag{1.1}$$

is also called the *(Levi) nondegenerate hyperquadric*.

Notice that the pair $(\ell, n - 1 - \ell)$ is completely determined by ℓ. Hence, in what follows, for brevity, we call ℓ the *signature* of the above hypersurface M.

When $\ell = 0$, we call $\partial\mathbb{H}^n_\ell$ *strongly pseudoconvex*. When $\ell > 0$, (1.1) is the model example of a real hypersurface which is Levi nondegenerate but is not strongly pseudoconvex.

2. We can also define

$$\mathbb{B}^n_\ell := \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} \mid |z|^2_\ell + |w|^2 < 1\}. \tag{1.2}$$

Its boundary is

$$\partial\mathbb{B}^n_\ell := \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} \mid |z|^2_\ell + |w|^2 = 1\}. \tag{1.3}$$

Cayley transformation By the *Cayley transformation*, we mean a biholomorphic map

$$\Psi_n : \mathbb{H}^n_\ell \to \mathbb{B}^n_\ell, \quad \Psi_n(z,w) = \left(\frac{2z}{1-iw}, \frac{1+iw}{1-iw}\right). \tag{1.4}$$

With Ψ_n we can identify $\mathbb{B}^n_\ell$ with $\mathbb{H}^n_\ell$ and identify $\partial\mathbb{H}^n_\ell$ with $\partial\mathbb{B}^n_\ell$. To verify this, it suffices to show that $\Psi_n : \partial\mathbb{H}^n_\ell \to \partial\mathbb{B}^n_\ell$, i.e., to verify

$$\left|\frac{2z}{i+w}\right|^2_\ell + \left|\frac{i-w}{i+w}\right|^2 = 1, \quad \forall\ Im(w) = |z|^2_\ell, \tag{1.5}$$

i.e. to verify that $\forall\ Im(w) = |z|^2_\ell$,

$$\begin{array}{cccc} 4|z|^2_\ell + & |i-w|^2 & = & |i+w|^2 \\ & \| & & \| \\ & (i-w)(-i-\overline{w}) & & (i+w)(-i+\overline{w}) \\ & \| & & \| \\ & 1+iw-i\overline{w}+|w|^2 & & 1-iw+i\overline{w}+|w|^2 \end{array},$$

i.e., to verify

$$4|z|^2_\ell + 2iw - 2i\overline{w} = 0, \quad \forall\ Im(w) = |z|^2_\ell,$$

i.e.,

$$4|z|^2_\ell - 4Im(w) = 0, \quad \forall\ Im(w) = |z|^2_\ell,$$

which is trivially true.

Automorphism group By an *automorphism*, we mean a biholomorphic map $F : \mathbb{B}^n_\ell \to \mathbb{B}^n_\ell$. Let us denote by $Aut(\mathbb{B}^n_\ell)$ the group of automorphisms of $\mathbb{B}^n_\ell$. Also we define $Aut(\partial\mathbb{B}^n_\ell)$ where $F \in Aut(\partial\mathbb{B}^n_\ell)$ if $F \in Aut(\mathbb{B}^n_\ell)$ such that it maps the boundary $\partial\mathbb{B}^n_\ell$ onto itself.

We can define $Aut(\mathbb{H}^n_\ell)$ and $Aut(\partial\mathbb{H}^n_\ell)$ similarly. By Cayley transformation, we can identify $Aut(\partial\mathbb{B}^n_\ell)$ with $Aut(\mathbb{H}^n_\ell)$, and identify $Aut(\partial\mathbb{B}^n_\ell)$ with $Aut(\partial\mathbb{H}^n_\ell)$.

The group $Aut(\partial\mathbb{H}^n_\ell)$ is transitive, i.e., for any two points $P, Q \in \partial\mathbb{H}^n_\ell$, there exists a map $F \in Aut(\partial\mathbb{H}^n)$ such that $F(Q) = P$. To prove this, we can assume $Q = 0$. We write $P = (z_0, w_0) \in \mathbb{C}^{n-1} \times \mathbb{C}$, and then we can take

$$F(z,w) = \big(z + z_0, w + w_0 + 2i\langle z, \overline{z_0}\rangle_\ell\big). \tag{1.6}$$

where $\langle z, w\rangle_\ell = -\sum_{j=1}^{\ell} z_j w_j + \sum_{j=\ell+1}^{n} z_j w_j$.

For simplicity, we only consider the case where $\ell = 0$.

Isotropic subgroup We define

$$Aut_0(\partial\mathbb{H}^n) = \{F \in Aut(\partial\mathbb{H}^n) \mid F(0) = 0\}.$$

which is called the *isotropic subgroup* of $Aut(\partial\mathbb{H}^n)$. It is known that any $F = (f, g) \in Aut_0(\partial\mathbb{H}^n)$ is of the form

$$f(z,w) = \frac{\lambda(z + \vec{a}w)U}{1 - 2i\langle z, \overline{\vec{a}}\rangle - (r + \langle \vec{a}, \overline{\vec{a}}\rangle)w},$$

$$g(z,w) = \frac{\sigma\lambda^2 w}{1 - 2i\langle z, \overline{\vec{a}}\rangle - (r + \langle \vec{a}, \overline{\vec{a}}\rangle)w}$$

where $\sigma = \pm 1, \lambda > 0, r \in \mathbb{R}, \vec{a} \in \mathbb{C}^{n-1}$, U is an $(n-1)\times(n-1)$ matrix satisfying $\langle zU, \overline{z}\overline{U}\rangle = \sigma\langle z, \overline{z}\rangle$, $\forall z \in \mathbb{C}^n$.

Here we verify such $(f, g) \in Aut_0(\partial\mathbb{H}^n)$, i.e., to verify

$$Im(g) = |f|^2, \quad \forall Im(w) = |z|^2,$$

i.e. to verify that for any $Im(w) = |z|^2$,

$$\begin{aligned}&\frac{\sigma\lambda^2 w}{1 - 2i\langle z, \overline{\vec{a}}\rangle - (r + i\langle \vec{a}, \overline{\vec{a}}\rangle)w} - \frac{\sigma\lambda^2 \overline{w}}{1 + 2i\langle \overline{z}, \vec{a}\rangle - (r - i\langle \vec{a}, \overline{\vec{a}}\rangle)\overline{w}} \\ &= 2i\left|\frac{\lambda(z + \vec{a}w)U}{1 - 2i\langle z, \overline{\vec{a}}\rangle - (r + i\langle \vec{a}, \overline{\vec{a}}\rangle)w}\right|^2,\end{aligned}$$

i.e., to verify

$$\begin{aligned}&\sigma\lambda^2 w\big[1 + 2i\langle \overline{z}, \vec{a}\rangle - (r - i\langle \vec{a}, \overline{\vec{a}}\rangle)\overline{w}\big] - \sigma\lambda^2\overline{w}\big[1 - 2i\langle z, \overline{\vec{a}}\rangle - (r + i\langle \vec{a}, \overline{\vec{a}}\rangle)w\big] \\ &= 2i\big|\lambda(z + \vec{a}w)U\big|^2, \qquad \forall\ Im(w) = |z|^2.\end{aligned} \tag{1.7}$$

Notice

$$\big|\lambda(z + \vec{a}w)U\big|^2 = \langle \lambda(z + \vec{a}w)U,\ \lambda(\overline{z} + \overline{\vec{a}}\overline{w})\overline{U}\rangle.$$

Motivated from the equation $Im(w) = |z|^2$, we define the *weighted degree*:

$$deg(z^j) = deg(\overline{z}^j) = j \quad and \quad deg(w^k) = deg(\overline{w}^k) = 2k. \tag{1.8}$$

To prove the equality in (1.7), we first prove the equality involving all terms of weighted degree 2 (i.e., the $z^2, z\overline{z}, \overline{z}^2, w$ and $\overline{w}$ terms) in (1.7):

$$\sigma\lambda^2 w - \sigma\lambda^2\overline{w} = 2i\lambda^2\langle zU,\ \overline{z}\overline{U}\rangle, \quad \forall\ Im(w) = |z|^2.$$

Since U is unitary, we need to show

$$\sigma\lambda^2 w - \sigma\lambda^2\overline{w} = 2i\lambda^2\sigma\langle z,\ \overline{z}\rangle, \quad \forall\ Im(w) = |z|^2,$$

which is true.

Secondly, we prove the equality involving all terms of weighted degree 3 (i.e., the $zw, \overline{zw}$, $\overline{z}w$ and $z\overline{w}$ terms) in (1.7):

$$\sigma\lambda^2 w 2i\langle\overline{z},\vec{a}\rangle - \sigma\lambda^2\overline{w}(-2i)\langle z,\overline{\vec{a}}\rangle = 2i\langle\lambda z U, \lambda\overline{\vec{a}}\overline{w}\overline{U}\rangle + 2i\langle\lambda\vec{a}wU,\ \lambda\overline{z}\overline{U}\rangle, \ \ \forall\ Im(w) = |z|^2,$$

Since U is unitary, the above is equivalent to

$$\sigma\lambda^2 w 2i\langle\overline{z},\vec{a}\rangle - \sigma\lambda^2\overline{w}(-2i)\langle z,\overline{\vec{a}}\rangle = 2i\lambda^2\sigma\langle z,\overline{\vec{a}}\overline{w}\rangle + 2i\lambda^2\sigma\langle\vec{a}w,\ \overline{z}\rangle, \ \ \forall\ Im(w) = |z|^2,$$

which is true.

Finally we prove the equality involving all terms of weighted degree 4(i.e., the $w\overline{w}$ terms) in (1.7), which is the highest weighted degree case:

$$-\sigma\lambda^2(r - i|\vec{a}|^2)|w|^2 + \sigma\lambda^2(r + i|\vec{a}|^2)|w|^2 = 2i\langle\lambda\vec{a}wU, \lambda\overline{\vec{a}}\overline{w}\overline{U}\rangle, \quad \forall\ Im(w) = |z|^2_\ell,$$

Since U is unitary, divided by $|w|^2$, the above is equivalent to

$$-\sigma\lambda^2(r - i|\vec{a}|^2) + \sigma\lambda^2(r + i|\vec{a}|^2) = 2i\sigma\lambda^2|\vec{a}|^2 \quad \forall\ Im(w) = |z|^2,$$

which is true.

1.2 Levi nondegenerate real hypersurfaces

Defining functions Let M be a smooth real hypersurface of $\mathbb{C}^n$, i.e., M is a subset of $\mathbb{C}^n$ such that for any point $p \in M$ there exists a neighborhood U of p and a smooth real-valued function r defined in U such that

$$M \cap U = \{(z, w) \in U \cap (\mathbb{C}^{n-1} \times \mathbb{C}) \mid r(z, w, \overline{z}, \overline{w}) = 0\}$$

with $dr \neq 0$ in U. The function r is called a *defining function* of M at p. Notice that defining function is not unique. Any hr is also a defining function where h is smooth real-valued function without zero.

[Example 1.2 A] For $\partial\mathbb{H}^n$, we can take a defining function

$$r(z, w, \overline{z}, \overline{w}) = Im(w) - |z|^2 = \frac{w - \overline{w}}{2i} - \sum_{j=1}^{n-1} z_j \overline{z_j}. \quad \Box$$

[Example 1.2 B] If $r(z, \overline{z})$ is real analytic near $0 \in \mathbb{C}^n$, we can write it as a power series

$$r(z, \overline{z}) = \sum_{\alpha,\beta} c_{\alpha,\beta} z^\alpha \overline{z^\beta}$$

where $\alpha = (\alpha_1, ..., \alpha_n)$, $z^\alpha = z_1^{\alpha_1} \cdot ... \cdot z_n^{\alpha_n}$, and $\overline{z}^\alpha = \overline{z_1}^{\alpha_1} \cdot ... \cdot \overline{z_n}^{\alpha_n}$. Then r is real-valued if and only if

$$\sum_{\alpha,\beta} c_{\alpha,\beta} z^\alpha \overline{z^\beta} = \sum_{\alpha,\beta} \overline{c_{\alpha,\beta}}\, \overline{z^\alpha} z^\beta, \quad \forall z \ near \ 0$$

i.e.,

$$r(z, \overline{z}) = \overline{r}(\overline{z}, z), \quad \forall z \ near \ 0,$$

i.e.,

$$c_{\alpha\beta} = \overline{c_{\beta,\alpha}}, \quad \forall \alpha, \beta. \tag{1.9}$$

$\Box$

We denote by TM the *tangent bundle* of M, and by $\mathbb{C}TM = \mathbb{C} \otimes TM$ the *complexification of the tangent bundle* of M. We define

$$T^{1,0}M = \mathbb{C}TM \cap T^{1,0}\mathbb{C}^n,$$

which is called the *bundle of* $(1, 0)$ *vectors on* M. Similarly we can define $T^{0,1}M = \mathbb{C}TM \cap T^{0,1}\mathbb{C}^n$.

First, after a local change of coordinates, we assume that

$$p = 0, \ T_0M = \{v = 0\}, \ T_0^{(1,0)}M = \{w = 0\},$$

where we use $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ for the coordinates of $\mathbb{C}^n$ and write $w = u + iv$. Then M near 0 is the graph of the function

$$v = \rho(z, \overline{z}, u) \quad with \ \ \rho(0) = 0 \ and \ d\rho(0) = 0.$$

Since ρ is real-valued, by (1.9), we can write ρ as Taylor series

$$\rho = \sum_{k,\ell=1}^{n-1} a_{k\bar{\ell}} z_k \overline{z_\ell} + \sum_{k,\ell=1}^{n-1} b_{k\ell} z_k z_\ell + \sum_{k,\ell=1}^{n-1} \overline{b_{k\ell}}\, \overline{z_k z_\ell} + \sum_{k=1}^{n-1} b_k z_k u + \sum_{k=1}^{n-1} \overline{b_k z_k} u + cu^2 + O(3),$$

where $c \in \mathbb{R}$, $O(3) = O(|(z,w)|^3) = O(|(z,u)|^3)$, $A := (a_{k\overline{\ell}})$ is a Hermitian matrix. Then $v = \rho(z, \overline{z}, u)$ can be written as:

$$Im(w) = 2Re\left(\sum_{k,\ell=1}^{n-1} b_{k\ell} z_k z_\ell + \sum_{k=1}^{n-1} b_k z_k u\right) + \sum_{k,\ell=1}^{n-1} a_{k\overline{\ell}} z_k \overline{z_\ell} + cu^2 + O(3).$$

Since $Re(z) = Im(iz)$, the above becomes

$$Im(w) = 2Im\left(i\sum_{k,\ell=1}^{n-1} b_{k\ell} z_k z_\ell + i\sum_{k=1}^{n-1} b_k z_k u\right) + \sum_{k,\ell=1}^{n-1} a_{k\overline{\ell}} z_k \overline{z_\ell} + cu^2 + O(3),$$

i.e.,

$$Im\left(w - 2i\sum_{k,\ell=1}^{n-1} b_{k\ell} z_k z_\ell - 2i\sum_{k=1}^{n-1} b_k z_k u\right) - cu^2 = \sum_{k,\ell=1}^{n-1} a_{k\overline{\ell}} z_k \overline{z_\ell} + O(3).$$

Since $w = u + iv = u + i\rho(z, \overline{z}, u) = u + O(2)$, we have $u = w + O(2) = \overline{w} + O(2)$ and $u^2 = u(w + O(2)) = uw + O(3) = w^2 + O(3) = \overline{w}^2 + O(3)$ so that $u^2 = \frac{w^2 + \overline{w}^2}{2} + O(3) = Re(w^2) + O(3) = Im(iw^2, O(3))$ and that

$$Im\left(w - 2i\sum_{k,\ell=1}^{n-1} b_{k\ell} z_k z_\ell - 2i\sum_{k=1}^{n-1} b_k z_k w - icw^2\right) = \sum_{k,\ell=1}^{n-1} a_{k\overline{\ell}} z_k \overline{z_\ell} + O(3),$$

Then we define a local holomorphic coordinate change

$$\begin{cases} z' = z, \\ w' = w - 2i\sum_{k,\ell=1}^{n-1} b_{k\ell} z_k z_\ell - 2i\sum_{k=1}^{n-1} b_k z_k w - ciw^2, \end{cases}$$

In the (z', w') coordinates, M can be expressed as the graph of the following function:

$$v' = \sum a_{k'\overline{l'}} z'_k \overline{z'_l} + O(|(z', w')|^3) = z' A \overline{z'}^t + O(|(z', w')|^3)$$

where $A = (a_{k'\overline{l'}}) = \overline{A}^t$ is a Hermitian $(n-1) \times (n-1)$ matrix and $w' = u' + iv'$. Write

$$A = \Gamma \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_{n-1} \end{pmatrix} \overline{\Gamma}^t = \Gamma \Lambda \overline{\Gamma}^t$$

where Γ is a certain non-singular $(n-1)\times(n-1)$ matrix and Λ is a diagonal matrix. Then

$$v' = z'\Gamma\Lambda\overline{(z'\Gamma)}^t + O(|(z',w')|^3).$$

Let

$$\begin{cases} z'' = z'\Gamma, \\ w'' = w'. \end{cases}$$

We have

$$v'' = \sum_{j=1}^{n-1} \lambda_j |z_j''|^2 + O(|(z'',w'')|^3)$$

where $w'' = u'' + iv''$. We say that $p = 0$ is a *Levi nondegenerate* point of M if $\lambda_j \neq 0$ for each j (cf. Example 1.2 B). [1]

Assume in what follows that M is Levi nondegenerate at 0. Then without loss of generality, we can assume that

$$v'' = \sum_{j=1}^{n-1} \epsilon_j \left| \sqrt{|\lambda_j|} z_j'' \right|^2 + O(|(z'',w'')|^3),$$

where $\epsilon_j = -1$ if $j \leq \ell$; and $\epsilon_j = 1$ if $j \geq \ell+1$. Let

$$\begin{cases} z_j''' = \sqrt{|\lambda_j|} z_j'', \\ w''' = w''. \end{cases}$$

Then in the (z''', w''') coordinates, M is the graph of the following function:

$$v''' = \sum_{j=1}^{n-1} \epsilon_j |z_j'''|^2 + O(3).$$

Still write z for z''' and w for w'''. Then by changing some order of indices, M is defined by:

$$v = |z|_\ell^2 + O(|(z,w)|^3), \tag{1.10}$$

where

$$|z|_\ell^2 := -\sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n-1} |z_j|^2. \tag{1.11}$$

[1] The most simplest real hypersurface is a hyperplane $M = \{(z,w) \in \mathbb{C}^{n-1}\times\mathbb{C} \mid Im(w) = 0\}$, but it is not interesting. We will focus on Levi nondegenerate real hypersurfaces.

In the above expression and for the rest of this section, when $\ell = 0$, we regard the first term after the equality sign to be zero. Replacing (z, w) by $(z_{\ell+1}, \cdots, z_{n-1}, z_1, \cdot, z_\ell, -w)$ if necessary, we can assume that $\ell \leq \frac{n-1}{2}$. The integer ℓ (sometimes the pair $(\ell, n-1-\ell)$) is called the *signature* of M at 0, which is a holomorphic invariant.

Therefore, among all Levi nondegenerate real hypersurfaces in $\mathbb{C}^n$, the nondegenerate hyperquadrics $\partial\mathbb{H}^n_\ell$ in Example 1.1 B above are the most simplest one.

1.3 Segre family and Segre variety

Let $M \subset \mathbb{C}^n$ be a local real analytic hypersurface containing 0. Let U be a small neighborhood of 0 in $\mathbb{C}^n$, and $M = \{z \in U \mid r(z, \overline{z}) = 0\}$, with dr never vanish, where $r(z, \overline{z})$ is a real analytic function defined on U.

Let $\mathcal{M} = \{(z, \zeta) \in U \times Conj(U) \mid r(z, \zeta) = 0\}$ be *Segre family* of M —– the complexification of M—–which is also complex manifold of complex dimension $2n-1$ in $\mathbb{C}^n \times \mathbb{C}^n$, where $Conj(U) = \{\overline{z} \mid z \in U\}$. Here we may shrink U if necessary, so that the power series $r(z, \zeta)$ is convergent. Sometimes, we denote it as $\mathcal{M} = \{(z, w) \in U \times U \mid r(z, \overline{w}) = 0\}$.

Write r as a local power series near 0:

$$r(z, \overline{z}) = \sum_{I,J} r_{IJ} z^I \overline{z}^J. \tag{1.12}$$

Since r is real-valued, we have

$$r(z, \overline{z}) = \overline{r(z, \overline{z})} = \overline{r}(\overline{z}, z), \quad \forall z \tag{1.13}$$

which implies

$$r_{IJ} = \overline{r_{JI}}, \quad \forall I, J. \tag{1.14}$$

and then

$$r(z, \overline{w}) = \overline{r(w, \overline{z})} = \overline{r}(\overline{w}, z). \tag{1.15}$$

Lemma 2.3 *(i) $\mathcal{M}$ is independent of the choice of the defining function r of M.*

(ii) A function holomorphic on $\mathcal{M}$ which vanishes on M also vanishes on any connected open subset of $\mathcal{M}$ which contains a point of M.

(iii) Let $f : U \to U'$ be a biholomorphic map where $U, U' \subset \mathbb{C}^{n+1}$ are open subsets. Suppose f maps M into another real hypersurface M' with real analytic defining function r'. Denote $\mathcal{M}$ and $\mathcal{M}'$ be the corresponding Segre families of M and M', respectively.

Denote $F(z,\overline{w}) := (f(z), \overline{f}(\overline{w}))$, *called the analytic continuation. Then* $F(\mathcal{M}) \subseteq \mathcal{M}'$. *When* $f(M) = M'$, *we have* $F(\mathcal{M}) = \mathcal{M}'$.

Proof (i) If r' is another defining function of M, then $r'(z,\overline{z}) = s(z,\overline{z})r(z,\overline{z})$, where s is some real analytic function on U which never vanish on U, where U is sufficiently small.

(ii) Consider the power series of $r(z,\overline{z})$ and $r(z,\zeta)$ and use the property of real analytic functions.

(iii) We have

$$r'(f(z), \overline{f}(\overline{w})) = s(z,\overline{w})r(z,\overline{w}) \tag{1.16}$$

with $s \neq 0$ as in the proof of (i). From this (iii) follows. □

[**Example 1.3 A**] Let $\partial\mathcal{H}^n$ be the Segre family of $\partial\mathbb{H}^n$. Let us consider the automorphism group $Aut(\partial\mathcal{H}^n)$. It is proved in [HJ07] that if Φ is a local holomorphic Segre self-isomorphism of $(\partial\mathcal{H}^n, 0)$, then Φ is of the following form:

$$\begin{aligned} F(z,\xi) &= (S(z), T(\xi) = (S_1(z), ..., S_{n-1}(z), S_n(z), T_1(\xi), ..., T_{n-1}(\xi), T_n(\xi)) \\ &= (\widetilde{S}(z), S_n(z), \widetilde{T}(\xi), T_n(\xi)) \end{aligned}$$

where

$$\widetilde{S}(z) = \frac{\lambda(z' + \vec{a}w)U}{1 - 2i\langle z', \vec{e}\rangle + e_n w}, \quad S_n(z) = \frac{\lambda w}{1 - 2i\langle z', \vec{e}\rangle + e_n w}, \tag{1.17}$$

$$\widetilde{T}(\xi) = \frac{(\xi' + \vec{e}\eta)V}{1 + 2i\langle \xi', \vec{a}\rangle + (e_n + 2i\langle \vec{e}, \vec{a}\rangle)\eta}, \tag{1.18}$$

$$T_n(\xi) = \frac{\lambda\eta}{1 + 2i\langle \xi', \vec{a}\rangle + (e_n + 2i\langle \vec{e}, \vec{a}\rangle)\eta}; \tag{1.19}$$

where U, V are non-singular $(n-1)\times(n-1)$ matrices of complex numbers with $U \cdot V^t = \mathrm{Id}$, $\vec{a} = (a_1, ..., a_{n-1})$, $\vec{e} = (e_1, ..., e_{n-1}) \in \mathbb{C}^{n-1}$, $\lambda \in \mathbb{C}^*$, $e_n \in \mathbb{C}$, $\langle \vec{x}, \vec{y}\rangle = \vec{x}\cdot\vec{y}^t$ for any $\vec{x}, \vec{y} \in \mathbb{C}^{n-1}$. Also, F is uniquely determined by the data $\lambda, \vec{a}, \vec{e}, e_n, U$. □

Let $M \subset \mathbb{C}^n$ be a local smooth real hypersurface such that $M \cap U = \{z \in U \mid r(z,\overline{z}) = 0\}$ where r is a defining function. For any $w \in U$, we define its *Segre variety* with respect to M by

$$Q_w := \{z \in U \mid r(z,\overline{w}) = 0\}.$$

[**Example 1.3 B**] Consider Heisenberg hypersurface $M = \partial\mathbb{H}^n$ which is defined by

$$r(z,\overline{z}) := \frac{w - \overline{w}}{2i} - \sum_{j=1}^{n-1} |z_j|^2.$$

Let $p=(z_0,w_0)\in\partial\mathbb{H}^n$. Then the Segre variety

$$Q_p=\Big\{(z,w)\in\mathbb{C}^{n-1}\times\mathbb{C}\mid \frac{w-\overline{w_0}}{2i}-\sum_{j=1}^{n-1}z_j\overline{z_{0j}}=0\Big\}.$$

Q_p is a complex hyperplane, which can be identified with the holomorphic tangent space to $\partial\mathbb{H}^n$ at p. When $p=0$, $Q_0=\{(z,0)\in\mathbb{C}^{n-1}\times\mathbb{C}\}$. Locally p determines Q_p; conversely, Q_p determines p uniquely.

The most important property for Q_w is its invariance property.

Proposition 3.3 *(1)* $r(z,\overline{w})=\overline{r}(\overline{w},z)=\overline{r(w,\overline{z})}$.

(2) $z\in Q_w\Leftrightarrow w\in Q_z$.

(3) $z\in M\Leftrightarrow z\in Q_z$.

(4) Q_z is invariant under local biholomorphisms, i.e., if f is biholomorphic map such that $f(M)=M'$, then $f(Q_w)=Q'_{f(w)}$.

Proof (1) Since r is real, $r(z,\overline{w})=\sum a_{IJ}z^I\overline{w}^J=\sum\overline{a_{IJ}z^I}w^J$, where $a_{IJ}=\overline{a_{JI}},\forall I,J$. Then $\overline{r}(\overline{w},z)=\sum\overline{a_{IJ}\overline{w^I}z^J})=\sum a_{JI}z^J\overline{w^I}=\sum a_{IJ}z^I\overline{w^J}$ and $\overline{r(w,\overline{z})}=\overline{\sum a_{IJ}w^I\overline{z^J}}=\sum\overline{a_{IJ}w^I}z^J=\sum a_{JI}z^J\overline{w^I}=\sum a_{IJ}z^I\overline{w^J}$.

(2) We apply (1) to see $z\in Q_w\Leftrightarrow r(z,\overline{w})=0=\overline{r(w,\overline{z})}\Leftrightarrow w\in Q_z$.

(3) $z\in M\Leftrightarrow r(z,\overline{z})=0\Leftrightarrow z\in Q_z$.

(4) Write $M=\{z\mid r(z,\overline{z})=0\}$, $M'=\{z'\mid r'(z',\overline{z'})=0\}$, $z'=f(z)$ and $w'=f(w)$. Assume that $r=f\circ r'$ is a defining function of M. Then $Q_{f(w)}=\{z'\mid r'(z',\overline{f(w)})=0\}=\{f(z)\mid r'(f(z),\overline{f(w)})=0\}=f(\{z\mid r(z,\overline{w})=0\})=f(Q_w)$. □

1.4 CR manifolds

Foundation of CR geometry CR geometry originated from a work by Poincaré in 1907 below. N. Tanaka [T62] extended this result to high dimensional case.

Theorem 1.4.1 *(Poincaré [P07]) Any non-constant holomorphic map $f:U\to V$ satisfying $f(U\cap\partial\mathbb{B}^2)\subset V\cap\partial\mathbb{B}^2$ is a map in $Aut(\partial\mathbb{B}^2)$, where U,V are open subsets of $\mathbb{C}^2$.*

Proof:(Sketch) Assume the local map f is biholomorphic, otherwise shrinking U. Since $f(Q_w) \subset Q_{f(w)}$ where Q_w is the Segre variety of $\partial\mathbb{B}^n$, f maps hyperplanes into hyperplanes. By the fundamental theorem of classical projective geometry, f must be projective linear transformation between $\mathbb{CP}^n$. Therefore f must be linear fractional. □

Poincaré-Tanaka theorem could be regarded as a CR analogue of the following classical Liouville's Conformality Theorem. In the Euclidean space $\mathbb{E}^n$ with $n \geq 3$, the only conformal mappings are inversions, similarity transformations, and congruence transformations. More precisely, let U, V be open subsets in $\mathbb{R}^n$ with $n \geq 3$, equipped with the flat metric ω, and $f : U \to V$ a smooth map. Then f is conformal (i.e., if $f^*(\omega) = e^u\omega$ for some continuous function u) if and only if f is a Mobius transformation: A composition of the following type of transformations: (i) translations, (ii) rotations, (iii) scalings and inversions.

By E. Cartan [Ca32]-Chern-Moser[CM74]'s work, complete invariants for local Levi non-degenerate real hypersurfaces are constructed.

These two pieces of work laid down the foundation of CR geometry.

CR manifolds

[Example 1.4 A] Let M be a smooth real hypersurface in $\mathbb{C}^n$. For any $p \in M$, we define a complex vector space

$$\mathcal{V}_p := \mathbb{C}T_pM \cap T_p^{0,1}\mathbb{C}^n.$$

The complex dimension $\dim_{\mathbb{C}} \mathcal{V}_p = n - 1$ for any point $p \in M$. Then $\mathcal{V} = \cup_{p\in M}\mathcal{V}_p$ defines a subbundle of $\mathbb{C}TM$ satisfying $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$ and $\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}$ where $\overline{\mathcal{V}} =: \mathbb{C}TM \cap T^{1,0}$.

Such M is called a CR manifold in $\mathbb{C}^n$ with CR dimension $n - 1$. The bundle $\mathcal{V}$ is called a *CR structure (bundle)* on the manifold M. The complex dimension $\dim_{\mathbb{C}} \mathcal{V}_p$, independent of p, is called the *CR dimension*. A section of $\mathcal{V}$ is called a *CR vector field* over M.

Let us find a basis of CR vectors fields over M as follows.

Recall a real hypersurface M in $\mathbb{R}^n$ defined by $\rho(x) = 0$. Let $\gamma : [0, 1] \to M$, $t \mapsto \gamma(t) = (\gamma_1(t), ..., \gamma_n(t))$, be any curve insider M. Then $\rho(\gamma(t)) = 0$, $\forall t \in [0, 1]$. By the chain rule, $\sum_{j=1}^n \frac{\partial\rho}{\partial x_j}\frac{d\gamma_j}{dt} = 0$, $\forall t \in [0, 1]$. Then the vector $(\frac{\partial\rho}{\partial x_1}, ..., \frac{\partial\rho}{\partial x_n}) \perp T(M)$, a normal vector. Let $L = \sum_{j=1}^n b_j \frac{\partial}{\partial x_j}$. Then $\sum_{j=1}^n b_j \frac{\partial\rho}{\partial x_j} = 0$ iff $(b_1, ..., b_n) \perp (\frac{\partial\rho}{\partial x_1}, ..., \frac{\partial\rho}{\partial x_n})$ iff L is a tangent vector of M.

Now consider a real hypersurface M in $\mathbb{C}$ defined by $\rho(z, \overline{z}) = 0$. We regard $\mathbb{C}^n = \mathbb{R}^{2n}$ and $(z, \overline{z})$ as a basis of vectors of $\mathbb{R}^{2n}$ over the field $\mathbb{R}$. Let $L = \sum_{j=1}^n b_j \frac{\partial}{\partial z_j} + \sum_{k=1}^n c_k \frac{\partial}{\partial \overline{z_k}}$. Then L is a tangent vector of M if and only if

$$\sum_{j=1}^n b_j \frac{\partial\rho}{\partial z_j} + \sum_{k=1}^n c_k \frac{\partial\rho}{\partial \overline{z_k}} = 0.$$

Consequently, for a (1,0)-vector $L_1 = \sum_{j=1}^n b_j \frac{\partial}{\partial z_j}$, it is a tangent vector of M if and only if

$$\sum_{j=1}^n b_j \frac{\partial \rho}{\partial z_j} = 0.$$

For a (0,1)-vector $L_2 = \sum_{k=1}^n c_k \frac{\partial}{\partial \overline{z_k}}$, it is a tangent vector of M if and only if

$$\sum_{k=1}^n c_k \frac{\partial \rho}{\partial \overline{z_k}} = 0. \tag{1.20}$$

Let M locally be defined by $\rho = v - \phi(z, \overline{z}, u) = 0$ near 0 where (z, w) is holomorphic coordinates of $\mathbb{C}^n$ and $w = u + iv$. Define

$$\overline{L_j} = \frac{\partial}{\partial \overline{z}_j} - 2i \frac{\phi_{\overline{z_j}}}{1 + i\phi_u} \frac{\partial}{\partial \overline{w}}, \quad 1 \le j \le n \tag{1.21}$$

where we denote $\phi_{\overline{z_j}} = \frac{\partial \phi}{\partial \overline{z_j}}$ and $\phi_u = \frac{\partial \phi}{\partial u}$. In fact, as we did in (1.20), we just need to verify that $\overline{L_j}(\rho) = 0$ where $\rho = \frac{w - \overline{w}}{2i} - \phi(z, \overline{z}, \frac{w + \overline{w}}{2})$. Then

$$\begin{aligned} \overline{L_j}(\rho) &= \left(\frac{\partial}{\partial \overline{z}_j} - 2i \frac{\phi_{\overline{z_j}}}{1 + i\phi_u} \frac{\partial}{\partial \overline{w}} \right) \left(\frac{w - \overline{w}}{2i} - \phi(z, \overline{z}, \frac{w + \overline{w}}{2}) \right) \\ &= -\phi_{\overline{z_j}} - 2i \frac{\phi_{\overline{z_j}}}{1 + i\phi_u} \left(-\frac{1}{2i} - \phi_u \frac{1}{2} \right) = 0. \end{aligned}$$

$\{\overline{L}_1, ..., \overline{L_{n-1}}\}$ form a basis for the CR bundle $\mathcal{V}$. □

A *CR manifold* is a differentiable manifold together with a geometric structure modeled on that of a real hypersurface in $\mathbb{C}^n$. More precisely, a CR manifold is a differentiable manifold M together with a subbundle $\mathcal{V}$ of the complexified tangent bundle $\mathbb{C}TM = TM \otimes \mathbb{C}$ such that

$$[\mathcal{V}, \mathcal{V}] \subseteq \mathcal{V}, \quad and \quad \mathcal{V} \cap \overline{\mathcal{V}} = \{0\}.$$

The bundle $\mathcal{V}$ is called a *CR structure* on the manifold M. $\mathcal{V} \oplus \overline{\mathcal{V}}$ is called the *complex tangent bundle* of M. The complex dimension $\dim_{\mathbb{C}} \mathcal{V}_p$, independent of p, is called the *CR dimension*. A section of $\mathcal{V}$ is called a *CR vector field* over M. A C^1 - smooth function f is called a *CR function* if it locally annihilated by any CR vector field. A *CR mapping* is a smooth mapping F between CR manifolds $(M, \mathcal{V}_M)$ and $(N, \mathcal{V}_N)$ such that $df(\mathcal{V}_M) \subseteq (\mathcal{V}_N)$.

[Example 1.4 B] Let $M = \partial\mathbb{H}^n \subset \mathbb{C}^n$ be the Heisenberg hypersurface. We can take a defining function

$$\rho(z) = Im(w) - |z|^2 = \frac{w - \overline{w}}{2i} - \sum_{j=1}^{n-1} |z_j|^2$$

of $\partial\mathbb{H}^n$. From Example 1.4 A, $\phi = |z|^2$ so that $\phi_{\overline{z_j}} = z_j$ and $\phi_u = 0$, and that

$$\overline{L_j} := \frac{\partial}{\partial \overline{z_j}} - 2iz_j \frac{\partial}{\partial \overline{w}}, \quad 1 \le j \le n-1$$

be a basis of $\mathcal{V} = \mathbb{C}T^{0,1}(\partial\mathbb{H}^n)$, and

$$L_j := \frac{\partial}{\partial z_j} + 2i\overline{z_j} \frac{\partial}{\partial w}, \quad 1 \le j \le n-1$$

be a basis of $\overline{\mathcal{V}} = \mathbb{C}T^{1,0}(\partial\mathbb{H}^n)$.

Also, from Example 1.4 A, the following vector field

$$T = \frac{\partial}{\partial w} + \frac{\partial}{\partial \overline{w}}$$

is a tangent vector field of M. Such T is a real vector, i.e., $\overline{T} = T$. T is called a Reeb vector field. The vector fields $\{L_j, \overline{L_j}, T\}_{1 \le j \le n-1}$ form a basis of the tangent vector bundle $T(M)$. $\square$

[Example 1.4 C] Let M be a smooth real submanifold in $\mathbb{C}^n$ of real codimension d. If $d = 1$, it is the hypersurface case (see Example 1.4 A). Let us consider $d > 1$. Then for any point $p \in M$, there is an open subset U of $\mathbb{C}^n$ such that

$$M \cap U = \{z \in U \mid \rho_1(z, \overline{z}) = 0, ..., \rho_d(z, \overline{z}) = 0\}$$

where $\rho = (\rho_1, ..., \rho_d)$ is a real-valued smooth function defined on U such that $d\rho_1(z)$, ..., $d\rho_d(z)$ are linearly independent $\forall z \in U$. [2]

We define a complex vector space

$$\mathcal{V}_p := \mathbb{C}T_pM \cap T_p^{0,1}\mathbb{C}^n.$$

[2] To apply the Implicit Function Theorem, one needs the condition " $d\rho_1(z)$, ..., $d\rho_d(z)$ are linearly independent", i.e., $d\rho_1(z) \wedge ... \wedge d\rho_d(z) \neq 0$, $\forall z$. Notice

$$d\rho_1 \wedge ... \wedge d\rho_d \neq 0 \quad \Leftarrow \ but \ \not\Rightarrow \quad \partial\rho_1 \wedge ... \wedge \partial\rho_d \neq 0$$

If $\partial\rho_1 \wedge ... \wedge \partial\rho_d \neq 0$ holds, we say that M is *generic*.

When $d = 1$, the complex dimension $\dim_{\mathbb{C}} \mathcal{V}_p = n - 1$ is independent of $p \in M$. However, when $d > 1$, $\dim_{\mathbb{C}} \mathcal{V}_p$ may depend on $p \in M$. Let us put a condition that $\dim_{\mathbb{C}} \mathcal{V}_p = constant$. Then $\mathcal{V} = \cup_{p \in M} \mathcal{V}_p$ defines a subbundle of $\mathbb{C}TM$ satisfying $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$ and $\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}$. Then M is a CR manifold in $\mathbb{C}^n$ with CR dimension $\dim_{\mathbb{C}} \mathcal{V}_p$. $\square$

Remarks:

1. A CR manifold defined in Example 1.4 A or 1.4 C is called an *embedded CR manifold*, while a CR manifold (without mention of $\mathbb{C}^n$) is called an *abstract CR manifold*.

2. For an abstract CR manifold M, when the CR dimension $= n - 1$, or codimension 1, M is called a *CR manifold of hypersurface type*.

3. For any CR manifold, the complex tangent bundle $\mathcal{V} \oplus \overline{\mathcal{V}}$ is a subbundle of complex codimensional d in $\mathbb{C}TM$.

4. For a CR manifold $M \subset \mathbb{C}^n$ as in Example 1.4 A, its CR dimension can be calculated by the following formula:

$$\dim_{\mathbb{C}} \mathcal{V}_p = n - rank_{\mathbb{C}} \left(\frac{\partial \rho_k}{\partial \overline{z_j}}(p, \overline{p}) \right)_{1 \le j \le n, 1 \le k \le d}. \tag{1.22}$$

   In particular, by the formula above, for a real hypersurface $M = \{\rho(z, \overline{z}) = 0\}$, its CR dimension

$$\dim_{\mathbb{C}} \mathcal{V}_p = n - rank_{\mathbb{C}} \left(\frac{\partial \rho}{\partial \overline{z_j}}(p, \overline{p}) \right)_{1 \le j \le n} = n - 1$$

   always holds.

[Example 1.4 D] Let M be a complex manifold. Let $\mathcal{V} = T^{0,1}M$ be a subbundle of $\mathbb{C}TM$. Then the CR dimension $= n$, $\mathcal{V} + \overline{\mathcal{V}} = \mathbb{C}TM$ and

1. $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$

2. $\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}$

hold so that M is a CR manifold with CR dimension n. f is a CR function $\Leftrightarrow \overline{T}(f) = 0$ for any CR vector filed $T \Leftrightarrow \frac{\partial f}{\partial \overline{z}_j} = 0 \ \forall j \Leftrightarrow f$ is a holomorphic function. $\square$

[Example 1.4 E] A CR manifold M with CR dimension 0 is called *totally real*. For example, $M = \mathbb{R} \times \mathbb{R} \subset \mathbb{C}^2$. Its defining functions cane taken as

$$\rho_1 = y_1 = \frac{z_1 - \overline{z_1}}{2i}, \quad \rho_2 = y_2 = \frac{z_2 - \overline{z_2}}{2i}.$$

Then $\frac{\partial \rho_1}{\overline{z_1}} = -\frac{1}{2i}$, $\frac{\partial \rho_1}{\overline{z_2}} = 0$, $\frac{\partial \rho_2}{ovz_1} = 0$, $\frac{\partial \rho_2}{\overline{z_2}} = -\frac{1}{2i}$ so that its CR dimension can be calculated by

$$\dim_{\mathbb{C}} \mathcal{V}_p = 2 - rank_{\mathbb{C}} \begin{pmatrix} -\frac{1}{2i} & 0 \\ 0 & -\frac{1}{2i} \end{pmatrix} = 2 - 2 = 0.$$

By the definition, any C^1 function over M is CR function.

Contact form and Reeb vector field A real nonvanishing 1-form θ over M is called a *contact form* if $\theta \wedge (d\theta)^n \neq 0$. Let M be as above given by a defining function r. Then the 1-form $\theta = i\partial r$ is a contact form of M.

Associated with a contact form θ one has the *Reeb vector field* R_θ, defined by the equations: (i) $d\theta(R_\theta, \cdot) \equiv 0$, (ii) $\theta(R_\theta) \equiv 1$. As a skew-symmetric form of maximal rank $2n$, the form $d\theta|_{T_pM}$ has a 1- dimensional kernel for each $p \in M^{2n+1}$. Hence equation (i) defines a unique vector field R_θ on M. The unique real vector field is defined by the normalization condition (ii).

[Example 1.4 F] Let $M = \partial \mathbb{H}^n \subset \mathbb{C}^n$ be the Heisenberg hypersurface with the defining function $\rho(z) = -Im(w) + |z|^2 = -\frac{w - \overline{w}}{2i} + \sum_{j=1}^{n-1} |z_j|^2$. We can take a contact form θ to be [3]

$$\theta = -i\partial\rho = \frac{1}{2}dw - i\sum_{j=1}^{n-1} \overline{z_j} dz_j.$$

□

[3] The defining function ρ can have two choices with $\pm$ signs, and we chose it so that $\mathbb{H}^n = \{\rho < 0\}$. Also contact form θ can have two choices with $\pm$ signs, and we choose it so that $d\theta = ih_{\alpha\overline{\beta}} dz^\alpha \wedge \overline{dz^\beta}$ where $(h_{\alpha\overline{\beta}})$ is positive definite.

1.5 Levi forms

Levi form For a CR manifold $(M, \mathcal{V})$ and a point $p \in M$, its *Levi form* at p is a map (cf. [Bog91])

$$\begin{array}{rccc} h_p: & \overline{\mathcal{V}_p} & \rightarrow & \{T_p(M) \otimes \mathbb{C}\}/(\mathcal{V}_p \oplus \overline{\mathcal{V}}_p) \\ & v_p & \mapsto & \frac{1}{2i}\pi_p\{[v, \overline{v}]\} \end{array}$$

where v is any vector filed in $\overline{\mathcal{V}}$ that equals v_p at p, and $\pi_p : T_p(M)\otimes\mathbb{C} \rightarrow \{T_p(M)\otimes\mathbb{C}\}/(\mathcal{V}\oplus\overline{\mathcal{V}})$ is the natural projection. The definition of h_p is independent of choice of v.

If M is an embedded CR manifold, we can take $\overline{\mathcal{V}} = T^{1,0}(M)$ and identify the quotient space $\{T_p(M) \otimes \mathbb{C}\}/(\mathcal{V}_p \oplus \overline{\mathcal{V}}_p)$ with X_p, the complexified totall real part of the tangent bundle.

$$\begin{array}{rccc} h_p: & H_p^{1,0}(M) & \rightarrow & X_p(M) \\ & v_p & \mapsto & \frac{1}{2i}\pi_p\{[v, \overline{v}]\} \end{array}$$

It also regard the Levi form of an embedded CR manifold $M \subset \mathbb{C}^n$ as

$$\begin{array}{rccc} \widetilde{h}_p: & H_p^{1,0}(M) & \rightarrow & N_p(M) \\ & v_p & \mapsto & \frac{1}{2i}\widetilde{\pi}_p(J[\overline{v}, v])_p) \end{array}$$

where v is any $H^{1,0}(M)$-vector field extension of v_p, $N_p(M)$ is the normal space of M at p, J is the complex structure map for $T_p(\mathbb{C}^n)$, and $\widetilde{\pi}_p : T_p(\mathbb{C}^n) \mapsto N_p(M)$ is the orthogonal projection map.

Let $M = \{\rho = 0\}$ be a smooth real hypersurface. Let $p \in M$ and, by scaling, $|\bigtriangledown \rho(p)| = 1$ which is a unit base for $N_p(M)$. Then the Levi form is given by

$$\widetilde{h}_p(W) = -\sum_{j,k=1}^{n} \frac{\partial^2 \rho(p)}{\partial \zeta_j \partial \overline{\zeta_k}} w_j \overline{w_k} \bigtriangledown \rho(p), \quad \forall W = \sum_{k=1}^{n} w_k \frac{\partial}{\partial \zeta_k} \in H_p^{1,0}(M). \tag{1.23}$$

In this case, M is called *strictly pseudoconvex* at p if the Levi form at p is either positive definite or negative definite.

Levi form in terms of a contact form We could define Levi form in terms of a contact form θ.

Fixing a contact form θ, for (M, θ), we define the *Levi form*

$$h_\theta(v, w) := -d\theta(v, \overline{w}) = \theta([v, \overline{w}]), \quad \forall v, w \in \mathcal{V} \oplus \overline{\mathcal{V}}. \tag{1.24}$$

Here we used the Cartan formula

$$\langle d\theta, v \wedge \overline{w} \rangle = v\langle \theta, \overline{w} \rangle - \overline{w}\langle \theta, v \rangle - \langle \theta, [v, \overline{w}] \rangle.$$

and the fact that $\langle\theta, T\rangle = 0, \forall T \in \mathcal{V} \oplus \overline{\mathcal{V}}$ so that $\langle\theta, \overline{w}\rangle = \langle\theta, v\rangle = 0$. The Levi form of M can be regarded as a Hermitian 2-form, or a metric, on $\overline{\mathcal{V}} := T^{1,0}M$ defined by $h_\theta : T^{1,0}M \otimes T^{1,0}M \to \mathbb{C}$. (M, θ) is said to be *Levi nondegenerate at p* if $h_\theta(v_p, w_p) = 0$ for all w_p implies $v_p = 0$. (M, θ) is said to be *Levi nondegenerate* if h_θ is Levi nondegenerate at every point of M. (M, θ) is said to be *strongly pseudoconvex* if h_θ is positive definite (or *pseudoconvex* in case h_θ is positive semidefinite).

[**Example 1.5**] Let $M \subset \mathbb{C}^n$ be a smooth real analytic hypersurface. Locally we consider $M \cap U = \{z \in U \mid \rho(z, \overline{z}) = 0\}$ where U is an open subset of $\mathbb{C}^n$.

We choose a contact form θ to be

$$\theta := -i\partial\rho$$

so that from (1.24) we obtain $h_\theta(v, \overline{w}) = -\langle d\theta, v \wedge \overline{w}\rangle$, i.e.,

$$h_\theta = -d\theta = -i\overline{\partial}\partial\rho = i\partial\overline{\partial}\rho.$$

In particular, if $M = \partial\mathbb{H}^n$ and $\rho = -Im(w) + |z|^2$, we find

$$h_\theta = i\partial\overline{\partial}\big(- \frac{w - \overline{w}}{2} + \sum_{j=1}^{n-1} z_j\overline{z_j}\big) = i\partial(\frac{1}{2i}d\overline{w} + \sum_{j=1}^{n} z_j d\overline{z_j}) = i\sum_{j=1}^{n-1} dz_j \wedge d\overline{z_j}.$$

Then $(\partial\mathbb{H}^n, \theta)$ is strongly pseudoconvex.

1.6 Holomorphic extension of CR functions

Theorem 1.6.1 *(Bochner's Extension Theorem, 1943 [B43]) Let $\Omega \subset \mathbb{C}^n$ be a bounded open subset, $n > 1$, with C^∞ boundary $M := \partial\Omega$ and suppose that $\mathbb{C}^n - \overline{\Omega}$ is connected. If $f \in C^\infty(M)$ is a CR function, there is a unique function $F \in C^\infty(\overline{\Omega})$ such that $F|_M = f$ and F is holomorphic on Ω.*

Bochner's Extension Theorem is global. The first local version was proved by Lewy in 1956 (cf. [Bog91], p.198).

Let $M = \{z \in \mathbb{C}^n \mid \rho(z) = 0\}$ be a hypersurface where ρ is a C^k-smooth defining function with $d\rho \neq 0$ on M with $2 \leq k \leq \infty$. If ρ is scaled so that $| \bigtriangledown (p)\rho| = 1$, $\forall p \in M$. The Levi form of M at p is the map

$$W \mapsto \left(- \sum_{j,k=1}^{n} \frac{\partial^2\rho(p)}{\partial\zeta_j\partial\overline{\zeta_k}})w_j\overline{w_k}\right) \bigtriangledown \rho(p), \quad \forall W = \sum_{j=1}^{n} w_j\frac{\partial}{\partial\zeta_j} \in H_p^{1,0}(M).$$

When we speak of the eigenvalues of the Levi form of M at p, we are referring to the ones of the matrix $\left(\frac{\partial^2 \rho}{\partial \zeta_j \partial \overline{\zeta_k}}\right)$. Let $\Omega^+ = \{\rho > 0\}$ and $\Omega^- = \{\rho < 0\}$.

Theorem 1.6.2 *(Lewy extension theorem, [Bog91], p,198-199) Let $M \subset \mathbb{C}^n$ be a C^k-smooth real hypersurface with $3 \leq k \leq \infty$ and $n \geq 2$. Let $p \in M$ be a point.*

1. *If the Levi form of M at p has at least one positive eigenvalue, then for each open set ω in M with $p \in \omega$, there is an open set U in $\mathbb{C}^n$ with $p \in U$ such that for each C^1-smooth CR function f on ω, there is a unique function F which is holomorphic on $U \cap \Omega^+$ and continuous on $U \cap \overline{\Omega^+}$ such that $F|_{U \cap M} = f$.*
2. *If the Levi form of M at p has at least one negative eigenvalue, then the conclusion above holds with Ω^+ replaced by Ω^-.*
3. *If the Levi form of M at p has eigenvalues of opposite sign, then for each open set ω in M with $p \in \omega$, there is an open set U in $\mathbb{C}^n$ with $p \in U$ such that for each C^1-smooth CR function f on ω, there is a unique function F which is holomorphic on U such that $F|_{U \cap M} = f$.*

To illustrate the extension problem, we prove the following result.

Theorem 1.6.3 *Let $M \subset \mathbb{C}^{n+1}$ be a real analytic hypersurface, $p \in M$ and f a CR function in a neighborhood of p of M. Then the following two statements are equivalent:*
(1) f extends to a holomorphic function on a neighborhood of p in $\mathbb{C}^{n+1}$.
(2) f is real analytic in a neighborhood of p in M.

Proof: Locally we assume that M is given by the equation

$$v = \phi(z, \overline{z}, u)$$

where $z = (z_1, ..., z_n)$, $w = u + iv$, ϕ is real analytic with

$$\phi(0) = 0, d\phi(0) = 0.$$

We know that the map

$$(z, \overline{z}, u) \mapsto (z, w) = (z, u + i\phi(z, \overline{z}, u)) \tag{1.25}$$

is a parametrization of M with parameters $(z, \overline{z}, u) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{R}$.

From Example 1.4 A, we know that a local basis of CR vector fields is given by

$$\overline{L_j} = \frac{\partial}{\partial \overline{z}_j} - 2i \frac{\phi_{\overline{z_j}}}{1 + i\phi_u} \frac{\partial}{\partial \overline{w}}, \quad 1 \leq j \leq n, \tag{1.26}$$

where we denote $\phi_{\overline{z_j}} = \frac{\partial \phi}{\partial \overline{z_j}}$ and $\phi_u = \frac{\partial \phi}{\partial u}$.

Now we define

$$F(z, w) = f(z, \overline{z}, \zeta)$$

where ζ satisfies $\zeta + i\phi(z, \overline{z}, \zeta) = w$. Hence $\zeta = \zeta(z, \overline{z}, w)$ is uniquely determined by the equation with Implicit function theorem. [4] Also, by taking differentiation on the both sides of the equation $\zeta + i\phi(z, \overline{z}, \zeta) = w$, we obtain

$$\frac{\partial \zeta}{\partial \overline{z_j}} + i\phi_{\overline{z_j}} + i\phi_\zeta \frac{\partial \zeta}{\partial \overline{z_j}} = 0. \tag{1.27}$$

Also, we see $F|_M = f$ because $F(z, u + i\phi(z, \overline{z}, u)) = f(z, \overline{z}, u)$ for any $(z, w) \in M$.

To complete the proof, is suffices to prove that F is a holomorphic function.

Since $\zeta = \zeta(z, \overline{z}, w)$ is real analytic function without $\overline{w}$ terms, F is holomorphic in w. Then it is sufficient to prove that F is holomorphic for each z_j, $1 \leq j \leq n$.

In fact, for any j,

$$\begin{aligned}
\frac{\partial F}{\partial \overline{z_j}} &= \frac{\partial f}{\partial \overline{z_j}} + \frac{\partial f}{\partial \zeta} \frac{\partial \zeta}{\partial \overline{z_j}} \\
&= \frac{\partial f}{\partial \overline{z_j}} - \frac{i\phi_{\overline{z_j}}}{1 + i\phi_\zeta} \frac{\partial f}{\partial \zeta} \qquad (by\ (1.27)) \\
&= \overline{L_j} f \qquad (by\ the\ formula\ of\ \overline{L_j}\ above) \\
&= 0. \qquad (because\ f\ is\ a\ CR\ function)
\end{aligned}$$

The proof is complete. □

Let $F = (F_1, ..., F_n) : M \to N$ be a real analytic CR map between real analytic hypersurfaces $M, N \subset \mathbb{C}^{n+1}$. Since each F_j is CR function, by Theorem above, F extends holomorphically on a neighborhood of M.

[4] In general ζ may not be real-valued. But when $w \in M$, then $\zeta = Re(w)$ is real-valued.

1.7 Hopf Lemma

Lemma 1.7.1 *(Hopf lemma) Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with C^2 boundary, $a \in \Omega$, and $v(a)$ the inward normal to $\partial\Omega$ at a. Then for any subharmonic function u on Ω with $u < 0$ on Ω must satisfy*

$$\overline{lim}\frac{u(z)}{|z-a|} \leq -c$$

for some constant $c > 0$, where the limit superior is as $z \to a$ along $v(a)$.

Proof: Since $\partial\Omega$ is of C^2 smoothness, we can take a ball $B_R(z_0)$ with center z_0 and radius R in $\mathbb{C}^n$ such that it is tangent to $\partial\Omega$ at a and $B_R(z_0) \subset \Omega$. Such z_0 can be chosen in a fixed compact subset of Ω.

For any $0 < r < R$, define a function on $B_R(z_0) - \overline{B_r(z_0)}$:

$$g(z) := e^{-\lambda|z-z_0|^2} - e^{-\lambda R^2}.$$

When λ is sufficiently large compared to r, this function g is a subharmonic function. In fact, $\frac{\partial^2}{\partial\overline{z}_j\partial z_j}g = \frac{\partial}{\partial\overline{z}_j}\left(-\lambda(\overline{z}_j-\overline{z}_{0j})e^{-\lambda|z-z_0|^2}\right) = \lambda(\lambda|z_j-z_{0j}|^2-1)e^{-\lambda|z-z_0|^2} > 0$ holds $B_R(z_0)-\overline{B_r(z_0)}$ as $\lambda >> 0$.

Clearly $g = 0$ holds for $|z - z_0| = R$.

Since $u < 0$ on Ω, by taking sufficiently small $\varepsilon > 0$, $u + \varepsilon g \leq 0$ on the boundary of $B_R(z_0) - \overline{B_r(z_0)}$. Thus we can apply the maximum principle to conclude $u(z) \leq -\varepsilon g(z)$, i.e., $\frac{u(z)}{g(z)} \leq -\varepsilon$, for $r < |z - z_0| < R$. It remains to show that

$$\frac{u(z)}{|z-a|} \leq constant \cdot \frac{u(z)}{g(z)}$$

as $z \to a$ along the vector $v(a)$. Since $u < 0$, it is enough to prove $g(z) \leq constant \cdot |z - a|$. This can be done by Taylor series expansion of g. □

Corollary 1.7.2 *Let Ω, a and u be as above. If $u \leq 0$ on $\partial\Omega$ and $\overline{\lim}_{z\to a}\frac{u(z)}{z-a} = 0$, where z goes to a along the normal vector direction, then $u \equiv 0$.*

Proof Suppose $u \not\equiv 0$, by applying the maximun principle, $u < 0$ holds on Ω. By Hopf lemma above, $\overline{\lim}\frac{u(z)}{z-a} \leq -\varepsilon < 0$, which is a contradiction. □

As application, we have the following result.

Theorem 1.7.3 *(Burns-Krantz [BK 98]) Let $g(z) : \mathbb{H}^1 \to \mathbb{H}^1$ be a holomorphic function such that $g(w) = w + o(|w|^3)$ as $w \to 0$. Then $g(w) \equiv w$.*

Proof: Consider the harmmonic function $h(w) := Im\left(\frac{1}{w} - \frac{1}{g(w)}\right)$ defined on $\mathbb{H}^1$. Clearly $h(w) = o(|w|)$ as $w \to 0$.

We claim

$$\underline{lim}_{w\to x\in(\mathbb{R}^+\cup\infty)} h(w) \geq 0.$$

In fact, when $x \in \mathbb{R}$ with $x \neq 0$, we write $g(w) = U(w) + iV(w)$ and $w = u + iv$. Then

$$h(w) = Im\left(\frac{1}{u+iv} - \frac{1}{U+iV}\right) = -\frac{v}{u^2+v^2} + \frac{V}{U^2+V^2}$$

converges to $0 + \frac{Im\ g}{U^2+V^2} \geq 0$, as $w \to x \in \mathbb{R}$ with $x \neq 0$.

When $x = 0$, we have

$$Im\left(\frac{1}{w} - \frac{1}{g(w)}\right) = Im(\frac{g(w)-w}{wg(w)}) = Im(\frac{o(|w|^3)}{wg(w)}) = o(|w|), \quad as\ w \to 0.$$

When $x = \infty$, $h(w) = Im\left(\frac{1}{w} - \frac{1}{g(w)}\right) = -\frac{v}{u^2+v^2} + \frac{Img}{U^2+V^2} \to 0 + \frac{Im\ g}{U^2+V^2} \geq 0$ as $w \to \infty$. Claim is proved.

Take a linear fractional biholomorphic map $f : \mathbb{B}^1 \to \mathbb{H}^1$. By the maximun principle and the above Claim, the harmonic (hence subharmonic) function $-h \circ f \leq 0$ on $\mathbb{B}$. By Corollary 1.7.2, since $lim_{w\to x}\frac{(-h\circ f)(w)}{w-x} = 0$ by above calculation, one concludes $-h \circ f \equiv 0$ so that $h \equiv 0$, i.e., $\frac{1}{w} - \frac{1}{g(w)} \equiv 0$. Hence $g(w) \equiv w$. □

1.8 Three classes of CR submanifolds

$$\{\text{CR submanifolds in hyperquadratic}\} \subsetneq \{\text{Embeddable CR manifolds}\} \subsetneq \{\text{CR manifolds}\}$$

It has long been known that generic 3-dimensional CR manifolds are locally not embeddable, and that all strictly pseudoconvex CR manifolds of dimension 7 and higher are locally embeddable, but the 5- dimensional strictly pseudoconvex case remains open.

Forstnerič [Fo86b] and Faran [Fa88] proved the existence of real analytic strictly pseudoconvex hypersurfaces $M^{2n+1} \subset \mathbb{C}^{n+1}$ which do not admit any germ of holomorphic mapping taking M into sphere $\partial\mathbb{B}^{N+1}$ for any N. We may compare this with the Cartan-Janet theorem which asserted that for any analytic Riemannian manifold (M^n, g), there exist local isometric embeddings of M^n into Euclidean space $\mathbb{E}^N$ as N is sufficiently large.

On the other hand, by Webster [W78b], any Levy nondegenerate real-algebraic hypersurface is holomorphically embeddable into a nondegenerate hyperquadric $\partial\mathbb{H}^n_\ell$.

From above, it leads us to concentrate on a subclass of the set of all CR manifolds:

$$\{\text{CR submanifolds in a sphere } \partial\mathbb{B}^{N+1}\}$$

S.-Y. Kim and J.-W. Oh [KO06] gave a necessary and sufficient condition for local embeddability into a sphere $\partial\mathbb{B}^{N+1}$ of a generic strictly pseudoconvex pseudohermitian CR manifold (M^{2n+1}, θ) in terms of its Chern-Moser curvature tensors and their derivatives.

Zaitsev [Za08] constructed explicit examples for the Forstnerič and Faran phenomenon above.

Ebenfelt, Huang and Zaitsev [EHZ04] proved rigidity of CR embeddings of general M^{2n+1} into spheres with CR co-dimension $< \frac{n}{2}$, which generalizes a result of Webster that was for the case of co-dimension 1 [W79]. Here by *rigidity*, we mean that for any two smooth CR immersions f and $\widetilde{f} : M^{2n+1} \to \partial\mathbb{B}^{n+d+1}$ with $d < \frac{n}{2}$, there exists $\phi \in Aut(\partial\mathbb{B}^{n+1+d})$ such that $\widetilde{f} = \phi \circ f$.

Very recently, Ji and Yuan [JY09] proved that if a CR submanifold M with hypersurface type of $\partial\mathbb{B}^N$ and with zero CR second fundamental form, then M is the image of a sphere by a linear map.

The most basic and non-trivial example of CR submanifolds in a sphere $\partial\mathbb{B}^N$ is the image $M = F(\partial\mathbb{B}^n)$ where

$$F : \partial\mathbb{B}^n \to \partial\mathbb{B}^N$$

is a proper holomorphic map that is C^2-smooth up to the closed ball $\overline{\mathbb{B}^n}$. Here the C^2-smooth condition allows the map F restricted on the sphere to become a CR mapping

$$F : \partial\mathbb{B}^n \to \partial\mathbb{B}^N.$$

1.9 Proper Holomorphic Maps Between Balls

Recall that a continuous map $f : X \to Y$ where X and Y are topological spaces is called *proper* if for any compact subset $K \subset Y$, $f^{-1}(K)$ is compact in X.

Proposition 1.9.1 *Let $D, D' \subset \mathbb{C}^n$ be bounded domains and $f : D \to D'$ a holomorphic map. Then f is proper if and only if for any sequence z_v which converges to a point in ∂D, the image sequence $\{f(z_v)\}$ tends to $\partial D'$.*

Proof: ($\Rightarrow$) Suppose that $\{f(z_v)\}$ does not tend to $\partial D'$. Then there is a subsequence $\{z_{v_k}\}$ such that $\{f(z_{v_k})\}$ is relatively compact in D', which is a contradiction to the properness of f.

($\Leftarrow$) Suppose there is a compact subset $K \subset D'$ such that $f^{-1}(K)$ is not compact in D. Then there is a sequence $\{z_v\}$ converging to ∂D but $\{f(z_v)\} \subset K$ does not tend to $\partial D'$. $\square$.

From the last section, it leads us to concentrate on a subclass of the set of CR submanifolds in a sphere:

$$Prop(\mathbb{B}^n, \mathbb{B}^N) := \{\text{proper holomorphic map } F : \mathbb{B}^n \to \mathbb{B}^N\},$$

$$Prop_k(\mathbb{B}^n, \mathbb{B}^N) := Prop(\mathbb{B}^n, \mathbb{B}^N) \cap C^k(\overline{\mathbb{B}^n}),$$

$$Rat(\mathbb{B}^n, \mathbb{B}^N) := Prop(\mathbb{B}^n, \mathbb{B}^N) \cap \{rational\ maps\}.$$

$$Poly(\mathbb{B}^n, \mathbb{B}^N) := Prop(\mathbb{B}^n, \mathbb{B}^N) \cap \{polynomial\ maps\}.$$

We say that $F, G \in Prop(\mathbb{B}^n, \mathbb{B}^N)$ are *equivalent*, denoted as $F \cong G$, if there are automorphisms $\sigma \in Aut(\mathbb{B}^n)$ and $\tau \in Auto(\mathbb{B}^N)$ such that $F = \tau \circ G \circ \sigma$, i.e., the following diagram commutes

$$\begin{array}{ccc} \mathbb{B}^n & \xrightarrow{G} & \mathbb{B}^N \\ \uparrow \sigma & \circlearrowright & \downarrow \tau \\ \mathbb{B}^n & \xrightarrow{F} & \mathbb{B}^N. \end{array}$$

Theorem 1.9.2 *(H. Alexander [A77]) Any proper holomorphic map from $\mathbb{B}^n$ onto $\mathbb{B}^n$ must be an automorphism when $n \geq 2$.*

The condition that $n \geq 2$ is crucial. In fact, when $n = 1$, we have

Proposition 1.9.3

$$Prop(\mathbb{B}^1, \mathbb{B}^1) = \left\{ F(z) = e^{i\theta} \prod_{j=1}^{m} \frac{z - a_j}{1 - \overline{a_j} z}, \quad with\ |a_j| < 1 \right\}.$$

Proof: If f is proper, $f^{-1}(0)$ is compact: $f^{-1}(0) = \sum_{j=1}^{N} m_j [a_j]$ where $a_j \in \mathbb{B}^1$ and $m_j \in \mathbb{Z}^+$. Let

$$g(z) = \prod_{j=1}^{N} \left(\frac{z - a_j}{1 - \overline{a_j} z} \right)^{m_j}.$$

To show: $\frac{f}{g} = constant$ and $\left|\frac{f}{g}\right| \equiv 1$, which implies $f \equiv e^{i\theta} g$.

In fact, both $\frac{f}{g}$ and $\frac{g}{f}$ are meromorphic and have only removable singularities. Then both

$$\frac{f}{g},\ \frac{g}{f}\ are\ holomorphic\ in\ \mathbb{B}^1.$$

We apply Proposition 1.9.1 to know that for any $\epsilon > 0$, there is $\delta > 0$ such that

$$1 - \epsilon \leq \left|\frac{f(z)}{g(z)}\right| \leq \frac{1}{1 - \epsilon}, \quad \forall |z| > 1 - \delta.$$

By applying the maximum principle,

$$1 - \epsilon \leq \left|\frac{f(z)}{g(z)}\right| \leq \frac{1}{1 - \epsilon}, \quad \forall |z| \leq 1 - \delta.$$

Hence $\frac{f}{g} \equiv constant$. By letting $\epsilon \to 0$, $\left|\frac{f(z)}{g(z)}\right| \equiv 1$. □

Bochner and Martin [BM48] found a necessary and sufficient condition for mappings in $Prop(\mathbb{B}^n, \mathbb{B}^N)$ in terms of its power series centered at the origin. More precisely, if $F = (f_1, ..., f_h)$ is written as power series

$$f_j(z) = \sum a^{(j)}_{n_1 \cdots n_k} z_1^{n_1} \cdots z_k^{n_k}, \quad j = 1, ..., h,$$

then F maps $\partial \mathbb{B}^k$ into $\partial \mathbb{B}^h$ if and only if

$$\sum_{j=1}^{h} a^{(j)}_{m_1 \cdots m_k} \overline{a^{(j)}_{n_1 \cdots n_k}} = 0,\ for\ (m_1 - n_1)^2 + ... + (m_k - n_k)^2 > 0,$$

and

$$\sum_{j=1}^{h} \left| a^{(j)}_{n_1 \cdots n_k} \right|^2 = \frac{(n_1 + ... + n_k)!}{n_1! \cdots n_k!} A_{n_1 + \cdots + n_k},$$

where A_N are suitable nonnegative numbers.

It was discovered in the early 80's (cf. [Fo93][H99]) that $Prop(\mathbb{B}^n, \mathbb{B}^N)$ is much larger than $Prop_k(\mathbb{B}^n, \mathbb{B}^N)$ in general. In fact, there are some mappings $F \in Prop(\mathbb{B}^n, \mathbb{B}^{n+1}) \cap C^0(\overline{\mathbb{B}^n})$ but they are neither in $Prop_2(\mathbb{B}^n, \mathbb{B}^{n+1})$ nor in $Rat(\mathbb{B}^n, \mathbb{B}^{n+1})$.

For any $F \in Prop_2(\mathbb{B}^{n+1}, \mathbb{B}^{N+1})$, it induces a C^2 smooth CR map from $\partial\mathbb{B}^{n+1}$ into $\partial\mathbb{B}^{N+1}$.

Webster was the first to investigate the geometric structure of proper holomorphic maps between balls in complex spaces of different dimensions. In 1979, he showed [W79] that a proper holomorphic map $F \in Prop_3(\mathbb{B}^n, \mathbb{B}^{n+1})$ with $n > 2$ is indeed a linear fractional embedding.

Forstnerič shown [Fo86] that

$$Prop_{N-n+1}(\mathbb{B}^n, \mathbb{B}^N) = Rat(\mathbb{B}^n, \mathbb{B}^N).$$

Moreover, such F has no poles on $\partial\mathbb{B}^n$ by Cima-Suffridge [CS90].

J.P. D'Angelo did lots of work on polynomial and monomial mappings in $Prop_k(\mathbb{B}^n, \mathbb{B}^N)$ [DA88][DA92][DA93], in particular he found the structure of proper holomorphic polynomial mappings between balls.

Chapter 2

Earlier Result: The First Gap Theorem

2.1 The First Gap Theorem

Theorem 2.1.1 *(The First Gap Theorem) For $N < 2n-1$, any map $F \in Prop_2(\mathbb{B}^n, \mathbb{B}^N)$ is equivalent to the linear map $(z, 0, w)$.*

This theorem is a result by many mathematicians over 20 years.

In 1979, S. Webster proved [W79] that any mapping in $Prop_3(\mathbb{B}^n, \mathbb{B}^{n+1})$ with $n \geq 3$ must be equivalent to a linear map $(z, 0, w)$.

In 1982, J. Faran [Fa82] proved that there are exactly four maps in $Prop_3(\mathbb{B}^2, \mathbb{B}^3)$, up to equivalence class.

Next year, A. Cima and T.J. Suffridge [CS83] improved the above results of Webster and Faran by replacing "$Prop_3$" with "$Prop_2$". In the same paper [CS83], A. Cima and T. J. Suffridge conjectured that any mapping in $Prop_2(\mathbb{B}^n, \mathbb{B}^N)$ with $n \geq 3$ and $N \leq 2n-2$ should be equivalent to the linear map $(z, 0, w)$.

In 1986, Faran [Fa86] proved the Cima-Suffridge's conjecture under the assumption that F is holomorphic in a neighborhood of $\overline{\mathbb{B}^n}$.

In the same year, F. Forstnerič [Fo86] proved $Prop_{N-n+1}(\mathbb{B}^n, \mathbb{B}^N) = Rat(\mathbb{B}^n, \mathbb{B}^N)$ and later Cima and Suffridge [CS90] shown that any mapping in $Rat(\mathbb{B}^n, \mathbb{B}^N)$ must be holomorphic on the boundary. As a consequence, the First Gap Theorem is proved for any $F \in Prop_{N-n+1}(\mathbb{B}^n, \mathbb{B}^N)$ with $N < 2n-1$.

In 1999 X. Huang [Hu99] proved that any mapping in $Prop_2(\mathbb{B}^n, \mathbb{B}^N)$ with $N \leq 2n-2$ is equivalent to the linear map $(z, 0, w)$.

Outline of the Proof for the First Gap Theorem:

Step 1. if $N < 2n-1$, it implies that its geometric rank $\kappa_0 = 0$.

- (*analytic proof*) Use Uniqueness theorem (see Corollary 2.11.1 and Theorem 2.11.2 below).
- (*geometric proof*) Use the formula

$$N \geq n + \frac{(2n - \kappa_0 - 1)\kappa_0}{2}$$

  for any $F \in Prop_2(\mathbb{B}^n, \mathbb{B}^N)$ with geometric rank κ_0. In fact, if $N < 2n-1$, the above inequality forces $\kappa_0 = 0$.

Step 2. Show: $\kappa_0 = 0 \Longleftrightarrow F$ is a linear fractional map.

- (*analytic proof*) The first order PDE argument (see Theorem 2.10.1 below).
- (*geometric proof*) $\kappa_0 = 0 \Longleftrightarrow$ the CR second fundamental form $II_M = 0 \Longleftrightarrow F$ is a linear fractional map. □

We need to explain the following:

1. What is the geometric rank κ_0 of a map F ? (see (2.74) below, or [HJ01])

2. Why $N \geq n + \frac{(2n-\kappa_0-1)\kappa_0}{2}$? (see Corollary 4.2.2, or [H03])

3. Why κ_0 if and only if $II_M = 0$? (see Corollary 5.7.3, [JY09][HJ09])

4. Why $II_M = 0$ if and only if F is a linear fractional map (see Theorem 5.2.1, [JY09]).

2.2 Passing from $\partial\mathbb{B}^n$ to $\partial\mathbb{H}^n$

Recall the Heisenberg hypersurface

$$\partial\mathbb{H}^n := \{(z,w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \ \mathrm{Im}(w) = |z|^2\}$$

and the Cayley transformation

$$\rho_n : \mathbb{H}^n \to \mathbb{B}^n, \quad \rho_n(z,w) = \left(\frac{2z}{1-iw}, \frac{1+iw}{1-iw}\right).$$

We can define the space $Prop(\mathbb{H}^n, \mathbb{H}^N)$, $Prop_k(\mathbb{H}^n, \mathbb{H}^N)$ and $Rat(\mathbb{H}^n, \mathbb{H}^N)$.

We can identify a map $F \in Prop_k(\mathbb{B}^n, \mathbb{B}^N)$ or $Rat(\mathbb{B}^n, \mathbb{B}^N)$ with $\rho_N^{-1} \circ F \circ \rho_n$ in the space $Prop_k(\mathbb{H}^n, \mathbb{H}^N)$ or $Rat(\mathbb{H}^n, \mathbb{H}^N)$, respectively.

We say that F and $G \in Prop(\mathbb{H}^n, \mathbb{H}^N)$ are *equivalent* if there are automorphisms $\sigma \in Aut(\mathbb{H}^n)$ and $\tau \in Aut(\mathbb{H}^N)$ such that $F = \tau \circ G \circ \sigma$.

$$\begin{array}{ccc} \mathbb{B}^n & \xrightarrow{F} & \mathbb{B}^N \\ \uparrow \rho_n & \circlearrowright & \downarrow \rho_N^{-1} \\ \mathbb{H}^n & \xrightarrow{\rho_N^{-1}\circ F\circ\rho_n} & \mathbb{N}^N. \end{array}$$

2.3 Differential Operators on $\partial\mathbb{H}^n$

The vector fields $\{L_1, ..., L_{n-1}\}$, where $L_j := 2i\overline{z}_j\frac{\partial}{\partial w}+\frac{\partial}{\partial z_j}$, form a global basis for the complex tangent bundle $\mathbb{C}T^{1,0}\partial\mathbb{H}^n$ over $\partial\mathbb{H}^n$, and their conjugates $\{\overline{L_1}, ..., \overline{L_{n-1}}\}$, called *CR vector fields*, form a global basis for the complex tangent bundle $\mathbb{C}T^{0,1}\partial\mathbb{H}^n$ over $\partial\mathbb{H}^n$. Recall that for $z_j = x_j + iy_j$ and for $w = u + iv$, we have

$$\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}),\ \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}).$$

and

$$\frac{\partial}{\partial w} = \frac{1}{2}(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}),\ \frac{\partial}{\partial \overline{w}} = \frac{1}{2}(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}).$$

There is a real vector field which is transversal to $\mathbb{C}T^{(1,0)}\partial\mathbb{H}^n + \mathbb{C}T^{(0,1)}\partial\mathbb{H}^n$

$$T = \frac{\partial}{\partial Re(w)} = \frac{\partial}{\partial u} = \frac{\partial}{\partial w} + \frac{\partial}{\partial \overline{w}}. \tag{2.1}$$

which is the *Reeb vector field*.

The vector fields $\{L_1, ..., L_{n-1}, \overline{L_1}, ..., \overline{L_{n-1}}, T\}$ forms a basis of $\mathbb{C}T\partial\mathbb{H}_n$.

Lemma 2.3.1 *(i) $TL_j = L_jT$, $T\overline{L_j} = \overline{L_j}T$, and $L_jL_k = L_kL_j$ for all $1 \le j, k \le n-1$.*

(ii) For any continuous CR function h over an open subset $M_1 \subset \partial\mathbb{H}^n$, $T\ h$ is a CR distribution over M_1. For any $1 \le j, k \le n-1$, $\overline{L_k}(L_jh) = -[L_j, \overline{L_k}]h = 2i\delta_{kj}T\ h$.

(iii) Let h be a C^2 CR function over $\partial\mathbb{H}^n$ and χ a C^1 function over $\partial\mathbb{H}^n$. Then for any integer $k > 0$, we have

$$\overline{L_k}(L_k^2(h)\chi) = 4iL_k(T(h))\chi + L_k^2(h)\overline{L_k}(\chi),$$
$$\overline{L_k}(L_k(T(h))\chi) = 2iT^2(h)\chi + L_k(T(h))\overline{L_k}(\chi)$$

in the sense of distribution.

(iv) For any k, l, j and any C^2 CR function h, we have

$$\overline{L_k}L_lL_jh = 2i\delta_{k\ell}TL_jh + 2i\delta_{kj}TL_\ell h$$

in the sense of distribution. In particular, we have

$$\overline{L_k}L_lL_jh = \begin{cases} 0, & \textit{when } k \neq l \textit{ and } k \neq j; \\ 2iT(L_lh), & \textit{when } k = j \neq l; \\ 2iT(L_jh), & \textit{when } k = l \neq j; \\ 4iT(L_kh), & \textit{when } k = j = l. \end{cases}$$

Proof of Lemma 2.3.1 : (i) For any differentiable function $f(z, \overline{z}, w, \overline{w})$,

$$T(L_jf) = (\frac{\partial}{\partial w} + \frac{\partial}{\partial \overline{w}})(\frac{\partial f}{\partial z_j} + 2i\overline{z_j}\frac{\partial f}{\partial w}) = \frac{\partial^2 f}{\partial w \partial z_j} + 2i\overline{z_j}\frac{\partial^2 f}{\partial w^2} + \frac{\partial^2 f}{\partial \overline{w}\partial z_j} + 2i\overline{z_j}\frac{\partial^2 f}{\partial w \partial \overline{w}}.$$

$$L_j(Tf) = (\frac{\partial}{\partial z_j} + 2i\overline{z_j}\frac{\partial}{\partial w})(\frac{\partial f}{\partial w} + \frac{\partial f}{\partial \overline{w}}) = \frac{\partial^2 f}{\partial w \partial z_j} + 2i\overline{z_j}\frac{\partial^2 f}{\partial w^2} + \frac{\partial^2 f}{\partial \overline{w}\partial z_j} + 2i\overline{z_j}\frac{\partial^2 f}{\partial w \partial \overline{w}}.$$

Then $TL_j = L_jT$ and hence $T\overline{L_j} = \overline{L_j}T$. Similarly, $L_jL_k = L_kL_j$, $\forall 1 \leq j, k \leq n-1$.

(ii) The first statement follows from (i): Th is CR because $\overline{L_j}Th = T\overline{L_j}h = 0$. The second statement follows from the following calculation:

$$\begin{aligned}&[L_j, \overline{L_k}] = \big(\tfrac{\partial}{\partial z_j} + 2i\overline{z_j}\tfrac{\partial}{\partial w}\big)\big(\tfrac{\partial}{\partial \overline{z_k}} - 2iz_k\tfrac{\partial}{\partial \overline{w}}\big) - \big(\tfrac{\partial}{\partial \overline{z_k}} - 2iz_k\tfrac{\partial}{\partial \overline{w}}\big)\big(\tfrac{\partial}{\partial z_j} + 2i\overline{z_j}\tfrac{\partial}{\partial w}\big)\\ &= -2i\delta_{jk}\tfrac{\partial}{\partial \overline{w}} - 2i\delta_{jk}\tfrac{\partial}{\partial w} = -2i\delta_{kj}T.\end{aligned}$$

(iii) It is sufficient to prove (iii) for any holomorphic polynomial h by a lemma below. By (ii), we know that Th is CR and that $\overline{L_k}L_kh = 2iTh$. This follows the second identity. To prove the first identity, it is sufficient to prove

$$\overline{L_k}L_k^2h = 4iL_kTh, \;\; \forall\, C^2 \; CR \; function \; h. \tag{2.2}$$

In fact, $\overline{L_k}L_k^2h$ equals to

$$([\overline{L_k}, L_k] + L_k\overline{L_k})L_kh = 2iTL_kh + L_k([\overline{L_k}, L_k] + L_k\overline{L_k})h = 2iTL_kh + 2iL_kTh + 0 = 4iTL_kh.$$

(iv) It is sufficient to prove (iv) for any holomorphic polynomial h as above.

Consider

$$\begin{aligned}
&\overline{L_k}L_\ell L_j h = \big([\overline{L_k}, L_\ell] + L_\ell \overline{L_k}\big)L_j h \\
&= 2i\delta_{k\ell}TL_jh + L_\ell(\overline{L_k}L_j)h = 2i\delta_{k\ell}TL_jh + L_\ell\big([\overline{L_k}, L_j] + L_j\overline{L_k}\big)h \\
&= 2i\delta_{k\ell}TL_jh + L_\ell 2i\delta_{kj}Th + 0 = 2i\delta_{k\ell}TL_jh + 2i\delta_{kj}TL_\ell h \\
&= \begin{cases} 0, & if\ k \neq j, k \neq \ell, \\ 2iTL_\ell h & if\ k = j, j \neq \ell, \\ 2iTL_j h & if\ k = \ell \neq j, \\ 4iTL_k h & k = j = \ell. \end{cases}
\end{aligned}$$

by using the similar computation. □

Let h be a C^v-smooth function and then $D_1(h)$ is a C^0-smooth function for any differential operator D_1 of degree v. Let D_2 be another differential operator. In general $D_2D_1(h)$ does not make sense. However if $D_2D_1(h)$ can be written as $D_3(h)$ where D_3 is of degree v. Then $D_2D_1(h)$ is still a C^0 function. This fact is presented by a lemma below. As an example, $\overline{L_j}L_l h = 2i\delta_{jl}Th$. It can also been seen in Lemma 2.3.1 (ii) and (iii).

Lemma 2.3.2 *Let h be a C^v-smooth CR map from a neighborhood of M in $\partial\mathbb{H}_n$ into $\mathbb{C}^N$. Let $D_1(h) = H(p, \overline{p}, L^\alpha \overline{L^\beta} T^\gamma(h))_{|\alpha|+|\beta|+|\gamma|\leq v}$ with H holomorphic in its argument where $p \in \partial\mathbb{H}_n$. Let $D_2 = L^{\alpha_1}\overline{L^{\beta_1}}T^{\gamma_1}$ be a differential operator along M. Suppose that there is a certain holomorphic function H_0 in its argument such that for each polynomial map h^* from $\mathbb{C}^n$ into $\mathbb{C}^N$,*

$$D_2(D_1(h^*)) = H_0(p, \overline{p}, L^{\alpha_2}\overline{L^{\beta_2}}T^{\gamma_2}(h^*))_{|\alpha_2|+|\beta_2|+|\gamma_2|\leq v}$$

Then the distribution $D_2(D_1(h))$, acting on $C_0^\infty(M)$, coincides with the continuous function $D_3(h) := H_0(p, \overline{p}, L^{\alpha_2}\overline{L^{\beta_2}}T^{\gamma_2}(h))_{|\alpha_2|+|\beta_2|+|\gamma_2|\leq v}$.

Proof of Lemma 2.3.2: It is an immediate application of the Baouendi-Treves approximation theorem. Here we outline the proof. There is a sequence of holomorphic polynomial maps $\{h_m\}_{m=1}^\infty$ which converges to h in the C^v-norms over $\overline{M}$. Hence $D_1(h_m) \to D_1(h)$ in the C^0-norm over $\overline{M}$, and $D_2(D_1(h_m)) \to D_2(D_1(h))$ in the sense of distribution. By the assumption, $D_2(D_1(h_m))$ converges also to $H_0(p, \overline{p}, L^\alpha\overline{L^\beta}T^\gamma(G))_{|\alpha|+|\beta|+|\gamma|\leq v}$ in the C^0-norm over $\overline{M}$. □

2.4 Equations Associated with F

Let $F = (f, \phi, g) = (\widetilde{f}, g) : M_1 \cap \partial\mathbb{H}^n \to \partial\mathbb{H}^N$ be a non-constant C^2- smooth CR map with $F(0) = 0$, where M_1 is an open subset of $\partial\mathbb{H}^n$. We denote $f = (f_1, ..., f_{n-1})$, $\phi = (\phi_1, ..., \phi_{N-n})$ and $\widetilde{f} = (f, \phi)$. The basic equation is

$$Im\ g = \widetilde{f} \cdot \overline{\widetilde{f}}^t = \langle \widetilde{f}, \overline{\widetilde{f}} \rangle, \quad \forall (z, w) \in M_1$$

i.e.,

$$\frac{g - \overline{g}}{2i} = \sum_{j=1}^{n-1} |f_j|^2 + \sum_{j=1}^{N-n} |\phi_j|^2, \quad \forall (z, w) \in M_1 \ with\ Im(w) = |z|^2. \tag{2.3}$$

By the Lewy Extension Theorem (see Theorem 1.6.2), F extends holomorphically to a certain pseudoconvex side of M_1 denoted.

Let us differentiate (2.3) by L_j and T. First we consider the first order differential operators: L_l and T, $1 \leq l \leq n-1$: $\frac{L_l g}{2i} = L_l \widetilde{f} \cdot \overline{\widetilde{f}}^t$ where we denote by ^{t} the transport, i.e.,

$$\frac{L_l g}{2i} = \sum_j L_l f_j \cdot \overline{f_j}^t + \sum_j L_l \phi_j \cdot \overline{\phi_j}^t = L_l \widetilde{f} \cdot \overline{\widetilde{f}}^t, \ \forall (z, w) \in M_1, \tag{2.4}$$

$$\frac{Tg - \overline{Tg}}{2i} = T\widetilde{f} \cdot \overline{\widetilde{f}}^t + \widetilde{f} \cdot \overline{T\widetilde{f}}^t, \quad \forall (z, w) \in M_1. \tag{2.5}$$

We consider the second order differential operators $L_k L_l, TL_l$ and T^2, $1 \leq k, l \leq n-1$.

$$\frac{L_k L_l g}{2i} = L_k(L_l \widetilde{f}) \cdot \overline{\widetilde{f}}^t, \quad \forall (z, w) \in M_1. \tag{2.6}$$

$$\frac{1}{2i} T(L_l g) = T(L_l \widetilde{f}) \cdot \overline{\widetilde{f}}^t + L_l(\widetilde{f}) \cdot \overline{T\widetilde{f}}^t, \quad \forall (z, w) \in M_1 \tag{2.7}$$

$$Im(T^2 g) = 2\ Im(iT^2 \widetilde{f} \cdot \overline{\widetilde{f}}^t) + 2|T\widetilde{f}|^2, \quad \forall (z, w) \in M_1. \tag{2.8}$$

$$\frac{1}{2i} \overline{L_k} L_l g = \overline{L_k} L_l \widetilde{f} \cdot \overline{\widetilde{f}}^t + L_l \widetilde{f} \cdot \overline{L_k \widetilde{f}}^t, \quad \forall (z, w) \in M_1. \tag{2.9}$$

In particular, if $k = l$, by using $\overline{L_l} L_l = 2iT$, we obtain

$$Tg = 2i \langle T\widetilde{f}, \overline{\widetilde{f}} \rangle + |L_j \widetilde{f}|^2, \quad \forall (z, w) \in M_1. \tag{2.10}$$

Next we consider the third order differential operators $\overline{L_k}L_jL_l$, $1 \leq k, j, l \leq n-1$:

$$\frac{1}{2i}\overline{L_k}(L_j(L_l g)) = \overline{L_k}(L_j(L_l\widetilde{f})) \cdot \overline{\widetilde{f}}^t + L_j(L_l\widetilde{f}) \cdot \overline{L_k\widetilde{f}}^t. \tag{2.11}$$

When $k \neq j$ and $k \neq l$, by Lemma 2.3.1(iv), (2.11) becomes

$$L_j(L_l\widetilde{f}) \cdot \overline{L_k\widetilde{f}}^t = 0. \tag{2.12}$$

When $k = j \neq l$, by Lemma 2.3.1 (iv), (2.11) becomes

$$T(L_l g) = 2iT(L_l\widetilde{f}) \cdot \overline{\widetilde{f}}^t + L_j(L_l\widetilde{f}) \cdot \overline{L_j\widetilde{f}}^t. \tag{2.13}$$

When $k = l \neq j$, by Lemma 2.3.1 (iv), (2.11) becomes

$$T(L_j g) = 2iT(L_j\widetilde{f}) \cdot \overline{\widetilde{f}}^t + L_l(L_j\widetilde{f}) \cdot \overline{L_l\widetilde{f}}^t. \tag{2.14}$$

When $k = j = l$, by Lemma 2.3.1(iv) again, we have

$$2T(L_k g) = 4iT(L_k\widetilde{f}) \cdot \overline{\widetilde{f}}^t + L_k(L_k\widetilde{f}) \cdot \overline{L_k\widetilde{f}}^t. \tag{2.15}$$

Since $F(0) = 0$, by (2.4) and (2.6), we obtain

$$\frac{\partial g}{\partial z_j}|_0 = \frac{\partial^2 g}{\partial z_k \partial z_l}|0 = 0. \tag{2.16}$$

2.5 The Associated Map F^* of F

From (2.9), since $F(0) = 0$, we have

$$\frac{1}{2i}\overline{L_k}L_j g|_0 = L_j\widetilde{f}|_0 \cdot \overline{L_k\widetilde{f}}|_0.$$

By Lemma 2.3.1, we have

$$\frac{1}{2i}\overline{L_k}L_j g|_0 = \frac{1}{2i}2i\delta_{kj}Tg|_0 = \lambda\delta_{kj}$$

where

$$\lambda = Tg|_0 > 0. \tag{2.17}$$

In fact, by (2.10), $Tg|_0 = 2i\langle T\widetilde{f}, \overline{\widetilde{f}}\rangle|_0 + |L_l\widetilde{f}|^2|_0 = |L_l\widetilde{f}|^2\big|_0 > 0$.

<u>Remark</u> Another way to take look at the formula $Tg|_0 = \lambda > 0$ is to use Hopf lemma. We apply the maximum principle to the subharmonic function $-Im(g)+\sum_{j=1}^{n-1}|f_j|^2+\sum_{j=1}^{N-n}|\phi_j|^2 \le 0$ over Ω, we conclude $F(\Omega) \subset \mathbb{H}^N$. Then we apply Hopf lemma to obtain

$$\frac{\partial}{\partial\ Im(w)}\Big(- Im(g) + \sum_{j=1}^{n-1}|f_j|^2 + \sum_{j=1}^{N-n}|\phi_j|^2\Big)\Big|_0 = \frac{\partial}{\partial\ Im(w)}(-Im(g))|_0$$
$$= -i\big(\frac{\partial}{\partial w} - \frac{\partial}{\partial \overline{w}}\big)\frac{g-\overline{g}}{2i}\Big|_0 = -Tg|_0 = -\lambda < 0.$$

Then we have the orthogonal property:

$$L_j\widetilde{f}|_0 \cdot \overline{L_k\widetilde{f}}|_0 = \lambda\delta_{jk}.$$

Denoting

$$E_l = (\frac{\partial\widetilde{f}}{\partial z_l})|_0 = \big(\frac{\partial f_1}{\partial z_l}, ..., \frac{\partial f_{n-1}}{\partial z_l}, \frac{\partial \phi_1}{\partial z_l}, ..., \frac{\partial \phi_{N-n}}{\partial z_l}\big)\big|_0,$$

and

$$E_w = (\frac{\partial\widetilde{f}}{\partial w})|_0 = \big(\frac{\partial f_1}{\partial w}, ..., \frac{\partial f_{n-1}}{\partial w}, \frac{\partial \phi_1}{\partial w}, ..., \frac{\partial \phi_{N-n}}{\partial w}\big)\big|_0.$$

Then it has orthogonal property:

$$\frac{E_j}{\sqrt{\lambda}}\overline{\frac{E_k^t}{\sqrt{\lambda}}} = \delta_{jk}. \tag{2.18}$$

We extend $\{\frac{E_1}{\sqrt{\lambda}}, ..., \frac{E_{n-1}}{\sqrt{\lambda}}\}$ to a certain orthonormal basis of $\mathbb{C}^{N-1}$:

$$\Big\{\frac{E_1}{\sqrt{\lambda}}, ..., \frac{E_{n-1}}{\sqrt{\lambda}}, C_1, ..., C_{N-n}\Big\}. \tag{2.19}$$

Now we define a new map $F^* = (f_l^*, \phi_k^*, g^*) = H \circ F$ where $H \in Aut(\mathbb{H}^N)$, which is equivalent to F, defined by

$$f_l^* = \frac{1}{\lambda}\widetilde{f} \cdot \overline{E_l}^t, \quad \phi_k^* = \frac{1}{\sqrt{\lambda}}\widetilde{f} \cdot \overline{C_k}^t, \ g^* = \frac{1}{\lambda}g. \tag{2.20}$$

F^* satisfies some initial conditions at 0:

$$F^*(0) = 0, \frac{\partial f_j^*}{\partial z_l}\bigg|_0 = \delta_j^l, \frac{\partial \phi_j^*}{\partial z_l}\bigg|_0 = 0, \frac{\partial g^*}{\partial z_l}\bigg|_0 = 0, \frac{\partial g^*}{\partial w}\bigg|_0 = 1. \tag{2.21}$$

In fact, for example,

$$\frac{\partial f_j^*}{\partial z_l}|_0 = L_l f_j^*|_0 = L_l(\frac{1}{\lambda}\widetilde{f}\cdot \overline{E_j}^t)|_0 = \frac{1}{\lambda}L_l\widetilde{f}\cdot\overline{E_j}^t = \frac{1}{\lambda}E_l|_0\cdot\overline{E_j}^t = \frac{1}{\lambda}\lambda\delta_{lj} = \delta_{lj}.$$

It is not good enough because we need to take care of the terms $\frac{\partial f_j^*}{\partial w}\big|_0$ and $\frac{\partial \phi_j^*}{\partial w}\big|_0$. We need further normalization.

Since $L_j|_0 = \frac{\partial}{\partial z_j}|_0$ and $T|_0 = \frac{\partial}{\partial w}|_0$, by taking differential and by the chain rule, we have

$$\begin{aligned}
\Big(f_l^*\Big)'_{z_k}|_0 &= \frac{1}{\lambda}L_k\widetilde{f}\cdot\overline{E_l}^t|_0 = \frac{1}{\lambda}L_k(\widetilde{f})\cdot\overline{L_l(\widetilde{f})}^t|_0 = \delta_l^k,\\
\Big(f_l^*\Big)'_{w}|_0 &= \frac{1}{\lambda}E_w\cdot\overline{E_l}^t|_0 = \frac{1}{\lambda}T(\widetilde{f})\cdot\overline{L_l(\widetilde{f})}^t|_0,\\
\Big(\phi_l^*\Big)'_{z_k}|_0 &= \frac{1}{\sqrt{\lambda}}L_k\widetilde{f}\cdot\overline{C_l}^t|_0 = 0,\\
\Big(\phi_k^*\Big)'_{w}|_0 &= \frac{1}{\sqrt{\lambda}}E_w\cdot\overline{C_k}^t|_0 = \frac{1}{\sqrt{\lambda}}T(\widetilde{f})\cdot\overline{C_k}^t|_0,
\end{aligned}$$

$$\begin{aligned}
\Big(g^*\Big)'_{z_l}|_0 &= \frac{1}{\lambda}\Big(L_l g - 2iL_l\widetilde{f}\cdot\overline{\widetilde{f}}^t\Big)|_0 = 0, \quad (By\ (2.4))\\
\Big(g^*\Big)'_{w}|_0 &= \frac{1}{\lambda}\Big(Tg - 2iT\widetilde{f}\cdot\overline{\widetilde{f}}^t\Big)|_0 = 1, \quad (By\ (2.10))
\end{aligned}$$

Besides, other formulas up to degree 2 are given as follows.

$$\begin{aligned}
&\Big(f_j^*\Big)''_{z_kz_l}|_0 = \frac{1}{\lambda}L_kL_l\widetilde{f}\cdot\overline{L_j\widetilde{f}}^t|_0,\ \Big(f_l^*\Big)''_{z_jw}|_0 = \frac{1}{\lambda}L_jT(\widetilde{f})\cdot\overline{L_l(\widetilde{f})}^t|_0,\\
&\Big(f_j^*\Big)''_{w^2}|_0 = \frac{1}{\lambda}T^2\widetilde{f}\cdot\overline{L_j\widetilde{f}}^t|_0,\ \Big(\phi_j^*\Big)''_{z_kz_l}|_0 = \frac{1}{\sqrt{\lambda}}L_kL_l\widetilde{f}\cdot\overline{C_j}^t|_0,
\end{aligned}$$

$$\left(\phi_j^*\right)''_{z_k w}|_0 = \frac{1}{\sqrt{\lambda}} T L_k \widetilde{f} \cdot \overline{C_j}^t|_0, \quad \left(\phi_j^*\right)''_{w^2}|_0 = \frac{1}{\sqrt{\lambda}} T^2 \widetilde{f} \cdot \overline{C_j}^t|_0,$$

$$\left(g^*\right)''_{z_l z_k}|_0 = \frac{1}{\lambda}\left(L_l L_k g - 2i L_l L_k \widetilde{f} \cdot \overline{\widetilde{f}}^t\right)|_0 = 0, \quad (By (2.6))$$

$$\left(g^*\right)''_{z_l w}|_0 = \frac{1}{\lambda} L_l \left(Tg - 2iT\widetilde{f} \cdot \overline{\widetilde{f}}^t\right) = \frac{2i}{\lambda} L_l \widetilde{f} \cdot \overline{T\widetilde{f}}^t|_0,$$

$$\left(g^*\right)''_{w^2}|_0 = \frac{1}{\lambda}\left(T^2 g - 2iT^2 \widetilde{f} \cdot \overline{\widetilde{f}}^t - 2iT\widetilde{f} \cdot \overline{T\widetilde{f}}^t\right)|_0.$$

2.6 The Associated Map F^{**} of F

We want to define $F^{**} = (\widetilde{f}^{**}, g^{**}) = (f^{**}, \phi^{**}, g^{**}) = (f_l^{**}, \phi_k^{**}, g^{**}) = G \circ F^*$, for some $G \in Aut(\partial \mathbb{H}^N)$, such that this normalization F^{**} satisfies the following properties:

$$F^{**}(0) = 0, \frac{\partial f_l^{**}}{\partial z_j}|_0 = \delta_{lj}, \frac{\partial f^{**}}{\partial w}|_0 = 0, \frac{\partial \phi_k^{**}}{\partial z_l}|_0 = 0, \frac{\partial \phi_k^{**}}{\partial w}|_0 = 0, \frac{g^{**}}{\partial z_l}|_0 = 0, \frac{g^{**}}{\partial w}|_0 = 1, \quad (2.22)$$

and

$$\frac{\partial^2 g^{**}}{\partial z_j \partial z_k}|_0 = \frac{\partial^2 g^{**}}{\partial w^2}|_0 = 0. \quad (2.23)$$

This can be done by defining (cf. [H99])

$$G(z^*, w^*) = \frac{(z^* - \mathbf{a} w^*, w^*)}{1 + 2i\langle z^*, \overline{\mathbf{a}}\rangle + (r - i|\mathbf{a}|^2) w^*} \in Aut_0(\partial \mathbb{H}^N) \quad (2.24)$$

where

$$\mathbf{a} := \left(\widetilde{f}^*\right)'_w\Big|_0 = \left(\cdots, \frac{T\widetilde{f} \cdot \overline{L_j \widetilde{f}}^t}{\lambda}, \cdots; \cdots, \frac{T\widetilde{f} \cdot \overline{C_j}^t}{\sqrt{\lambda}}, \cdots\right)\Big|_0 = (a_1, ..., a_{n-1}, b_1, ..., b_{N-n}),$$

$$r := \frac{1}{2} Re \left(g^*\right)''_{w^2}|_0 = \frac{1}{2\lambda} Re\left(T^2 g - 2iT^2 \widetilde{f} \cdot \overline{\widetilde{f}}^t\right)\Big|_0. \quad (2.25)$$

The the normalization is defined by $F^{**} := G \circ F^*$.

$$f_j^{**} = \frac{f_j^* - a_j g^*}{1 + 2i\langle \widetilde{f}^*, \overline{\mathbf{a}}\rangle + (r - i|\mathbf{a}|^2)g^*}, \tag{2.26}$$

$$\phi_j^{**} = \frac{\phi_j^* - b_j g^*}{1 + 2i\langle \widetilde{f}^*, \overline{\mathbf{a}}\rangle + (r - i|\mathbf{a}|^2)g^*}. \tag{2.27}$$

$$g^{**} = \frac{g^*}{1 + 2i\langle \widetilde{f}^*, \overline{\mathbf{a}}\rangle + (r - i|\mathbf{a}|^2)g^*}. \tag{2.28}$$

It implies (2.22).

To prove (2.23), by taking differential and by the chain rule, we continue to calculate

$$\begin{aligned} \Big(f_j^{**}\Big)''_{z_k z_l}|_0 &= \Big(f_j^{*}\Big)''_{z_k z_l}|_0 - 2i\delta_j^k\overline{a_l} - 2i\delta_j^l\overline{a_k} \\ = \ & \tfrac{1}{\lambda}L_kL_l\widetilde{f}\cdot\overline{L_j\widetilde{f}}^t|_0 - \tfrac{2i\delta_j^k}{\lambda}\overline{T\widetilde{f}}\cdot L_l\widetilde{f}^t|_0 - \tfrac{2i\delta_j^l}{\lambda}\overline{T\widetilde{f}}\cdot L_k\widetilde{f}^t|_0. \end{aligned} \tag{2.29}$$

$$\begin{aligned} \Big(f_l^{**}\Big)''_{z_j w}|_0 &= \Big(f_l^{*}\Big)''_{z_j w}|_0 - a_l\Big(g^*\Big)''_{z_j w}|_0 - \delta_j^l\Big[2i(\widetilde{f}^*)'_w|_0\cdot\overline{\mathbf{a}} + (r - i|\mathbf{a}|^2)\Big]|_0 \\ = \ & \Big(f_l^{*}\Big)''_{z_j w}|_0 - a_l\Big(g^*\Big)''_{z_j w}|_0 - \delta_j^l[i|\mathbf{a}|^2 + r]|_0 \\ = \ & \frac{1}{\lambda}L_jT\widetilde{f}\cdot\overline{L_l\widetilde{f}}^t|_0 - \frac{2i}{\lambda^2}\Big(T\widetilde{f}\cdot\overline{L_l\widetilde{f}}^t\Big)\Big(L_j\widetilde{f}\cdot\overline{T\widetilde{f}}^t\Big)|_0 \\ & -\frac{i\delta_{jl}}{\lambda}|T\widetilde{f}|^2|_0 - \frac{\delta_{jl}}{2\lambda}Re\Big(T^2g - 2iT^2\widetilde{f}\cdot\overline{\widetilde{f}}^t\Big)|_0. \end{aligned}$$

We can say more about this important formula which will be used to define geometric rank κ_0. Applying T^2 to the basic equation $Im(g) = |\widetilde{f}|^2$, we get $0 = 2iIm(iT^2\widetilde{f}\cdot\overline{\widetilde{f}^t}) + 2i|T\widetilde{f}|^2 - i\ Im(T^2g)$ on $\partial\mathbb{H}^n$ by (2.8), i.e.,

$$|T\widetilde{f}|^2 = \frac{1}{2}Im\Big(T^2g - 2iT^2\widetilde{f}\cdot\overline{\widetilde{f}}^t\Big) \tag{2.30}$$

Combining this to the above, we get

$$\begin{aligned}
&\left(f_l^{**}\right)''_{z_j w}|_0 = \tfrac{1}{\lambda} L_j T\widetilde{f}\cdot \overline{L_l\widetilde{f}}^t|_0 - \tfrac{2i}{\lambda^2}\left(T\widetilde{f}\cdot\overline{L_l\widetilde{f}}^t\right)\left(L_j\widetilde{f}\cdot\overline{T\widetilde{f}}^t\right)|_0 \\
&-\tfrac{\delta_{jl}}{2\lambda}\left(T^2 g - 2iT^2\widetilde{f}\cdot\overline{\widetilde{f}}^t\right)|_0.
\end{aligned} \tag{2.31}$$

$$\begin{aligned}
&\left(f_l^{**}\right)''_{w^2}|_0 = \left(f_l^{*}\right)''_{w^2}|_0 - a_l\left(g^*\right)''_{w^2}|_0 \\
= \;& \tfrac{1}{\lambda}T^2\widetilde{f}\cdot\overline{L_l\widetilde{f}}^t|_0 - \tfrac{1}{\lambda^2}\left(T\widetilde{f}\cdot\overline{L_l\widetilde{f}}^t\right)\left(T^2 g - 2iT^2\widetilde{f}\cdot\overline{\widetilde{f}}^t - 2i|T\widetilde{f}|^2\right)|_0.
\end{aligned} \tag{2.32}$$

$$\left(\phi_l^{**}\right)''_{z_j z_k}|_0 = \left(\phi_l^{*}\right)''_{z_j z_k}|_0 - b_l(g^*)''_{z_j z_k} = \left(\phi_l^{*}\right)''_{z_j z_k}|_0 = \tfrac{1}{\sqrt{\lambda}}L_j L_k\widetilde{f}\cdot\overline{C_l}^t|_0. \tag{2.33}$$

Here we used the fact that $(g^*)''_{z_j z_k}|_0 = 0$.

$$\begin{aligned}
&\left(\phi_l^{**}\right)''_{z_j w}|_0 = \left(\phi_l^{*}\right)''_{z_j w}|_0 - b_l(g^*)''_{z_j w}|_0 \\
= \;& \tfrac{1}{\sqrt{\lambda}}TL_j\widetilde{f}\cdot\overline{C_l}^t|_0 - \tfrac{1}{\lambda^{3/2}}\left(T\widetilde{f}\cdot\overline{C_l}^t\right)L_j\left(Tg - 2iT\widetilde{f}\cdot\overline{\widetilde{f}}^t\right)|_0 \\
= \;& \tfrac{1}{\sqrt{\lambda}}TL_j\widetilde{f}\cdot\overline{C_l}^t|_0 - \tfrac{2i}{\lambda^{3/2}}\left(T\widetilde{f}\cdot\overline{C_l}^t\right)\left(L_j\widetilde{f}\cdot T\overline{\widetilde{f}}^t\right)|_0.
\end{aligned} \tag{2.34}$$

$$\begin{aligned}
&\left(\phi_l^{**}\right)''_{w^2}|_0 = \left(\phi_l^{*}\right)''_{w^2}|_0 - b_j\left(g^*\right)''_{w^2}|_0 \\
= \;& \tfrac{1}{\sqrt{\lambda}}T^2\widetilde{f}\cdot\overline{C_l}^t|_0 - \tfrac{1}{\lambda^{3/2}}\left(T\widetilde{f}\cdot\overline{C_l}^t\right)\left(T^2 g - 2iT^2\widetilde{f}\cdot\overline{\widetilde{f}}^t - 2i|T\widetilde{f}|^2\right)|_0.
\end{aligned} \tag{2.35}$$

$$\left(g^{**}\right)''_{z_j z_k}|_0 = 0,$$

$$\left(g^{**}\right)''_{z_j w}|_0 = \left(g^{*}\right)''_{z_j w}|_0 - 2i\overline{a_j} = \frac{2i}{\lambda}L_j\widetilde{f}\cdot\overline{T\widetilde{f}}^t|_0 - \frac{2i}{\lambda}\overline{T\widetilde{f}\cdot\overline{L_j\widetilde{f}}^t}|_0,$$

$$
\begin{aligned}
&\Big(g^{**}\Big)''_{w^2}|_0 = \Big(g^*\Big)''_{w^2}|_0 - 2\Big[i|\mathbf{a_j}|^2 + r\Big]|_0 \\
= \;& \frac{1}{\lambda}\Big(T^2 g - 2iT^2\widetilde{f}\cdot\overline{\widetilde{f}}^t\Big)|_0 - \frac{2}{\lambda}\Big[i|T\widetilde{f}|^2 + \frac{1}{2}Re\Big(T^2 g - 2iT^2\widetilde{f}\cdot\overline{\widetilde{f}}^t\Big)\Big]|_0 \\
= \;& \frac{1}{\lambda}Im(T^2 g - 2iT^2\widetilde{f}\cdot\overline{\widetilde{f}}^t)|_0 = \frac{2}{\lambda}|T\widetilde{f}|^2\Big|_0 = 0.
\end{aligned}
$$

This implies $\big(g^{**}\big)''_{w^2}|_0 = 0$. Here we used (2.30). Then (2.23) are proved.

2.7 The Chern-Moser Operator

If $F = F^{**} \in Prop_2(\partial\mathbb{H}^n, \partial\mathbb{H}^N)$, then we have

$$
f = z + \hat{f},\ g = w + \hat{g}\ \ with\ \hat{f}, \hat{g}, \phi = O(|(z,w)|^2),\ \frac{\partial^2 \hat{g}}{\partial z_l \partial z_k}|_0 = \frac{\partial^2 \hat{g}}{\partial w^2}|_0 = 0. \tag{2.36}
$$

Then we obtain

$$
Im(w + \hat{g}) = \sum_{j=1}^{n-1} |z_j + \hat{f}_j|^2 + \sum_{j=1}^{N-n} |\phi_j|^2,\ \ \forall (z,w) \in \partial\mathbb{H}^n. \tag{2.37}
$$

Let $M_1 \subset \partial\mathbb{H}^n$ be an open subset. For a function f on M_1, we denote $h \in o_{wt}(s)$ if

$$
\lim_{t\to 0^+} \frac{h(tz, t^2 w, t\overline{z}, t^2\overline{w})}{t^s} \to 0
$$

uniformly with respect to $(z,w) \approx (0', 0) \in \mathbb{C}^{n_1} \times \mathbb{C}$. In other words, we define *weighted degree* by (see also (1.8))

$$
deg_{wt}(z^k w^l) = k + 2l.
$$

We write F as

$$
\hat{f}_j = \sum_{s=2}^{m-1} f_j^{(s)} + o_{wt}(m-1),\ \hat{g} = \sum_{s=3}^{m} g^{(s)} + o_{wt}(m),\ \ \phi_j = \sum_{s=l}^{m-l} \phi_j^{(s)} + o_{wt}(m-l),\ \ l \ge 2, \tag{2.38}
$$

where we denote by $h^{(s)}$ the homogeneous polynomial of (z,w) of weighted degree s.

Substituting these into (2.37), we obtain

$$\begin{aligned} Im(w) + Im(\hat{g}) &= \sum_j (z_j + \hat{f}_j)(\overline{z_j} + \overline{\hat{f}_j}) + \sum_k \big(\sum_s \phi_k^{(s)}\big)\big(\sum_t \overline{\phi^{(t)}}_k\big) \\ &= |z|^2 + \sum_j (z_j \overline{\hat{f}_j} + \hat{f}_j \overline{z_j} + |\hat{f}_j|^2) + \sum_k \big(\sum_s \phi_k^{(s)}\big)\big(\sum_t \overline{\phi_k^{(t)}}\big) \\ &= |z|^2 + \sum_j Im(2i\langle \overline{z_j}, \hat{f}_j\rangle) + \sum_j |\hat{f}_j|^2 + \sum_k \big(\sum_s \phi_k^{(s)}\big)\big(\sum_t \overline{\phi_k^{(t)}}\big), \quad \forall Im(w) = |z|^2. \end{aligned}$$

Here we used the fact $a + \overline{a} = Im(2ia)$ for any $a \in \mathbb{C}$. Then

$$Im(\hat{g}) = Im(2i\langle \overline{z}, \hat{f}\rangle) + |\hat{f}|^2 + \sum_k \big(\sum_s \phi_k^{(s)}\big)\big(\sum_t \overline{\phi_k^{(t)}}\big), \quad \forall Im(w) = |z|^2.$$

Then for any $l \leq s \leq m$, we collect terms in the above equation of weighted degree s to obtain the following equation:

$$Im(g^{(s)} - 2i\langle \overline{z}, f^{(s-1)}\rangle) = \sum_{j=1}^{N-n} \sum_{p=l}^{s-l} \phi_j^{(s-p)} \overline{\phi_j^{(p)}} + G^{(s)}, \quad \forall (z,w) \in \partial\mathbb{H}^n \tag{2.39}$$

where $G^{(s)}$ is weighted homogeneous polynomial of weighted degree s contributed by $f^{(\sigma-1)}$ and $g^{(\sigma)}$, $\sigma \leq s-1$. Here we denote $\phi^{(s)} \equiv 0$ if $s < 0$. The operator

$$\mathcal{L}(f, g) := Im(\hat{g} - 2i\langle \overline{z}, \hat{f}\rangle)$$

is called the *Chern-Moser operator*.

We notice $G^{(s)} \equiv 0$ if $f^{(\sigma-1)} \equiv g^{(\sigma)} \equiv 0$ for $\sigma \leq s-1$. Let us consider the following two cases.

Case 1: $s = 2k$ We suppose $s = 2k \leq m$. If the following additional conditions are satisfied

$$f^{(\sigma-1)} \equiv \phi^{(\sigma)} \equiv 0, \quad for\ \sigma \leq 2k-1, \tag{2.40}$$

then

$$Im(g^{(2k)}(z,w) - 2i\langle \overline{z}, f^{(2k-1)}(z,w)\rangle) = \sum_{j=1}^{N-n} \phi_j^{(k)} \overline{\phi_j^{(k)}}, \quad \forall (z,w) \in M_1. \tag{2.41}$$

Case 2: $s = 2k+1$ We suppose $s = 2k+1 \leq m$. If the following conditions are satisfied

$$f^{(\sigma-1)} \equiv \phi^{(\sigma)} \equiv 0 \ \ for\ \sigma \leq 2k, \tag{2.42}$$

then

$$Im(g^{(2k+1)}(z,w) - 2i\langle \overline{z}, f^{(2k)}(z,w)\rangle) = 0, \ \ \forall (z,w) \in M_1. \tag{2.43}$$

Lemma 2.7.1 *Let $F = F^{**} \in Prop_2(\mathbb{H}^n, \mathbb{H}^N)$ be as above. Then*
(i) $f^{(2)} \equiv 0,\ f^{(3)} \equiv a^{(1)}(z)w,\ \phi^{(2)}(z,w) = \phi^{(2)}(z), g^{(3)} = g^{(4)} \equiv 0.$
(ii) $-2i\langle a^{(1)}(z), \overline{z}\rangle |z|^2 = \sum_{j=1}^{N-n} |\phi_j^{(2)}(z)|^2.$

Proof: Consider $s = 2$ and (2.39). Since both sides of the equality are zero, the equation (2.39) is trivially true.

Consider $s = 3$ and $m = 3$ in the identity (2.43):

$$Im(g^{(3)} - 2i\langle \overline{z}, f^{(2)}\rangle) \equiv 0 \ \ on\ \partial\mathbb{H}^n. \tag{2.44}$$

We claim

$$g^{(3)} \equiv 0\ and\ f^{(2)} \equiv 0. \tag{2.45}$$

In fact, write $f^{(2)}(z,w) = a^{(2)}(z)$ and $g^{(3)}(z,w) = c^{(3)}(z) + c^{(1)}(z)w$. Substituting into (2.43), we have

$$Im(c^{(3)}(z) + c^{(1)}(z)w - 2i\langle \overline{z}, a^{(2)}(z)\rangle) \equiv 0, \quad \forall Im(w) = |z|^2.$$

Since $w = u + i|z|^2$, it follows that $c^{(1)}(z) \equiv 0$, $c^{(3)}(z) \equiv 0$ and $a^{(2)}(z) \equiv 0$. Hence Claim is proved.

Consider $s = 4$ and $m = 4$ in (2.41):

$$Im(g^{(4)} - 2i\langle \overline{z}, f^{(3)}\rangle) = \sum_{j=1}^{N-n} |\phi_j^{(2)}|^2, \ \ \forall Im(w) = |z|^2. \tag{2.46}$$

We claim

$$g^{(4)} \equiv 0,\ \phi_j^{(2)} \equiv \phi_j^{(2)}(z),\ f^{(3)} \equiv a^{(1)}(z)w,$$

$$-2i\langle a^{(1)}(z), \overline{z}\rangle |z|^2 = \sum_{j=1}^{N-n} |\phi_j^{(2)}(z)|^2, \tag{2.47}$$

where $a^{(1)}(z)$ is a certain holomorphic homogeneous polynomial of weighted degree one. In fact, write

$$f^{(3)}(z,w) = a^{(1)}(z)w + a^{(3)}(z), \ \ \phi_j^{(2)}(z,w) = b_j^{(2)}(z)$$

and $g^{(4)}(z,w) = c^{(4)}(z) + c^{(2)}(z)w$. Here we used $\frac{\partial^2 g}{\partial w^2}|_0 = 0$. Substituting into (2.41),

$$Im(c^{(4)}(z) + c^{(2)}(z)w - 2i\langle\overline{z}, a^{(1)}(z)\rangle|z|^2 - 2i\langle\overline{z}, a^{(3)}(z)\rangle) = \sum_{j=1}^{N-n} |b_j^{(2)}(z)|^2, \quad \forall (z,w) \in M_1.$$

Since $w = u + i|z|^2$ and z, u are independent variables, we consider u^0 and u terms to get three identities:

$$Im(c^{(4)}(z) + ic^{(2)}(z)|z|^2 + 2\langle\overline{z}, a^{(1)}(z)\rangle w - 2i\langle\overline{z}, a^{(3)}(z)\rangle) = \sum_{j=1}^{N-n} |b_j^{(2)}(z)|^2,$$

$$Im(c^{(2)}(z) - 2i\langle\overline{z}, a^{(1)}(z)\rangle)u = 0,$$

Then $c^{(2)}(z) \equiv 0$ and $Im(2i\langle\overline{z}, a^{(1)}(z)\rangle) \equiv 0$. Thus from the first one, $c^{(4)}(z) \equiv 0$ and $a^{(3)}(z) \equiv 0$ so that the claim is proved. $\square$

By Lemma 2.7.1, we obtain:

Theorem 2.7.2 *([H99], Lemma 5.3) Let $F \in Prop_2(\mathbb{H}^n, \mathbb{H}^N)$, $2 \le n \le N$ with $F(0) = 0$. Then there is an automorphism $\tau^{**} \in Aut_0(\mathbb{H}^N)$ such that $F^{**} := \tau^{**} \circ F = (f^{**}, \phi^{**}, g^{**})$ satisfies the following normalization:*

$$f^{**} = z + \frac{i}{2}a^{**(1)}(z)w + o_{wt}(3), \;\; \phi^{**} = \phi^{**(2)}(z) + o_{wt}(2), \;\; g^{**} = w + o_{wt}(4), \tag{2.48}$$

$$\langle\overline{z}, a^{**(1)}(z)\rangle|z|^2 = |\phi^{**(2)}(z)|^2.$$

2.8 The Associated Map F_p of F

Let

$$F = (f, \phi, g) = (\widetilde{f}, g) = (f_1, ..., f_{n-1}, \phi_1, ..., \phi_{N-n}, g)$$

be a non-constant C^2 smooth CR map from $M_1 \subset \partial\mathbb{H}^n$ into $M_2 \subset \partial\mathbb{H}^N$ as above.

For any point $p \in M_1$, we have an associated CR map F_p from a small neighborhood of $0 \in \partial\mathbb{H}^n$ to $\partial\mathbb{H}^N$ with $F_p(0) = 0$, defined by

$$F_p = \tau_p^F \circ F \circ \sigma_p^0, \tag{2.49}$$

$$\begin{array}{ccc} p \in \partial\mathbb{H}^n & \xrightarrow{F} & \partial\mathbb{H}^N \ni F(p) \\ \uparrow \sigma_p^0 & & \downarrow \tau_p^F \\ 0 \in \partial\mathbb{H}^n & \xrightarrow{F_p := \tau_p^F \circ F \circ \sigma_p} & \partial\mathbb{H}^N \ni 0 \end{array}$$

where $\sigma_p^0 \in Aut(\mathbb{H}^n)$, $p = (z_0, w_0)$, given by

$$\sigma_p^0(z, w) = (z + z_0, w + w_0 + 2i\langle z, \overline{z_0}\rangle), \tag{2.50}$$

and $\tau_p^F \in Aut(\mathbb{H}^N)$ is given by

$$\tau_p^F(z^*, w^*) = (z^* - \widetilde{f}(z_0, w_0),\ w^* - \overline{g(z_0, w_0)} - 2i\langle z^*, \overline{\widetilde{f}(z_0, w_0)}\rangle). \tag{2.51}$$

Notice that $F(0)$ may not be 0, but we always have $F_p(0) = 0$. By the similar calculation of F^* and F^{**}, w have the following formulas.

$$\begin{aligned}
&\left(\widetilde{f}_p\right)'_{z_l}|_0 = L_l(\widetilde{f})(p) := E_l(p), \\
&\left(\widetilde{f}_p\right)'_{w}|_0 = T(\widetilde{f})(p) := E_w(p), \\
&\lambda(p) := |L_j\widetilde{f}|^2(p),\ \text{for any j} \in \{1, ..., n-1\}, \\
&(g_p)'_{z_l}|_0 = L_l g(p) - 2iL_l\widetilde{f}(p) \cdot \overline{\widetilde{f}(p)}^t = 0 \quad (because\ (2.4)), \\
&(g_p)'_{w}|_0 = Tg(p) - 2iT\widetilde{f}(p) \cdot \overline{\widetilde{f}(p)}^t = |L_j\widetilde{f}_p(0)|^2, \quad 1 \le j \le n-1,
\end{aligned}$$

$$\begin{aligned}
&\left(\widetilde{f}_p\right)''_{z_l z_k}|_0 = L_l L_k(\widetilde{f})(p), \\
&\left(\widetilde{f}_p\right)''_{z_l w}|_0 = TL_l(\widetilde{f})(p), \\
&\left(\widetilde{f}_p\right)''_{w^2}|_0 = T^2(\widetilde{f})(p), \\
&(g_p)''_{z_l z_k}|_0 = L_l L_k g(p) - 2iL_l L_k\widetilde{f}(p) \cdot \overline{\widetilde{f}(p)}^t = 0, \quad (By\ (2.6)) \\
&(g_p)''_{w z_l}|_0 = L_l\left(Tg(p) - 2iT\widetilde{f}(p) \cdot \overline{\widetilde{f}(p)}^t\right) = 2iL_l\widetilde{f}(p) \cdot \overline{T\widetilde{f}(p)}^t, \\
&(g_p)''_{w^2}|_0 = T^2 g(p) - 2iT^2\widetilde{f}(p) \cdot \overline{\widetilde{f}(p)}^t - 2iT\widetilde{f}(p) \cdot \overline{T\widetilde{f}(p)}^t.
\end{aligned}$$

Here for the second equality about $(g_p)''_{wz_l}$, we used the fact that $g - \overline{g} = 2i\widetilde{f} \cdot \overline{\widetilde{f}}^t$ and then $TL_l g = 2iTL_p\widetilde{f} \cdot \overline{\widetilde{f}}^t + 2iL_l\widetilde{f} \cdot T\widetilde{f}^t$. Notice that there are two formulas for $(g_p)''_{wz_l}|_0$.

We define $F_p^* = (\widetilde{f}_p^*, g_p^*)$ given by

$$F_p^* = (f_p^*, \phi_p^*, g_p^*) = \left(f_{p,l}^*, \phi_{p,k}^*, g_p^* \right) \tag{2.52}$$

where

$$f_{p,l}^* = \frac{1}{\lambda_p}\widetilde{f}_p \cdot \overline{E_l(p)}^t, \quad \phi_{p,k}^* = \frac{1}{\sqrt{\lambda_p}}\widetilde{f}_p \cdot \overline{C_k(p)}^t, \; g_p^* = \frac{1}{\lambda_p} g_p, \tag{2.53}$$

where $1 \leq l \leq n-1$ and $1 \leq k \leq N-n$. F_p^* satisfies the following properties:

$$F_p^*(0) = 0, \left(f_{p,j}^* \right)'_{z_l} |_0 = \delta_j^l, \left(\phi_{p,j}^* \right)'_{z_l} |_0 = 0, \left(g_p^* \right)'_{z_l} |_0 = 0, \left(g_p^* \right)'_{w} |_0 = 1. \tag{2.54}$$

As before, we can choose vectors $C_1(p), ..., C_{N-n}(p) \in \mathbb{C}^{N-1}$ so that

$$\left\{ \frac{E_1(p)^t}{\sqrt{\lambda}}, ..., \frac{E_{n-1}(p)^t}{\sqrt{\lambda}}, C_1(p)^t, ..., C_{N-n}(p)^t \right\} \tag{2.55}$$

form an $(N-1) \times (N-1)$ unitray matrix.

$$\left(f_{p,l}^* \right)'_{z_k} |_0 = \frac{1}{\lambda(p)} L_k\widetilde{f(p)} \cdot \overline{E_l(p)}^t = \frac{1}{\lambda(p)} L_k(\widetilde{f})(p) \cdot \overline{L_l(\widetilde{f})(p)}^t = \delta_l^k,$$
$$\left(f_{p,l}^* \right)'_{w} |_0 = \frac{1}{\lambda(p)} E_w(p) \cdot \overline{E_l(p)}^t = \frac{1}{\lambda(p)} T(\widetilde{f})(p) \cdot \overline{L_l(\widetilde{f})(p)}^t,$$

$$\left(\phi_{p,l}^* \right)'_{z_k} |_0 = \frac{1}{\sqrt{\lambda(p)}} L_k\widetilde{f(p)} \cdot \overline{C_l(p)}^t = 0,$$
$$\left(\phi_{p,k}^* \right)'_{w} |_0 = \frac{1}{\sqrt{\lambda(p)}} E_w(p) \cdot \overline{C_k(p)}^t = \frac{1}{\sqrt{\lambda(p)}} T(\widetilde{f})(p) \cdot \overline{C_k(p)}^t,$$

$$\left(g_p^* \right)'_{z_l} |_0 = \frac{1}{\lambda(p)} \left(L_l g(p) - 2iL_l\widetilde{f}(p) \cdot \overline{\widetilde{f}(p)}^t \right) = 0, \quad (By\ (2.4))$$
$$\left(g_p^* \right)'_{w} |_0 = \frac{1}{\lambda(p)} \left(Tg(p) - 2iT\widetilde{f}(p) \cdot \overline{\widetilde{f}(p)}^t \right) = 1, \quad (By\ (2.10))$$

$$\left(f^*_{p,j}\right)''_{z_k z_l}|_0 = \frac{1}{\lambda(p)} L_k L_l \widetilde{f}(p)\cdot\overline{L_j\widetilde{f}(p)}^t,\ \left(f^*_{p,l}\right)''_{z_j w}|_0 = \frac{1}{\lambda(p)} L_j T(\widetilde{f})(p)\cdot\overline{L_l(\widetilde{f})(p)}^t,$$
$$\left(f^*_{p,j}\right)''_{w^2}|_0 = \frac{1}{\lambda(p)} T^2\widetilde{f}(p)\cdot\overline{L_j\widetilde{f}(p)}^t,\ \left(\phi^*_{p,j}\right)''_{z_k z_l}|_0 = \frac{1}{\sqrt{\lambda(p)}} L_k L_l \widetilde{f}(p)\cdot\overline{C_j(p)}^t,$$

$$\left(\phi^*_{p,j}\right)''_{z_k w}|_0 = \frac{1}{\sqrt{\lambda(p)}} T L_k \widetilde{f}(p)\cdot\overline{C_j(p)}^t,\ \left(\phi^*_{p,j}\right)''_{w^2}|_0 = \frac{1}{\sqrt{\lambda(p)}} T^2 \widetilde{f}(p)\cdot\overline{C_j(p)}^t,$$

$$\left(g^*_p\right)''_{z_l z_k}|_0 = \frac{1}{\lambda(p)}\left(L_l L_k g(p) - 2i L_l L_k \widetilde{f}(p)\cdot\overline{\widetilde{f}(p)}^t\right) = 0, \quad (By (2.6))$$
$$\left(g^*_p\right)''_{z_l w}|_0 = \frac{1}{\lambda(p)} L_l\left(T g(p) - 2i T\widetilde{f}(p)\cdot\overline{\widetilde{f}(p)}^t\right) = \frac{2i}{\lambda(p)} L_l\widetilde{f}(p)\cdot\overline{T\widetilde{f}(p)}^t,$$
$$\left(g^*_p\right)''_{w^2}|_0 = \frac{1}{\lambda(p)}\left(T^2 g(p) - 2i T^2\widetilde{f}(p)\cdot\overline{\widetilde{f}(p)}^t - 2i T\widetilde{f}(p)\cdot\overline{T\widetilde{f}(p)}^t\right).$$

We define

$$G_p = \frac{(z^* - \mathbf{a}(p)w^*, w^*)}{1 + 2i\langle z^*, \overline{\mathbf{a}(p)}\rangle + (r(p) - i|\mathbf{a}(p)|^2)w^*} \tag{2.56}$$

where

$$\mathbf{a}(p) := \left(\widetilde{f}^*_p\right)'_w|_0 = (a(p), b(p)) = (a_1(p), .., a_{n-1}(p), b_1(p), ..., b_{N-n}(p)) =$$

$$= \left(\cdots, \frac{T\widetilde{f}(p)\cdot\overline{L_j\widetilde{f}(p)}^t}{\lambda(p)}, \cdots; \cdots, \frac{T\widetilde{f}(p)\cdot\overline{C_j(p)}^t}{\sqrt{\lambda(p)}}, \cdots\right), \tag{2.57}$$

$$r(p) := \frac{1}{2} Re \left(g^*_p\right)''_{w^2}|_0 = \frac{1}{2\lambda(p)} Re\left(T^2 g(p) - 2i T^2\widetilde{f}(p)\cdot\overline{\widetilde{f}(p)}^t\right). \tag{2.58}$$

In particular, because $A = (\frac{E_j}{\sqrt{\lambda}}, C_k)$ is a unitary matrix,

$$|\mathbf{a}(p)|^2 = \frac{1}{\lambda(p)}|E_w(p)|^2 = \frac{1}{\lambda(p)}|T\widetilde{f}(p)|^2. \tag{2.59}$$

We then define the normalization

$$F_p^{**} = (\widetilde{f}_p^{**}, g_p^{**}) = (f_p^{**}, \phi_p^{**}, g_p^{**}) := G_p \circ F_p^{*}. \tag{2.60}$$

$$f_{p,j}^{**} = \frac{f_{p,j}^{*} - a_j(p) g_p^{*}}{1 + 2i\langle \widetilde{f}_p^{*}, \overline{\mathbf{a}(p)}\rangle - (-r(p) + i|\mathbf{a}(p)|^2) g_p^{*}}, \tag{2.61}$$

$$\phi_{p,j}^{**} = \frac{\phi_{p,j}^{*} - b_j(p) g_p^{*}}{1 + 2i\langle \widetilde{f}_p^{*}, \overline{\mathbf{a}(p)}\rangle - (-r(p) + i|\mathbf{a}(p)|^2) g_p^{*}}. \tag{2.62}$$

$$g_p^{**} = \frac{g_p^{*}}{1 + 2i\langle \widetilde{f}_p^{*}, \overline{\mathbf{a}(p)}\rangle - (-r(p) + i|\mathbf{a}(p))|^2) g_p^{*}}. \tag{2.63}$$

The purpose of this normalization is that F_p^{**} must satisfy the following properties:

$$F_p^{**}, \left(f_p^{**} - z\right)'_{z_l}, \left(f_p^{**}\right)'_{w}, \left(\phi_p^{**}\right)'_{z_l}, \left(\phi_p^{**}\right)'_{w}, \left(g_p^{**}\right)'_{z_l}, \left(g_p^{**} - w\right)'_{w}, \left(g_p^{**}\right)''_{z_l z_k},$$

$$\text{and } \left(g_p^{**}\right)''_{w^2} \text{all vanish at } (z,w) = 0. \tag{2.64}$$

From (2.61) (2.62) and (2.63), we have

$$\begin{aligned}
&(f_{p,j}^{**})'_{z_l}|_0 = \delta_j^l, \quad (f_{p,j}^{**})'_{w}|_0 = (f_{p,j}^{*})'_{w}|_0 - a_j(p) = 0,\\
&(\phi_{p,j}^{**})'_{z_l}|_0 = 0, \quad (\phi_{p,j}^{**})'_{w}|_0 = (\phi_{p,j}^{*})'_{w} - b_j(p) = 0,\\
&(g_p^{*})'_{z_l}|_0 = 0, \quad (g_p^{*})'_{w}|_0 = 0.
\end{aligned}$$

$$\begin{aligned}
&\left(f_{p,j}^{**}\right)''_{z_k z_l}|_0 = \left(f_{p,j}^{*}\right)''_{z_k z_l}|_0 - 2i\delta_j^k \overline{a_l(p)} - 2i\delta_j^l \overline{a_k(p)}\\
= \ &\tfrac{1}{\lambda(p)} L_k L_l \widetilde{f}(p) \cdot \overline{L_j \widetilde{f}(p)}^t - \tfrac{2i\delta_j^k}{\lambda(p)} \overline{T\widetilde{f}(p)} \cdot L_l \widetilde{f}(p)^t - \tfrac{2i\delta_j^l}{\lambda(p)} \overline{T\widetilde{f}(p)} \cdot L_k \widetilde{f}(p)^t = 0.
\end{aligned} \tag{2.65}$$

Here we used the fact that $(g_p^{*})''_{z_j z_k}|_0 = 0$. The last equality holds because of Lemma 2.7.1 (i).

$$\begin{aligned}
\left(f^{**}_{p,l}\right)''_{z_jw}|_0 &= \left(f^{*}_{p,l}\right)''_{z_jw}|_0 - a_l(p)\left(g^{*}_{p}\right)''_{z_jw}|_0 - \delta^l_j\left[2i(\widetilde{f^*_p})'_w|_0\cdot\overline{\mathbf{a}} + (r(p) - i|\mathbf{a}(p)|^2)\right]\\
&= \left(f^{*}_{p,l}\right)''_{z_jw}|_0 - a_l(p)\left(g^{*}_{p}\right)''_{z_jw}|_0 - \delta^l_j[i|\mathbf{a}(p)|^2 + r(p)]\\
&= \frac{1}{\lambda(p)}L_jT\widetilde{f}(p)\cdot\overline{L_l\widetilde{f}(p)}^t - \frac{2i}{\lambda(p)^2}\left(T\widetilde{f}(p)\cdot\overline{L_l\widetilde{f}(p)}^t\right)\left(L_j\widetilde{f}(p)\cdot\overline{T\widetilde{f}(p)}^t\right)\\
&\quad -\frac{i\delta_{jl}}{\lambda(p)}|T\widetilde{f}(p)|^2 - \frac{\delta_{jl}}{2\lambda(p)}Re\left(T^2g(p) - 2iT^2\widetilde{f}(p)\cdot\overline{\widetilde{f}(p)}^t\right).
\end{aligned}$$

We can say more about this important formula which will be used to define geometric rank κ_0. Applying T^2 to the basic equation $Im(g) = |\widetilde{f}|^2$, we get $0 = 2iIm(iT^2\widetilde{f}\cdot\overline{\widetilde{f}}^t) + 2i|T\widetilde{f}|^2 - i\ Im(T^2g)$ on $\partial\mathbb{H}^n$ by (2.8). Combining this to the above, we get

$$\begin{aligned}
\left(f^{**}_{p,l}\right)''_{z_jw}|_0 &= \tfrac{1}{\lambda(p)}L_jT\widetilde{f}(p)\cdot\overline{L_l\widetilde{f}(p)}^t - \tfrac{2i}{\lambda(p)^2}\left(T\widetilde{f}(p)\cdot\overline{L_l\widetilde{f}(p)}^t\right)\left(L_j\widetilde{f}(p)\cdot\overline{T\widetilde{f}(p)}^t\right)\\
&\quad -\tfrac{\delta_{jl}}{2\lambda(p)}\left(T^2g(p) - 2iT^2\widetilde{f}(p)\cdot\overline{\widetilde{f}(p)}^t\right).
\end{aligned} \tag{2.66}$$

$$\begin{aligned}
\left(f^{**}_{p,l}\right)''_{w^2}|_0 &= \left(f^{*}_{p,l}\right)''_{w^2}|_0 - a_l(p)\left(g^{*}_{p}\right)''_{w^2}|_0\\
&= \tfrac{1}{\lambda(p)}T^2\widetilde{f}(p)\cdot\overline{L_l\widetilde{f}(p)}^t - \tfrac{1}{\lambda(p)^2}\left(T\widetilde{f}\cdot\overline{L_l\widetilde{f}}^t\right)\left(T^2g - 2iT^2\widetilde{f}\cdot\overline{\widetilde{f}}^t - 2i|T\widetilde{f}|^2\right)(p).
\end{aligned} \tag{2.67}$$

$$\left(\phi^{**}_{p,l}\right)''_{z_jz_k}|_0 = \left(\phi^{*}_{p,l}\right)''_{z_jz_k}|_0 - b_l(g^*_p)''_{z_jz_k} = \left(\phi^{*}_{p,l}\right)''_{z_jz_k}|_0 = \tfrac{1}{\sqrt{\lambda(p)}}L_jL_k\widetilde{f}(p)\cdot\overline{C_l(p)}^t. \tag{2.68}$$

Here we used the fact that $(g^*_p)''_{z_jz_k}|_0 = 0$.

$$\begin{aligned}
&\left(\phi^{**}_{p,l}\right)''_{z_jw}|_0 = \left(\phi^{*}_{p,l}\right)''_{z_jw}|_0 - b_l(p)(g^*_p)''_{z_jw}|_0 \\
=\ & \tfrac{1}{\sqrt{\lambda(p)}} TL_j\widetilde{f}(p)\cdot\overline{C_l(p)}^t - \tfrac{1}{\lambda(p)^{3/2}}\left(T\widetilde{f}(p)\cdot\overline{C_l(p)}^t\right)L_j\left(Tg(p)-2iT\widetilde{f}(p)\cdot\overline{\widetilde{f}(p)}^t\right) \\
=\ & \tfrac{1}{\sqrt{\lambda(p)}} TL_j\widetilde{f}(p)\cdot\overline{C_l(p)}^t - \tfrac{2i}{\lambda(p)^{3/2}}\left(T\widetilde{f}(p)\cdot\overline{C_l(p)}^t\right)\left(L_j\widetilde{f}(p)\cdot T\overline{\widetilde{f}}^t(p)\right).
\end{aligned} \tag{2.69}$$

$$\begin{aligned}
&\left(\phi^{**}_{p,l}\right)''_{w^2}|_0 = \left(\phi^{*}_{p,l}\right)''_{w^2}|_0 - b_j(p)\left(g^*_p\right)''_{w^2}|_0 \\
=\ & \tfrac{1}{\sqrt{\lambda(p)}} T^2\widetilde{f}(p)\cdot\overline{C_l(p)}^t - \tfrac{1}{\lambda(p)^{3/2}}\left(T\widetilde{f}(p)\cdot\overline{C_l(p)}^t\right)\left(T^2g(p)-2iT^2\widetilde{f}(p)\cdot\overline{\widetilde{f}(p)}^t-2i|T\widetilde{f}(p)|^2\right).
\end{aligned} \tag{2.70}$$

$$\left(g^{**}_p\right)''_{z_jz_k}|_0 = 0,$$

$$\left(g^{**}_p\right)''_{z_jw}|_0 = \left(g^{*}_p\right)''_{z_jw}|_0 - 2i\overline{a_j(p)} = \frac{2i}{\lambda(p)}L_j\widetilde{f}(p)\cdot\overline{T\widetilde{f}(p)}^t - \frac{2i}{\lambda(p)}\overline{T\widetilde{f}(p)\cdot\overline{L_j\widetilde{f}(p)}^t} = 0,$$

$$\begin{aligned}
&\left(g^{**}_p\right)''_{w^2}|_0 = \left(g^{*}_p\right)''_{w^2}|_0 - 2\left[i|a_j(p)|^2 + r(p)\right] \\
=\ & \frac{1}{\lambda(p)}\left(T^2g(p)-2iT^2\widetilde{f}(p)\cdot\overline{\widetilde{f}(p)}^t\right) \\
& -\frac{2}{\lambda(p)}\left[i|T\widetilde{f}(p)|^2+\frac{1}{2}Re\left(T^2g(p)-2iT^2\widetilde{f}(p)\cdot\overline{\widetilde{f}(p)}^t\right)\right] = 0.
\end{aligned}$$

The above two equalities equal to zero because of Lemma 2.7.1 (i).

By the similar calculation of F^* and F^{**}, we can define F^*_p and F^{**}_p with the following theorem.

Theorem 2.8.1 *([H99], Lemma 5.3) Let $F \in Prop_2(\mathbb{H}^n, \mathbb{H}^N)$, $2 \le n \le N$ with $F(0) = 0$. For each $p \in \partial\mathbb{H}^n$, there is an automorphism $\tau_p^{**} \in Aut_0(\mathbb{H}^N)$ such that $F_p^{**} := \tau_p^{**} \circ F_p = (f_p^{**}, \phi_p^{**}, g_p^{**})$ satisfies the following normalization:*

$$f_p^{**} = z + \frac{i}{2} a_p^{**(1)}(z)w + o_{wt}(3), \;\; \phi_p^{**} = \phi_p^{**(2)}(z) + o_{wt}(2), \;\; g_p^{**} = w + o_{wt}(4), \tag{2.71}$$

$$\langle \overline{z}, a_p^{**(1)}(z) \rangle |z|^2 = |\phi_p^{**(2)}(z)|^2. \tag{2.72}$$

2.9 Geometric Rank of F

We denote $a_p^{**(1)}(z) = z\mathcal{A}(p)$ where

$$\mathcal{A}(p) = -2i \left(\frac{\partial^2 f_{p,l}^{**}}{\partial z_j \partial w}|_0 \right)_{1 \le j,l \le n-1}$$

is an $(n-1) \times (n-1)$ matrix. $\mathcal{A}(p)$ is Hermitian. In fact, (2.72) can be written as $z\mathcal{A}(p)\overline{z}^t|z|^2 = |\phi^{**(2)}(z)|^2$, $\forall z$. Then $z\overline{\mathcal{A}(p)}^t \overline{z}^t |z|^2 = |\phi^{**(2)}(z)|^2$ so that $z\big(\mathcal{A}(p) - \overline{\mathcal{A}(p)}^t\big)\overline{z}^t = 0$, $\forall z$. This implies that $\mathcal{A}(p) = \overline{\mathcal{A}(p)}^t$, i.e., $\mathcal{A}(p)$ is Hermitian. Also, from (2.72), the matrix $\mathcal{A}(p)$ is semi-positive.

We define [H03]

$$Rk_F(p) := Rank(\mathcal{A}(p)), \tag{2.73}$$

which is called the *geometric rank of F at p* and is a lower semi-continuous function on p. We also define

$$\kappa_0 = \kappa_0(F) := max_{p \in \partial\mathbb{H}^n} Rk_F(p) \tag{2.74}$$

which is called the *geometric rank of F*.

<u>Remarks</u> (i) $\kappa_0(F)$ is an invariant.

(ii) $0 \le \kappa_0(F) \le n-1$.

(iii) $\kappa_0(F) = \kappa_0$ if and only if at a generic point $p \in \partial\mathbb{H}^n$, $F \cong F_p^{**}$ that satisfies

$$\begin{cases} f_{j,p}^{**} = z_j + \frac{i\mu_j(p)}{2} z_j w + o_{wt}(3), \quad 1 \le j \le \kappa_0, \mu_j(p) > 0 \\ f_{j,p}^{**} = z_j + o_{wt}(3), \quad \kappa_0 + 1 \le j \le n-1, \\ \phi_p^{**} = \phi_p^{(2)**}(z) + o_{wt}(2), \\ g_p^{**} = w + o_{wt}(4). \end{cases}$$

(iv) When $\kappa_0(F) = n-1$, the image submanifold $F(\partial\mathbb{H}^n)$ "occupies more room" in the target space $\partial\mathbb{H}^N$ so that it is the most complicated case. In fact, when $\kappa_0(F) \le n-2$, F has "semi-linearity" properties.

2.10 Maps with Geometric Rank $\kappa_0 = 0$

Theorem 2.10.1 *(Linearity Criterion, [H99])*

$$\kappa_0 = 0 \Longleftrightarrow F \text{ is equivalent to the linear map.}$$

To prove this theorem, let us first prove two lemmas.

Lemma 2.10.2 *Let m and n be any positive integers. Let $X = (f_1, ..., f_m)$ be a vector-valued differentiable function defined in a neighborhood of 0 in $\mathbb{R}^n$ satisfying*

$$DX = A(x)X^t, \quad X(0) = 0,$$

where $D = (\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n})$ and $A(x)$ is a matrix of continuous functions. Then $X \equiv 0$ holds in some neighborhood of 0 in $\mathbb{R}^n$.

Proof of Lemma 2.10.2: $\forall p \in \mathbb{R}^n$ near 0, we denote $X_p(t) := X(tp) = (f_1(tp), ..., f_m(tp))$ for $0 \le t \le 1$. Then

$$\frac{dX_p}{dt} = \big(\frac{d}{dt}f_1(tp), ..., \frac{d}{dt}f_m(tp)\big) = \big(\sum_{j=1}^{n}\frac{\partial f_1}{\partial x_j}p_j, ..., \sum_{j=1}^{n}\frac{\partial f_m}{\partial x_j}p_j\big) = pDX_p = pA(tp)X_p(t)^t.$$

Since $X_p(0) = 0$, we get $X_p(t) = \int_0^t pA(\tau p)X_p(\tau)^t d\tau$. Hence $\|X_p\| \le C\|p\|\|X_p\|$ for some constant $C > 0$ which is independent of p. It follows that $X_p \equiv 0$ once $\|p\| < \frac{1}{c}$. $\square$

Lemma 2.10.3 *We have*

(i) For any $p \in \partial\mathbb{H}^n$,

$$L_kL_l\widetilde{f}(p)\cdot\overline{L_j\widetilde{f}(p)}^t = 2\sqrt{-1}\delta_k^j\Big(\overline{T\widetilde{f}(p)}\cdot L_l\widetilde{f}(p)^t\Big) + 2\sqrt{-1}\delta_l^j\Big(\overline{T\widetilde{f}(p)}\cdot L_k\widetilde{f}(p)^t\Big).$$

(ii) For any fixed j and k, if $(\phi_p^{**})''_{z_jz_k}|_0 = 0$ for any $p \in \partial\mathbb{H}^n$[1]*, then*

$$L_jL_k\widetilde{f}(p) = \frac{2\sqrt{-1}}{\lambda}\left(\overline{T\widetilde{f}(p)} \cdot L_j\widetilde{f}(p)^t\right)L_k\widetilde{f}(p) + \frac{2\sqrt{-1}}{\lambda}\left(\overline{T\widetilde{f}(p)} \cdot L_k\widetilde{f}(p)^t\right)L_j\widetilde{f}(p).$$

Proof (i) By the construction of F^{**}, we know that $\left(f_{p,l}^{**}\right)''_{z_jz_k}|_0 = 0$. By (2.65), we have

$$\left(f_{p,l}^{**}\right)''_{z_jz_k}|_0 = \frac{1}{\lambda(p)}L_kL_l\widetilde{f}(p) \cdot \overline{L_j\widetilde{f}(p)}^t - \frac{2i\delta_j^k}{\lambda(p)}\overline{T\widetilde{f}(p)} \cdot L_l\widetilde{f}(p)^t - \frac{2i\delta_j^l}{\lambda(p)}\overline{T\widetilde{f}(p)} \cdot L_k\widetilde{f}(p)^t = 0.$$

Then (i) follows.

(ii) By the formula (2.68), we see that $(\phi_p^{**})''_{z_kz_l}|_0 \equiv 0$ if and only if $L_kL_l\widetilde{f}(p) \cdot \overline{C(p)}^t = 0$. Then $L_kL_l\widetilde{f}(p)$ is perpendicular to the subspace $span\{C(p)\}$ so that they are linear combination of the vectors $E_s(p)$: $L_kL_l\widetilde{f}(p) = \sum_{s=1}^{n-1}\lambda_{kl}^sE_s(p)$, and hence $L_kL_l\widetilde{f}(p) \cdot \overline{E_j(p)}^t = \sum_{s=1}^{n-1}\lambda_{kl}^sE_s(p) \cdot \overline{E_j(p)}^t = \lambda\lambda_{kl}^j$. Here we have used the orthogonal property: $E_s \cdot \overline{E_j}^t = \lambda\delta_{sj}$ in (2.18). Finally we use (i) to obtain the desired identity. □

Proof of Theorem 2.10.1: By the normalization condition, we assume $F = F^{**}$.

If we can show $\phi \equiv 0$, then $(f, g) : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ is a C^2-smooth CR map. By Poincaré-Tanaka theorem, $(f, g) \in Aut(\partial\mathbb{H}^n) = Aut(\mathbb{H}^n)$ so that (f, g) must be linear fractional. This implies that $F(z, w)$ is a linear map.

Since $\phi(0) = 0$, it suffices to show $X\phi \equiv 0$ for any tangent vector field X over $\partial\mathbb{H}^n$. Since L_j, $\overline{L_j}$ and T form a basis for $T(\partial\mathbb{H}^n)$ and ϕ is CR, it suffices to show that $L_j\phi \equiv 0$ and $T\phi \equiv 0$ for all $1 \le j \le n-1$.

By applying Lemma 2.10.2, it is enough for us to prove

$$\begin{cases} L_j(L_k(\phi)) &= A_j(z,w)L_k(\phi) + A_k(z,w)L_j(\phi); \\ TL_k\phi &= B_{k,1}(z,w)L_k(\phi) + B_{k,2}(z,w)T(\phi); \\ T^2\phi &= C_{k,1}(z,w)L_k(\phi) + C_{k,2}(z,w)T(\phi), \end{cases} \tag{2.75}$$

where $A_k, B_{k,1}, B_{k,2}, C_{k,1}$ and $C_{k,2}$ are continuous function defined in a neighborhood of 0 in $\partial\mathbb{H}^n$.

[1] Notice that the geometric rank $\kappa_0 = 0$ if and only if $(\phi_p^{**})''_{z_jz_k}|_0 = 0$, $\forall j, k$. In fact, by (2.72), this condition implies $\mathcal{A}(p) = 0$.

Notice $\kappa_0 = 0 \Longleftrightarrow (\phi_p^{**})''_{z_j z_k} = 0$, $\forall p \in \partial\mathbb{H}^n$. From Lemma 2.10.3(ii), we obtain

$$L_j(L_k(\phi)) = A_j(z,w)L_k(\phi) + A_k(z,w)L_j(\phi), \tag{2.76}$$

where $A_k := \frac{2i\overline{T\widetilde{f}}\cdot(L_k\widetilde{f})^t}{\lambda(z,w)}$ which are $C^1(\partial\mathbb{H}^n)$. Then the first equality of (2.75) is proved.

Putting $j = k$ in (2.76), we get $L_k^2(\phi) = 2A_kL_k(\phi)$. Applying $\overline{L_k}$ and by Lemma 2.3.1(ii)(iii), we have

$$TL_k\phi = \frac{\overline{L_k}A_k}{2i}L_k(\phi) + A_kT(\phi) = B_{k,1}L_k(\phi) + B_{k,2}T(\phi), \tag{2.77}$$

where $B_{k,1} := \frac{\overline{L_k}A_k}{2i} \in C^0(\partial\mathbb{H}^n)$ and $B_{k,2} := A_k \in C^1(\partial\mathbb{H}^n)$. We have proved the second equality of (2.75).

Applying $\overline{L_k}$ again to (2.77), we obtain

$$2iT^2(\phi) = (\overline{L_k}B_{k,1})L_k(\phi) + (B_{k,1}2i + \overline{L_k}B_{k,2})T(\phi) = C_{k,1}L_k(\phi) + C_{k,2}T(\phi), \tag{2.78}$$

where $C_{k,2} := B_{k,1}2i + \overline{L_k}B_{k,2} \in C^0(\partial\mathbb{H}^n)$ because of $B_{k,2} \in C^1(\partial\mathbb{H}^n)$, and $C_{k,1} := \overline{L_k}B_{k,1}$.

It remains to prove the following claim: $C_{k,1}$ is continuous. In fact, when $j = k$, apply Lemma 2.10.2(ii) and take the component f_k, as we did for (2.76), we get $A_k = \frac{L_k^2(f_k)}{2L_k(f_k)}$. Then

$$\begin{aligned} B_{k,1} &= \tfrac{1}{2i}\overline{L_k}(A_k) = \tfrac{1}{4i}\overline{L_k}\left(\tfrac{1}{L_k(f_k)}\right)L_k^2(f_k) + \tfrac{1}{L_k(f_k)}TL_k(f_k) \\ &= -\tfrac{T(f_k)}{2(L_k(f_k))^2}L_k^2(f_k) + \tfrac{1}{L_k(f_k)}TL_k(f_k) \\ &= b_{k,1}L_k^2(f_k) + b_{k,2}TL_k(f_k), \end{aligned} \tag{2.79}$$

where $b_{k,1}, b_{k,2} \in C^1(\partial\mathbb{H}^n)$. Thus

$$\begin{aligned} C_{k,1} &= \overline{L_k}B_{k,1} = \overline{L_k}(b_{k,1}L_k^2(f_k) + b_{k,2}TL_k(f_k)) \\ &= \overline{L_k}b_{k,1}\cdot L_k^2 f_k + 4ib_{k,1}L_kTf_k + \overline{L_k}b_{k,2}\cdot TL_k(f_k) + 2ib_{k,2}T^2(f_k) \\ &\in C^0(\partial\mathbb{H}^n). \end{aligned} \tag{2.80}$$

Hence the claim is proved so that the third equality in (3.11) is proved. □

2.11 Analytic Proof of the First Gap Theorem

By Theorem 2.10.1, in order to complete the proof of the First Gap Theorem, we need to show

Corollary 2.11.1 *Let $F \in Prop_2(\mathbb{B}^n, \mathbb{B}^N)$ with $2 \le n \le N \le n-2$. Then F has geometric rank $\kappa_0 = 0$.*

Proof: Let $F \in Prop_2(\mathbb{B}^n, \mathbb{B}^N)$ with $2 \le n \le N \le n-2$. Then for any $p \in \partial\mathbb{H}^n$, F_p^{**} satisfies the normalization condition in (2.72) and

$$\langle \overline{z}, a_p^{**(1)}(z)\rangle |z|^2 = |\phi_p^{**(2)}(z)|^2.$$

Since $n \le 2n-2$, by a uniqueness theorem 2.11.2 below, it implies

$$\phi_p^{**(2)} \equiv 0 \;\; and \;\; a_p^{**(1)} \equiv 0. \tag{2.81}$$

Thus $\kappa_0(F) = 0$. □

Theorem 2.11.2 *([H99], [EHZ05]) Let ϕ_j, ψ_j be holomorphic function near the origin of $\mathbb{C}^n$, $1 \le j \le k$, $n > 1$. Suppose that $H(z, \overline{z})$ is a real analytic function defined in a neighborhood of $0 \in \mathbb{C}^n$ such that*

$$H(z, \overline{z})|z|^2 = \sum_{j=1}^{k} \phi_j(z)\overline{\psi_j(z)} \;\; for \; z \in \mathbb{C}^n \; near \; 0, . \tag{2.82}$$

Suppose $k \le n-1$. Then $H(z, \overline{z}) \equiv 0$ and $\sum_{j=1}^{k} \phi_j(z)\overline{\psi_j(z)} \equiv 0$.

Proof: Complexifying the identity, we have

$$H(z, \overline{\zeta})\langle z, \overline{\zeta}\rangle = \sum_{j=1}^{k} \phi_j(z)\overline{\psi_j(\zeta)} \tag{2.83}$$

where z, ζ are independent variables. Assume that $\phi_j \not\equiv 0$ for each $1 \le j \le k$. We can find a point z_0 near the origin such that $\phi_j(z_0) = \epsilon_j \ne 0$ for each j.

Consider the complex variety $V_{z_0} = \{z \mid \phi_j(z) = \phi_j(z_0), \; 1 \le j \le k\}$. Since $k \le n-1$, this variety V_{z_0} has complex dimension at least 1. For each $z^* \in V_{z_0}$, there exists a complex hyperplane $K_{z^*} = \{\zeta | \; \langle z^*, \overline{\zeta}\rangle = 0\}$. Then for any $\zeta \in K_{z^*}$, we have $\sum_j^k \epsilon_j \overline{\psi_j(\zeta)} = 0$. Since $\dim_{\mathbb{C}} V_{z_0} \ge 1$ and $\dim_{\mathbb{C}} K_{z^*} = n-1$, such ζ fills in an open subset of $\mathbb{C}^n$. Hence $\sum_j^k \epsilon_j \overline{\psi_j(\zeta)} = 0$, or $\overline{\psi_k(z)} + \sum_{j=1}^{k-1} \frac{\epsilon_j}{\epsilon_k}\overline{\psi_j(z)} = 0$. Multiplying with $\psi_k(z)$ and subtracting this to (2.82), we obtain

$$H(z, \overline{z})\langle z, \overline{z}\rangle = \sum_{j=1}^{k-1} \left(\phi_j(z) - \frac{\epsilon_j}{\epsilon_k}\phi_k(z)\right)\overline{\psi_j(z)}.$$

Then applying an induction argument, it follows easily that $\sum \phi_j \overline{\psi_j} \equiv 0$ and $H \equiv 0$. $\square$

Theorem 2.11.2 can be extended into a more general version by induction as follows.

Corollary 2.11.3 *Let ϕ_{jp}, ψ_{jp}, $1 \leq j \leq n-1, 0 \leq p \leq q$, be holomorphic functions near the origin of $\mathbb{C}^n$ with $n > 1$. Suppose that $H(z, \overline{z})$ is a real analytic function defined in a neighborhood of $0 \in \mathbb{C}^n$ such that*

$$H(z, \overline{\zeta})\langle z, \overline{\zeta}\rangle_\ell^{q+1} = \sum_{p=0}^{q} \left(\sum_{j=1}^{n-1} \phi_{jp}(z)\overline{\psi_{jp}(\zeta)} \right) \langle z, \overline{\zeta}\rangle_\ell^p, \quad for\ z \sim 0\ and\ \zeta \sim 0.$$

Then $H(z, \overline{\zeta}) \equiv 0$ and $\sum_{j=1}^{n-1} \phi_{jp}(z)\overline{\psi_{jp}(\zeta)} \equiv 0$, $1 \leq p \leq q$.

Chapter 3

Construction and Classification of Rational Maps

3.1 Gap Phenomenon

A map $F \in Prop(\mathbb{B}^n, \mathbb{B}^N)$ is called *minimum* if F is not equivalent to a map of the form $(G, 0)$ where $G \in Prop(\mathbb{B}^n, \mathbb{B}^{N'})$ with $N' < N$.

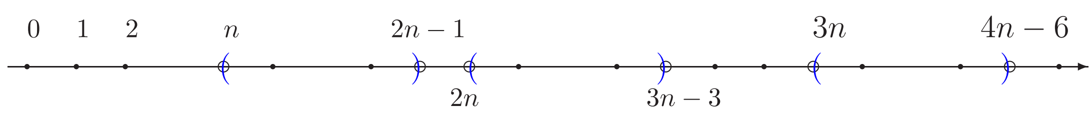
Recall the First Gap Theorem in Lecture 1:

> Any $F \in Prop_2(\mathbb{B}^n, \mathbb{B}^N)$ where $N < 2n - 1$ is equivalent to a linear map $(z, w) \mapsto (z, 0, w)$.

This theorem can be restated as

Theorem 3.1.1 *(The First Gap Theorem) There is no minimum map in* $Prop_2(\mathbb{B}^n, \mathbb{B}^N)$ *if*

$$N \in \mathcal{I}_1 = \{m \in \mathbb{Z}^+ \mid n < m < 2n - 1\}.$$



Furthermore, it is proved by Huang-Ji-Xu [HJX06] that if $F \in Prop_3(\mathbb{B}^n, \mathbb{B}^N)$ with $2n < N < 3n - 3$, then F is equivalent to another map $(G, 0)$ where $G \in Rat(\mathbb{B}^n, \mathbb{B}^{2n})$. As above, this theorem can be rewritten as

Theorem 3.1.2 *(The Second Gap Theorem) (Huang-Ji-Xu, [HJX06]) There is no minimum map in* $Prop_3(\mathbb{B}^n, \mathbb{B}^N)$ *if* $n \geq 4$ *and*

$$N \in \mathcal{I}_2 = \{m \in \mathbb{Z}^+ \mid 2n < m < 3n - 3\}.$$

Theorem 3.1.3 *(The Third Gap Theorem, Huang-Ji-Yin, preprint) There is no minimum map in* $Prop_3(\mathbb{B}^n, \mathbb{B}^N)$ *if* $n \geq 7$ *and*

$$N \in \mathcal{I}_3 = \{m \in \mathbb{Z}^+ \mid 3n < m < 4n - 6\}.$$

In general, we formulate the following: For the integer $n > 0$, let

$$K(n) := \max\{t \in \mathbb{Z}^+ \mid \frac{t(t+1)}{2} < n\}.$$

For integer k with $1 \leq k \leq K(n)$, let

$$\mathcal{I}_k := \left\{ m \in \mathbb{Z}^+ \mid kn < m < (k+1)n - \frac{k(k+1)}{2} \right\}$$

[Example]

If $n \geq 2$, then $K(n) \geq 1$. Take $k = 1$ and $\mathcal{I}_1 = \{m \in \mathbb{Z}^+ \mid n < m < 2n - 1\}$.
If $n \geq 4$, then $K(n) \geq 2$. Take $k = 2$ and $\mathcal{I}_2 = \{m \in \mathbb{Z}^+ \mid 2n < m < 3n - 3\}$.
If $n \geq 7$, then $K(n) \geq 3$. Take $k = 3$ and $\mathcal{I}_3 = \{m \in \mathbb{Z}^+ \mid 3n < m < 4n - 6\}$.

Theorem 3.1.4 *(Huang-Ji-Yin, [HJY09]) For* $n > 2$*, let* $K(n)$ *be as above. For each* k *with* $1 \leq k \leq K(n)$*, let* $\mathcal{I}_k$ *be as above. Then for each* $N > n$ *with*

$$N \notin \cup_{k=1}^{K(n)} \mathcal{I}_k,$$

there exists a minimum monomial map in $Rat(\mathbb{B}^n, \mathbb{B}^N)$.

Conjecture: *For* $n > 2$*, let* $K(n)$ *be as above. For each* k *with* $1 \leq k \leq K(n)$*, let* $\mathcal{I}_k$ *be as above. Then for each* $N > n$*, the following two statements are equivalent:*

(i) There exists no minimum maps in $Prop_2(\mathbb{B}^n, \mathbb{B}^N)$.
(ii) $N \in \mathcal{I}_k$ *for some* k *with* $1 \leq k \leq K(n)$.

Recently, D'Angelo and Lebl (2007) found out that there is no gap phenomenon for mappings in $Rat(\mathbb{B}^n, \mathbb{B}^N)$ when $N \geq T(n) = n^2 - 2n + 2$.

Based on the above conjecture, it would imply that there is no gap phenomenon for mappings in $Rat(\mathbb{B}^n, \mathbb{B}^N)$ when $N > n^{3/2}$.

3.2 Examples of Minimum Maps

Let us survey some important minimum maps.

• $N = n \geq 2$, Alexander's theorem. [A77], any map in $Prop_2(\mathbb{B}^n, \mathbb{B}^n) = Aut(\mathbb{B}^n)$ is equivalent to the identity map $F(z, w) = (z, w)$.

• $n < N < 2n - 1$, the first gap theorem, any map in $Prop_2(\mathbb{B}^n, \mathbb{B}^N)$ is equivalent to the linear map $F(z, w) = (z, 0, w)$.

• $N = 2n - 1$ with $n \geq 3$, Huang and Ji (2001) [HJ01], F is equivalent to the linear map $F(z, w) = (z, 0, w)$, or F is equivalent to Whitney map:

$$W_{n,1} = (z', wz) \quad where \;\; z = (z', w) \in \mathbb{C}^{n-1} \times \mathbb{C}.$$

• $N = 2n - 1 = 3$ with $n = 2$, Faran (1982) [Fa82], four equivalent classes of maps:

$$(z, w, 0); \quad (z, zw, w^2), \quad (z^2, \sqrt{2}zw, w^2); \quad (z^3, \sqrt{3}zw, w^3).$$

• $N = 2n$, *D'Angelo family* [DA88].

$$F_\theta = (z, wcos\theta, z_1 wsin\theta, ..., z_{n-1} wsin\theta, w^2 sin\theta), \quad with \;\; 0 \leq \theta \leq \frac{\pi}{2},$$

is a family of proper holomorphic monomial maps from $\mathbb{B}^n$ into $\mathbb{B}^{2n}$. Here F_θ is equivalent to $F_{\theta'}$ if and only if $\theta = \theta'$.

Denote $W_{n,1}(z; h, \lambda) = (z', \lambda z_n, \sqrt{1 - \lambda^2} z_n h(z))$ where $z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$, $\lambda \in [0, 1]$ and h is a holomorphic map from $\overline{\mathbb{B}^n}$ into $\mathbb{B}^{N'}$. In particular, when $h(z) = z$, the maps

$$W_{n,1}(z; z, \lambda) = (z', \lambda z_n, \sqrt{1 - \lambda^2} z_n z)$$

is the D'Angelo's family.

• $2n < N < 3n - 3$ with $n \geq 4$, by the second gap theorem [HJX06] any $F \in Prop_3(\mathbb{B}^n, \mathbb{B}^N)$ is equivalent to a map $\big(W_{n,1}(z; z, \lambda), 0\big)$ where $\lambda \in [0, 1]$.

The proof of Theorem 3.1.4 is based on the construction of the following minimum maps.

[Example A][HJY09] Let

$$\begin{cases} \psi_1 = (z_1, \sqrt{2}z_2, ..., \sqrt{2}z_k, z_{k+1}, ..., z_n), \\ \psi_2 = (z_2, \sqrt{2}z_3, ..., \sqrt{2}z_k, z_{k+1}, ..., z_n), \\ ......, \\ \psi_{k-1} = (z_{k-1}, \sqrt{2}z_k, z_{k+1}, ..., z_n), \\ \psi_k = (z_k, z_{k+1}, ..., z_n), \\ \psi_{k+1} = (z_{k+1}, ..., z_n). \end{cases}$$

Let

$$W_{n,k}(z) = W_{n,k}(z_1, ..., z_n) := \big(z_1\psi_1, ..., z_k\psi_k, \psi_{k+1}\big).$$

This map, called *generalized Whitney map*, is a quadratic polynomial minimum map in $Prop(\mathbb{B}^n, \mathbb{B}^N)$ where

$$N = (k+1)n - \frac{k(k+1)}{2}.$$

When $n = 1$, $W_{n,1} : \mathbb{B}^n \to \mathbb{B}^{2n-1}$ is given by

$$W_{n,1}(z_1, ..., z_n) = (z_1\psi_1, \psi_2) = \big(z_1(z_1, ..., z_n), (z_2, ..., z_n)\big).$$

We can verify $|W_{n,1}(z_1, ..., z_n)|^2 = 1$, $\forall |z_1|^2 + ... + |z_n|^2 = 1$. In fact,

$$\begin{array}{ccc} |z_1|^2(|z_1|^2 + .. + |z_n|^2) + (|z_2|^2 + ... + |z_n|^2) & ? & = 1, \quad \forall |z_1|^2 + ... + |z_n|^2 = 1, \\ \| & & \\ |z_1|^2 + (|z_2|^2 + ... + |z_n|^2) & = & 1 \end{array}$$

When $n = 2$, $W_{n,2} : \mathbb{B}^n \to \mathbb{B}^{3n-3}$ is given by

$$W_{n,2} = (z_1\psi_1, z_2\psi_2, \psi_3)$$

where $\psi_1 = (z_1, \sqrt{2}z_2, z_3, ..., z_n)$, $\psi_2 = (z_2, ..., z_n)$ and $\psi_3 = (z_3, ..., z_n)$. We can verify $|W_{n,2}(z_1, ..., z_n)|^2 = 1$, $\forall |z_1|^2 + ... + |z_n|^2 = 1$. In fact, $\forall |z_1|^2 + ... + |z_n|^2 = 1$, we have

$$\begin{array}{ccc} |z_1|^2(|z_1|^2 + 2|z_2|^2 + |z_3|^2 + ... + |z_n|^2) + |z_2|^2(|z_2|^2 + ... + |z_n|^2) + (|z_3|^2 + ... + |z_n|^2) & ? & = 1, \\ \| & & \\ |z_1|^2(1 + |z_2|^2) + |z_2|^2(|z_2|^2 + ... + |z_n|^2) + (|z_3|^2 + ... + |z_n|^2) & & \\ \| & & \\ |z_1|^2 + |z_2|^2(|z_1|^2 + |z_2|^2 + ... + |z_n|^2) + (|z_3|^2 + ... + |z_n|^2) & & \\ \| & & \\ |z_1|^2 + |z_2|^2 + |z_3|^2 + ... + |z_n|^2 & = & 1 \end{array}$$

$W_{n,1}$ can also be written as $(z,w) \mapsto (z, w(z,w))$ where $(z,w) \in \mathbb{C}^{n-1} \times \mathbb{C}$, which is the classical *Whitney map*.

[Example B] [HJY09] Let Let ψ_j be defined as above. Let τ be an integer with $1 \le \tau \le k$ and $\lambda_j \in (0,1)$ with $1 \le j \le \tau$. We define

$$W_{n,k}(z;\lambda_1,...,\lambda_\tau) = \left(z_1\widetilde{\psi_1},...,z_k\widetilde{\psi_k},\psi_{k+1},\lambda_1 z_1,...,\lambda_\tau z_\tau\right)$$

where

$$\begin{cases} \widetilde{\psi}_1 = (\sqrt{1-\lambda_1^2}z_1, \sqrt{1-\lambda_1^2+\mu_{12}^2}z_2, ..., \sqrt{1-\lambda_1^2+\lambda_{1k}^2}z_k, \sqrt{1-\lambda_1^2}z_{k+1}, ..., \sqrt{1-\lambda_1^2}z_n), \\ \widetilde{\psi}_2 = (\sqrt{1-\lambda_2^2}z_2, \sqrt{1-\lambda_2^2+\mu_{23}^2}z_3, ..., \sqrt{1-\lambda_2^2+\lambda_{2k}^2}z_k, \sqrt{1-\lambda_2^2}z_{k+1}, ..., \sqrt{1-\lambda_2^2}z_n), \\ ......, \\ \widetilde{\psi}_\tau = (\sqrt{1-\lambda_\tau^2}z_\tau, \sqrt{1-\lambda_\tau^2+\mu_{\tau(\tau+1)}^2}z_{\tau+1}, ..., \sqrt{1-\lambda_\tau^2+\lambda_{\tau k}^2}z_k, \sqrt{1-\lambda_\tau^2}z_{k+1}, ..., \\ \qquad , \sqrt{1-\lambda_\tau^2}z_n), \quad for\ \tau < k \\ \widetilde{\psi}_\tau = (\sqrt{1-\lambda_\tau^2}z_k, \sqrt{1-\lambda_\tau^2}z_{k+1}, ..., \sqrt{1-\lambda_\tau^2}z_n), \quad for\ \tau = k, \\ \widetilde{\psi}_j = \psi_j \quad if \quad \tau < j \le k. \end{cases}$$

where $\mu_{jl} = \sqrt{1-\lambda_l^2}$ for $j \le l \le \tau$ and $\mu_{jl} = 1$ for $l > \tau$.

This map is a quadratic polynomial minimum map in $Prop(\mathbb{B}^n, \mathbb{B}^N)$ where

$$N = (k+1)n - \frac{k(k+1)}{2} + \tau.$$

[Example C] [HJY09] Let $F : \mathbb{B}^n \to \mathbb{B}^{N^*}$ be a proper polynomial minimum map with $F(0) = 0$. Then we define a new map $W_{n,k}(z; F, \lambda_1, ..., \lambda_\tau)$ by modifying the map $W_{n,k}(z;\lambda_1,...,\lambda_\tau)$ in the following way: while keeping all other components the same, replacing $\widetilde{\psi}_1$ with

$$\widetilde{\psi}_1 = (\sqrt{1-\lambda_1^2}z_1 F, \sqrt{1-\lambda_1^2+\mu_{12}^2}z_2, ..., \sqrt{1-\lambda_1^2+\lambda_{1k}^2}z_k, \sqrt{1-\lambda_1^2}z_{k+1}, ..., \sqrt{1-\lambda_1^2}z_n).$$

This map is a polynomial minimum map in $Prop(\mathbb{B}^n, \mathbb{B}^N)$ where

$$N = N^* - 1 + (k+1)n - \frac{k(k+1)}{2} + \tau.$$

Lemma 3.2.1 *[HJY09] Let $F : \mathbb{B}^n \to \mathbb{B}^{n(k-k_0)}$ be a minimum proper polynomial map with $k > k_0 > 0$ and $F(0) = 0$. Then a new map*

$$W_{n,k_0}(z; F, \lambda_1, ..., \lambda_r) : \mathbb{B}^n \to \mathbb{B}^N,$$

with

$$N = (k+1)n - \frac{k_0(k_0+1)}{2}, \text{ and } 0 \le \tau \le k_0 \le n$$

is a proper polynomial minimum map.

Proof of Theorem 3.1.4: We need to construct minimum proper monomial map from $\mathbb{B}^n$ into $\mathbb{B}^N$ under the assumption that either $(k+1)n - k(k+1)/2 \le N \le (k+1)n$ with $k \le K(n)$ or $N \ge (K(n)+1)n - K(n)(K(n)+1)$. Apparently, $K(n) \le \sqrt{2n}$.

Let $k \le n$. By Example C, we see the existence of minimum proper monomial maps from $\mathbb{B}^n$ into $\mathbb{B}^N$ when $(k+1)n - k(k+1)/2 \le N \le (k+1)n - k(k-1)/2$. If $k - 1 > 0$, applying Lemma 3.2.1 with $\kappa_0 = k-1$ and $\tau = 0, ..., k-1$, we see the existence of minimum proper monomial maps from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $(k+1)n - k(k-1)/2 \le N \le (k+1)n - (k-1)(k-2)/2 - 1$. Again, applying Lemma 3.2.1 with $\kappa_0 = k-2$ (if $k-2 > 0$) and $\tau = 0, ..., k-2$, we see the existence of minimum proper monomial maps from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $(k+1)n - (k-1)(k-2)/2 - 1 \le N \le (k+1)n - (k-2)(k-3)/2 - 1$. By an inductive use of Lemma 3.2.1, we see the existence of the required maps for N with $(k+1)n - k(k+1)/2 \le N \le (k+1)n$ for $k \le n$.

Next, letting $k = n+1$ in Lemma 3.2.1 and inductively applying Lemma 3.2.1 with $\kappa_0 = n, n-1, ...,$, we conclude the existence of the required maps when $(n+2)n - n(n+1)/2 - 1 \le N \le (n+2)n$. In particular, this would give the existence of the required maps when $(n+1)n \le N \le (n+2)n$. Applying an induction argument, we easily conclude the existence of the required maps for any $N \ge (n+1)n$. □

3.3 Rational and Polynomial Map

All examples above are polynomial maps. Nevertheless, not every map in $Rat(\mathbb{B}^n, \mathbb{B}^N)$ can be equivalent to a polynomial map.

Let us introduce a criterion which tells whether or not a rational map can be equivalent to a polynomial one as follows.

Let $F = \frac{P}{q} = \frac{(P_1, ..., P_N)}{q} \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ where $(P_j)_{j=1}^N$, q are holomorphic polynomial functions and $(P_1, ..., P_N, q) = 1$. We define

$$deg(F) = max\{deg(P_j)N_{j=1}, deg(q)\}.$$

Then F induces a rational map from $\mathbb{CP}^n$ into $\mathbb{CP}^N$ given by

$$\hat{F}([z_1 : ... : z_n : t]) = \left[t^k P(\frac{z}{t}) : t^k q(\frac{z}{t})\right]$$

where $z = (z_1, ..., z_n) \in \mathbb{C}^n$ and $deg(F) = k > 0$. $\hat{F}$ may not be holomorphic in general. Denote by $Sing(\hat{F})$ the singular set of $\hat{F}$, namely, the collection of points where $\hat{F}$ fails to be (or fails to extend to be) holomorphic. Then $Sing(\hat{F})$ is an algebraic subvariety of codimension two or more in $\mathbb{CP}^n$. We denote $\mathbb{B}_1^n := \{[z_1 : ... : z_n : t] \in \mathbb{CP}^n \mid \sum_{j=1}^n |z_j|^2 < |t|^2\}$.

Theorem 3.3.1 *[FHJZ2010] Let F be a non-constant rational holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N, n \geq 1$. Then F is equivalent to a holomorphic polynomial map from $\mathbb{B}^n$ into $\mathbb{B}^N$, namely, there are $\sigma \in Aut(\mathbb{B}^n)$ and $\tau \in Aut(\mathbb{B}^N)$ such that $\tau \circ F \circ \sigma$ is a holomorphic polynomial map from $\mathbb{B}^n$ into $\mathbb{B}^N$, if and only if there exist (complex) hyperplanes $H \subset \mathbb{CP}^n$ and $H' \subset \mathbb{CP}^N$ such that $H \cap \overline{\mathbb{B}_1^n} = \emptyset$, $H' \cap \overline{\mathbb{B}_1^N} = \emptyset$ and*

$$\hat{F}(H \setminus Sing(\hat{F})) \subset H', \quad \hat{F}\left(\mathbb{CP}^n \backslash (H \cup Sing(\hat{F}))\right) \subset \mathbb{CP}^N \backslash H'.$$

Proof: If F is a non-constant holomorphic polynomial map, then $\hat{F} = [t^k F(\frac{z}{t}), t^k]$ with $deg(F) = k > 0$. Let $H = H_\infty$ and $H' = H'_\infty$. Then they satisfy the property described in the theorem.

If F is equivalent to a holomorphic polynomial map G, then there exist $\hat{\sigma} \in U(n+1, 1), \hat{\tau} \in U(n+1, 1)$ such that $\hat{F} = \hat{\tau} \circ \hat{G} \circ \hat{\sigma}$. Let $H = \hat{\sigma}^{-1}(H_\infty)$ and $H' = \hat{\tau}(H'_\infty)$. Then they are the desired ones.

Conversely, suppose that $\hat{F}$, H and H' are as in the theorem. By a lemma below, we can find $\hat{\sigma} \in U(n+1, 1)$ and $\hat{\tau} \in U(n+1, 1)$ such that $\hat{\sigma}(H) = H_\infty$ and $\hat{\tau}(H') = H'_\infty$. Let $\hat{Q} = \hat{\tau} \circ \hat{F} \circ \hat{\sigma}^{-1}$. Then $\hat{Q}$ induces a rational holomorphic map Q from $\mathbb{B}^n$ into $\mathbb{B}^N$. If $Q = \frac{P}{q}$ where $(P, q) = 1$ and $deg(Q) = k > 0$, then

$$\hat{Q} = [t^k P(\frac{z}{t}) : t^k q(\frac{z}{t})].$$

Suppose that $q \not\equiv constant$. Let $z_0 \in \mathbb{C}^n$ be such that $q(z_0) = 0$ but $P(z_0) \neq 0$. Then $[z_0 : 1] \notin Sing(\hat{Q}) \cup H_\infty$ and $\hat{Q}([z_0 : 1]) \in H'_\infty$. Notice that $\hat{Q}(H_\infty \setminus Sing(\hat{Q})) \subset H'_\infty$ and $\hat{Q}\left(\mathbb{CP}^n \backslash (H_\infty \cup Sing(\hat{Q}))\right) \subset \mathbb{CP}^N \backslash H'_\infty$. This is a contradiction. Thus, we showed that Q is a polynomial. □

Lemma 3.3.2 *For any hyperplane $H \subset \mathbb{CP}^n$ with $H \cap \overline{\mathbb{B}_1^n} = \emptyset$, there is a $\sigma \in U(n+1,1)$ such that $\sigma(H) = H_\infty = \{[z_1 : \cdots : z_n : 0] \in \mathbb{CP}^n\}$.*

Proof: Assume that $H : \sum_{j=1}^n a_j z_j - a_{n+1} t = 0$ with $\vec{a} = (a_1, ..., a_{n+1}) \neq 0$. Under the assumption that $H \cap \overline{\mathbb{B}_1^n} = \emptyset$, we have $a_{n+1} \neq 0$. Without loss of generality, we can assume that $a_{n+1} = 1$. Let U be an $n \times n$ unitary matrix such that

$$(a_1, ..., a_n)\overline{U} = (\lambda, 0, ..., 0),$$

for some $\lambda \in \mathbb{C}$. Let $\sigma = \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix}$. Then $\sigma(H) = \{[z : t] \in \mathbb{CP}^n \mid \lambda z_1 - t = 0\}$ with $|\lambda| < 1$. Let $T \in Aut(\mathbb{B}^n)$ be defined by

$$T(z_1, z') = \left(\frac{z_1 - \overline{\lambda}}{1 - \lambda z_1}, \frac{\sqrt{1 - |\lambda|^2} z'}{1 - \lambda z_1} \right)$$

with $z' = (z_2, ..., z_n)$. Then $\hat{T} \in U(n+1, 1)$ is defined by

$$\hat{T}([z_1 : z' : t]) = [z_1 - \overline{\lambda} t : \sqrt{1 - |\lambda|^2} z' : t - \lambda z_1].$$

Now, it is easy to see that $\hat{T} \circ \sigma$ maps H to H_∞. □

Example D[FHJZ2010] Let $G(z, w) = \left(z^2, \sqrt{2} zw, w^2 (\frac{z-a}{1-\overline{a}z}, \frac{\sqrt{1-|a|^2}w}{1-\overline{a}z}) \right)$, $|a| < 1$, be a map in $Rat(\mathbb{B}^2, \mathbb{B}^4)$. G is equivalent to a proper holomorphic polynomial map in $Poly(\mathbb{B}^2, \mathbb{B}^4)$ if and only if $a = 0$.

In fact, we have

$$\hat{G} = \left[(t - \overline{a}z) z^2 : (t - \overline{a}z) \sqrt{2} zw : w^2 (z - at) : w^2 \sqrt{1 - |a|^2} w : (t^3 - \overline{a} t^2 z) \right].$$

Suppose there exist hyperplanes $H = \{\mu_1 z_1 + \mu_2 w + \mu_0 t = 0\} \subset \mathbb{CP}^2$ and $H' = \{\sum_{j=1}^4 \lambda_j z'_j + \lambda_0 t' = 0\} \subset \mathbb{CP}^4$ such that

$$H \cap \overline{\mathbb{B}_1^2} = \emptyset, \ H' \cap \overline{\mathbb{B}_1^4} = \emptyset, \ \hat{G}(H \setminus Sing(\hat{G})) \subset H', \ \ \hat{G}\left(\mathbb{CP}^2 \backslash (H \cup Sing(\hat{G}))\right) \subset \mathbb{CP}^4 \backslash H'.$$

Then

$$\lambda_1(t-\overline{a}z)z^2+\lambda_2(t-\overline{a}z)\sqrt{2}zw+\lambda_3w^2(z-at)+\lambda_4w^2\sqrt{1-|a|^2}w$$
$$+\lambda_0(t^3-\overline{a}t^2z)=(\mu_1z+\mu_2w+\mu_0t)^3 \quad \forall[z:w:t]\in\mathbb{CP}^2.$$

Apparently $\lambda_0\neq 0$. Hence we can assume that $\lambda_0=1,\mu_0=1$. By comparing the coefficient of $z^3, w^3, wt^2, zt^2, z^2t, zwt, z^2w, zw^2, w^2t$, respectively, in the above equation, we get

$$\mu_1^3=-\overline{a}\lambda_1,\ \mu_2^3=\lambda_4\sqrt{1-|a|^2},\ 3\mu_2=0,\ 3\mu_1=-\overline{a},\ 3\mu_1^2=\lambda_1,$$
$$6\mu_1\mu_2=\sqrt{2}\lambda_2,\ 3\mu_1^2\mu_2=-\sqrt{2}\lambda_2\overline{a},\ 3\mu_1\mu_2^2=\lambda_3,\ 3\mu_2^2=-a\lambda_3.$$

We then have $\lambda_2=\lambda_3=\lambda_4=\mu_2=0$. If $a\neq 0$, then $\mu_1,\ \lambda_1\neq 0$. From $\mu_1^3=-\overline{a}\lambda_1$ and $3\mu_1^2=\lambda_1$, we get $\mu_1=-3\overline{a}$. Since $3\mu_1=-\overline{a}$, we get $\overline{a}=0$. This is a contradiction. Notice that when $a=0$, F is a polynomial. By Theorem 3.3.1, we see the conclusion. □

Example E[FHJZ2010] Let $F(z',w)=\left(z',wz',w^2(\frac{\sqrt{1-|a|^2}z'}{1-\overline{a}w},\frac{w-a}{1-\overline{a}w})\right)$ with $|a|<1$ be a map in $Rat(\mathbb{B}^n,\mathbb{B}^{3n-2})$. F is equivalent to a proper polynomial map in $Poly(\mathbb{B}^n,\mathbb{B}^{3n-2})$ if and only if $a=0$.

By the criterion in Theorem 3.3.1, it is also proved that

Theorem 3.3.3 *[FHJZ2010] A map $F\in Rat(\mathbb{B}^2,\mathbb{B}^N)$ of degree two is equivalent to a polynomial proper holomorphic map in $Poly(\mathbb{B}^2,\mathbb{B}^N)$.*

Recently, J. Lebl claimed in a preprint ([Le09], theorem 1.5):

Theorem 3.3.4 *Let $F\in Rat(\mathbb{B}^n,\mathbb{B}^N)$ with $n\geq 3$ and $deg(F)=2$. Then F is equivalent to a monomial map.*

[**Example F**][FHJZ2010] Let $F\in Rat(\mathbb{B}^2,\mathbb{B}^5)$ be a rational mapping given by $F=(f,\phi_1,\phi_2,\phi_3,g)$ defined as follows:

$$f(z,w)=\frac{z+(\frac{i}{2}-i)zw}{1-iw-\frac{1}{3}w^2},\ \phi_1(z,w)=\frac{z^2}{1-iw-\frac{1}{3}w^2},$$

$$\phi_2(z,w)=\frac{\sqrt{\frac{13}{12}}zw}{1-iw-\frac{1}{3}w^2},\ \phi_3(z,w)=\frac{\frac{\sqrt{3}}{3}w^2}{1-iw-\frac{1}{3}w^2},\ g(z,w)=\frac{w-iw^2}{1-iw-\frac{1}{3}w^2}.$$

Then this mapping F is indeed equivalent to the polynomial map

$$G(z,w)=\left(\frac{\sqrt{3}}{9}(-2+4z+z^2),-\frac{\sqrt{6}}{9}(1+z+z^2),\frac{\sqrt{3}}{12}(5+3z)w,\frac{\sqrt{6}}{6}w^2,\frac{\sqrt{13}}{12}i(1-z)w\right).$$

3.4 Degree of Rational Maps between Balls

In order to outline a proof for Faran's theorem (see next section), we need to introduce the degree problems for maps in $Rat(\mathbb{B}^n,\mathbb{B}^N)$.

For any rational map $H\not\equiv 0$, write $H=\frac{(P_1,\cdots,P_m)}{R}$, where P_j, R are holomorphic polynomials and $(P_1,\cdots,P_m,R)=1$. We then define

$$\deg(H)=\max(\deg(P_j)_{j=1,\cdots,m},\deg(R)).$$

(When $H\equiv 0$, we set $\deg(H)=-\infty$).

D'Angelo raised a conjecture [DKR 03]: For any $F\in Rat(\mathbb{B}^n,\mathbb{B}^N)$, does it satisfy

$$deg(F)\le\begin{cases}2N-3, & if\ n=2,\\ \frac{N-1}{n-1}, & if\ n\ge 3.\end{cases}\tag{3.1}$$

Both of the above bounds are sharp. In fact, when $n=2$, the degree bound $2N-3$ is achieved (see p.173 and p. 189 in [DA93]) for the polynomial map $F\in Rat(\mathbb{B}^2,\mathbb{B}^{2+r})$ defined by $F(z,w)=(z^{2r+1},...,c_sz^{2(r-s)_1}w^s,...,w^{2r+1})$ where c_s are certain constants. When $n\ge 3$, we consider the Whitney map $h(z,w)=(z,w(z,w)):\mathbb{B}^n\to\mathbb{B}^{2n-1}$ with degree 2. By letting $(z,w)\mapsto(z,wh)$, we get a proper polynomial map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N=3n-2$ of degree 3. Inductively, we can construct a proper polynomial map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N=kn-(k-1)$ of degree k. Hence $\frac{N-1}{n-1}=k$ so that the bound in (3.1) is sharp.

[Example] We can show that any $F\in Rat(\mathbb{B}^2,\mathbb{B}^5)$ has degree $deg(F)\le 7$. We have classified all degree 2 maps in $F\in Rat(\mathbb{B}^2,\mathbb{B}^5)$. For higher degree maps, the situation should be very complicated. D'Angelo classified all monomial maps in $F\in Rat(\mathbb{B}^2,\mathbb{B}^5)$. He find out

$$\begin{cases}degree\ 3: & \text{31 isolated maps or continuous families;}\\ degree\ 4: & \text{47 isolated maps or continuous families;}\\ degree\ 5: & \text{24 isolated maps or continuous families;}\\ degree\ 6: & \text{5 isolated maps or continuous families;}\\ degree\ 7: & \text{3 isolated maps;}\end{cases}$$

For example, maps with degree 7 in $Rat(\mathbb{B}^2,\mathbb{B}^5)$ are

1. $(z^7, w^7, \frac{\sqrt{7}}{\sqrt{2}}wz^5, \frac{\sqrt{7}}{\sqrt{2}}w^5z,\ \frac{\sqrt{7}}{\sqrt{2}}wz)$
2. $(z^7, w^7, \sqrt{7}wz^5, \sqrt{14}w^2z^3, \sqrt{7}w^3z)$
3. $(z^7, w^7, \sqrt{7}w^3z^3, \sqrt{7}wz^3, \sqrt{7}w^3z)$

• Forstnerič proved that for any $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$, its degree $\deg(F) \le N^2(N-n+1)$ in [Fo86].

• Huang-Ji-Xu proved [HJX06]: Let $F \in Rat(\mathbf{B}^n, \mathbf{B}^N)$ with geometric rank $\kappa_0 = 1$ and $n \ge 3$. Then $deg(F) \le \frac{N-1}{n-1}$. For the proof, see § 4.2.

To illustrate the idea how to deal with degree $deg(F)$, we present a lemma and a theorem below.

Lemma 3.4.1 *([HJ01], lemma 5.4) Let $H = \frac{(P_1,\cdots,P_m)}{R}$ be a rational map from $\mathbb{C}^n$ into $\mathbb{C}^m$, where P_j, R are holomorphic polynomials with $(P_1,\cdots,P_m,R)=1$ $(m > n > 1)$. Assume for each $p \in \partial\mathbb{H}^n$ close to the origin,*

$$\deg(H|_{Q_p}) \le k$$

with $k > 0$ a fixed integer, where $Q_{(\zeta,\eta)} = \{(z,w) \mid \frac{w-\overline{\eta}}{2i} = \sum_{j=1}^{n-1} z_j\overline{\eta_j}\}$ is the Segre variety of $\partial\mathbb{H}^n$. Then $\deg(H) \le k$.

Theorem 3.4.2 *Let $F \in Rat(\mathbb{B}^2, \mathbb{B}^3)$. Then $deg(F) \le 3$.*

Proof: By Cayley transformation, we consider $F \in Rat(\mathbb{H}^2, \mathbb{H}^3)$. By Lemma 3.4.1, it suffices to prove that $deg(F|_{Q_{p_0}}) \le 3$ for any $p_0 \in \partial\mathbb{H}^n$.

It is equivalent to show that for every $p \in \partial\mathbb{H}_2$, we have

$$deg\big(F_p^{**}|_{Q_0}\big) \le 3. \tag{3.2}$$

Here $Q_0 = \{w = 0\}$. In fact, $deg(F|_{Q_p}) = deg(F|_{\sigma_p(Q_0)}) = deg((F\circ\sigma_p)|_{Q_0}) = deg((\sigma\circ(F_p^{**})\circ\tau)|_{Q_0}) = deg((F_p^{**})|_{Q_0})$.

Write $F_p^{**} = (f, \phi, g)$ where $f = z + \sum_{j+k\ge 2} a_{jk}z^jw^k$, $\phi = \sum_{j+k\ge 2} b_{jk}z^jw^k$, and $g = w + \sum_{j+k\ge 4} c_{jk}z^jw^k$.

Applying L and L^2 to the basic equation $\frac{g-\overline{g}}{2} = f\overline{f} + \phi\overline{\phi}$, we get

$$\begin{cases} \frac{1}{2i}Lg = Lf\cdot\overline{f} + L\phi\cdot\overline{\phi}, \\ \frac{1}{2i}L^2g = L^2f\cdot\overline{f} + L^2\phi\cdot\overline{\phi}, \end{cases}$$

i.e.,

$$\frac{1}{2i}\begin{bmatrix} Lg \\ L^2 g \end{bmatrix} = \begin{bmatrix} Lf & L\phi \\ L^2 f & L^2\phi \end{bmatrix}\begin{bmatrix} \overline{f} \\ \overline{\phi} \end{bmatrix}, \quad \forall (z,w) \in \partial\mathbb{H}^2$$

where $L = \frac{\partial}{\partial z} + 2i\overline{z}\frac{\partial}{\partial w}$.

We complexify this identity so that

$$\frac{1}{2i}\begin{bmatrix} \mathcal{L}g(z,w) \\ \mathcal{L}^2 g(z,w) \end{bmatrix} = \begin{bmatrix} \mathcal{L}f(z,w) & \mathcal{L}\phi(z,w) \\ \mathcal{L}^2 f(z,w) & \mathcal{L}^2\phi(z,w) \end{bmatrix}\begin{bmatrix} \overline{f}(\zeta,\eta) \\ \overline{\phi}(\zeta,\eta) \end{bmatrix}$$

holds for any point $(z,w,\zeta,\eta) \in \partial\mathcal{H}^2$ where $\mathcal{L} = \frac{\partial}{\partial z} + 2i\zeta\frac{\partial}{\partial w}$ and $\partial\mathcal{H}^2 = \{(z,w,\zeta,\eta) \in \mathbb{C}^4 \mid \frac{w-\eta}{2i} = z\zeta\}$ is the Segre family of $\partial\mathbb{H}^2$.

Since $(0,0,\zeta,0) \in \partial\mathcal{H}^2$, we have

$$\begin{cases} \mathcal{L}f|_{(0,0,\zeta,0)} = 1, \\ \mathcal{L}\phi|_{(0,0,\zeta,0)} = 0, \\ \mathcal{L}g|_{(0,0,\zeta,0)} = 2i\zeta, \\ \mathcal{L}^2 f|_{(0,0,\zeta,0)} = -8a_{02}\zeta^2 + 4ia_{11}\zeta, \\ \mathcal{L}^2\phi|_{(0,0,\zeta,0)} = -8b_{02}\zeta^2 + 4ib_{11}\zeta + 2b_{02}, \\ \mathcal{L}^2 g|_{(0,0,\zeta,0)} = 0, \end{cases}$$

so that

$$det\begin{bmatrix} \mathcal{L}f & \mathcal{L}\phi \\ \mathcal{L}^2 f & \mathcal{L}^2\phi \end{bmatrix}\Bigg|_{(0,0,0,0)} = det\begin{bmatrix} 1 & 0 \\ 0 & 2b_{20} \end{bmatrix} = 2b_{02} \neq 0.$$

Then we obtain

$$\begin{bmatrix} \overline{f}(\zeta,0) \\ \overline{\phi}(\zeta,0) \end{bmatrix} = \frac{1}{2i}\begin{bmatrix} \mathcal{L}f & \mathcal{L}\phi \\ \mathcal{L}^2 f & \mathcal{L}^2\phi \end{bmatrix}^{-1}\Bigg|_{(0,0,\zeta,0)} \cdot \begin{bmatrix} \mathcal{L}g \\ \mathcal{L}^2 g \end{bmatrix}\Bigg|_{(0,0,\zeta,0)}$$

$$= \frac{1}{2i}\begin{bmatrix} 2i\zeta \\ \frac{2i\zeta(8a_{02}\zeta^2 - 4ia_{11}\zeta)}{-8b_{02}\zeta^2 + 4ib_{11}\zeta + 2b_{02}} \end{bmatrix} = \begin{bmatrix} \zeta \\ \frac{\zeta(8a_{02}\zeta^2 - 4ia_{11}\zeta)}{-8b_{02}\zeta^2 + 4ib_{11}\zeta + 2b_{02}} \end{bmatrix}$$

This implies

$$f(z,0) = z, \ \phi(z,0) = \frac{4\overline{a_{02}}z^3 + 2i\overline{a_{11}}z^2}{-4\overline{b_{02}}z^2 - 2i\overline{b_{11}}z + \overline{b_{02}}}.$$

Also we put $(0,0,\zeta,0)$ into the identity $\frac{g(z,w)-\overline{g}(\zeta,\eta)}{2i} = f(z,w)\overline{f}(\zeta,\eta) + \phi(z,w)\overline{\phi}(\zeta,\eta)$ to get $g(z,0) = 0$. Thus (3.2) is proved. $\square$

By similar argument, we are able to prove the following.

Theorem 3.4.3 *([HJ01], lemma 5.2) Let $F \in Prop_2(\mathbb{B}^n, \mathbb{B}^{2n-1})$ with $n \geq 3$. Then F is rational and $deg(F) \leq 2$.*

Proof: By Cayley transformation, we consider $F \in Prop_2(\mathbb{H}^n, \mathbb{H}^{2n-1})$. By Lemma 3.4.1, it suffices to prove that $deg(F|_{Q_{p_0}}) \leq 2$ for any $p_0 \in \partial\mathbb{H}^n$.

It is equivalent to show that for every $p \in \partial\mathbb{H}_n$, we have

$$deg\big(F_p^{***}|_{Q_0}\big) \leq 2. \tag{3.3}$$

Here $Q_0 = \{w = 0\}$.

By the normalization, for any $F \in Prop_2(\mathbb{H}_n, \mathbb{H}_{2n-1})$, we knew that $F_p^{**} = (f, \phi, g)$ satisfies

$$\begin{array}{l} F_p^{***}(0, w) = (0, w), \\ f_1 = z_1 + \frac{i}{2} z_1 w + z_1 \widetilde{a^{(1)}}(z) w + o_{wt}(4), \\ f_l = z_l + o_{wt}(4), \; 2 \leq l \leq n-1, \\ \phi_j = z_1 z_j + b_j z_1 w + b_j^{(3)}(z) + o_{wt}(3), \quad 1 \leq j \leq n-1, \\ g = w + o(|(z, w)|^3). \end{array} \tag{3.4}$$

$$\frac{g(z, w) - \overline{g(\zeta, \eta)}}{2i} = \sum_{l=1}^{n-1} f_l(z, w)\overline{f_l(\zeta, \eta)} + \sum_{l=1}^{n-1} \phi_l(z, w)\overline{\phi_l(\zeta, \eta)}. \tag{3.5}$$

Applying $\mathcal{L}_j$ and $\mathcal{L}_1\mathcal{L}_j$ to the above equation, using (3.4) and letting $(z, w) = 0, \; \eta = 0$, we get

$$\begin{pmatrix} \overline{\zeta_1} \\ \cdots \\ \overline{\zeta_{n-1}} \\ 0 \end{pmatrix} = \begin{pmatrix} I_{(n-1)\times(n-1)} & 0 \\ A_{(n-1)\times(n-1)} & B_{(n-1)\times(n-1)} \end{pmatrix} \begin{pmatrix} \overline{f(\zeta, 0)} \\ \overline{\phi(\zeta, 0)} \end{pmatrix}.$$

Here $I_{(n-1)\times(n-1)}$ is the identical $(n-1) \times (n-1)$ matrix,

$$A_{(n-1)\times(n-1)} = A = \begin{pmatrix} -2\overline{\zeta_1} & 0 & \cdots & 0 \\ -\overline{\zeta_2} & 0 & \cdots & 0 \\ \cdots & 0 & \cdots & 0 \\ -\overline{\zeta_{n-1}} & 0 & \cdots & 0 \end{pmatrix} \text{ and}$$

$$B_{(n-1)\times(n-1)} = B = \begin{pmatrix} 2 + 4ib_1\overline{\zeta_1} & 4ib_2\overline{\zeta_1} & \ldots & 4ib_{n-1}\overline{\zeta_1} \\ 2ib_1\overline{\zeta_2} & 1 + 2ib_2\overline{\zeta_2} & \ldots & 2ib_{n-1}\overline{\zeta_2} \\ \ldots & \ldots & \ldots & \ldots \\ 2ib_1\overline{\zeta_{n-1}} & 2ib_2\overline{\zeta_{n-1}} & \ldots & 1 + 2ib_{n-1}\overline{\zeta_{n-1}} \end{pmatrix}.$$

This implies

$$\widetilde{f}(z,0) = \left(z,\ \frac{z_1 z}{1 - 2i\sum_{j\geq 1}\overline{b_j} z_j}\right). \tag{3.6}$$

Finally, by putting $z = w = \eta = 0$, we get $\overline{g}(\zeta, 0) = 0$ by (3.4). Hence, it is clear that $F(z, 0)$ can be written as the quotient of a vector-valued quadratic polynomial with a linear function. Hence (3.3) is proved. □

By similar method, the following results are proved.

Theorem 3.4.4 *(1) [JX04] Let $F \in Rat(\mathbf{B}^n, \mathbf{B}^N)$ with geometric rank κ_0, $1 \leq \kappa_0 \leq n-2$, and with $N = n + \frac{(2n-\kappa_0-1)\kappa_0}{2}$. Then $deg(F) \leq \kappa_0 + 2$.*

(2) [HJX05] Let $F \in Rat(\mathbb{B}^3, \mathbb{B}^6)$ with geometric rank $\kappa_0(F) = 2$. Then $deg(F) \leq 4$.

3.5 Classification of Maps from $\mathbb{B}^2$ to $\mathbb{B}^3$

Theorem 3.3.3 is proved based on the classification of maps of $Rat(\mathbb{B}^2, \mathbb{B}^N)$ with degree 2 (see Theorem 3.6.1).

To illustrate techniques used to study the classification problem, we first give a proof for the following Faran's theorem [Fa82]:

Theorem 3.5.1 *(Faran, 1982) Any map $F \in Rat(\mathbb{B}^2, \mathbb{B}^3)$ must be equivalent to one of the following maps:*

$$\begin{cases} degree\ 1: & (z, w, 0); \\ degree\ 2: & (z, zw, w^2),\ and\ (z^2, \sqrt{2}zw, w^2); \\ degree\ 3: & (z^3, \sqrt{3}zw, w^3). \end{cases}$$

The proof here is given in [J09] which is different from Faran's original Proof. The difficulty to study $Rat(\mathbb{B}^2, \mathbb{B}^3)$, comparing study $Rat(\mathbb{B}^n, \mathbb{B}^N)$ with high n and N, is that we have less numbers of equations.

We already shown in Theorem 3.4.3) that $deg(F) \leq 3$. Since maps in $Rat(\mathbb{B}^2, \mathbb{B}^N)$ with degree ≤ 2 can be classified (see Theorem 3.6.1), it suffices to show: there exists exactly one map $F \in Rat(\mathbb{B}^2, \mathbb{B}^3)$ with degree 3.

The normal form F^{***} of F [1], still denoted as (f, ϕ, g), becomes

$$f = \frac{z - 2i\overline{b_{11}}z^2 + (ie_1 + i/2)zw - 4b_{02}z^3 + E_{11}z^2w + A_{12}zw^2 + A_{03}w^3}{1 - 2i\overline{b_{11}}z + ie_1w - 4b_{02}z^2 + E_{11}zw + E_{02}w^2 + E_{21}z^2w + E_{12}zw^2 + E_{03}w^3},$$
$$\phi = \frac{z^2 + b_{11}zw + b_{02}w^2 + B_{21}z^2w + B_{12}zw^2 + B_{03}w^3}{1 - 2i\overline{b_{11}}z + ie_1w - 4b_{02}z^2 + E_{11}zw + E_{02}w^2 + E_{21}z^2w + E_{12}zw^2 + E_{03}w^3},$$
$$g = \frac{w - 2i\overline{b_{11}}zw + ie_1w^2 - 4b_{02}z^2w + E_{11}zw^2 + C_{03}w^3}{1 - 2i\overline{b_{11}}z + ie_1w - 4b_{02}z^2 + E_{11}zw + E_{02}w^2 + E_{21}z^2w + E_{12}zw^2 + E_{03}w^3},$$

with $b_{02} > 0$ and $e_1 \in \mathbb{R}$.

Consider the basic equation: $Im(g) = |f|^2 + |\phi|^2$, $\forall Im(w) = |z|^2$, we obtain all algebraic equations about the parameters. Among these equations, we find

$$e_1 Im(b_{11}^2) = 0. \tag{3.7}$$

By (3.7), we consider

$$\begin{cases} Case\ A: \ e_1 \neq 0 \Rightarrow \begin{cases} Case\ A_1: \ Im(b_{11}) = 0; \\ Case\ A_2: \ Re(b_{11}) = 0. \end{cases} \\ Case\ B: e_1 = 0. \end{cases}$$

In Case A_1, we list all the equations about the parameters:

$$\begin{aligned}
&A_{12} = E_{02} - \frac{1}{8} - \frac{5}{4}e_1 - \frac{1}{2}b^2, \quad b_{11} = b \ is\ a\ real\ parameter, \\
&b_{02}\ determined\ by\ \frac{1}{2}e_1 + 4e_1b^2 + e_1^2 + 12b_{02}^2 + 4b_{02}b^2 = 0, \\
&B_{21} = i(\frac{1}{4} + \frac{3}{2}e_1 + b^2), \quad B_{12} = i(\frac{1}{4}b + \frac{3}{2}be + b^3), \\
&B_{03} = ib_{02}(\frac{1}{4} + \frac{3}{2}e_1 + b^2), \quad C_{03} = E_{02} - \frac{e_1}{2}, \quad e_1 \neq 0\ is\ a\ real\ parameter, \\
&E_{11} = \frac{1}{2}b + e_1b + 2b^3 - 8bb_{02}, \quad E_{12} = -i(eb + 2bb_{02}), \\
&E_{21} = -2ib_{02}, \quad E_{02} = \frac{1}{16} + \frac{5}{4}e_1 + \frac{1}{2}b^2 + 2b_{02}^2 + 3e_1b^2 + \frac{5}{4}e_1^2 + b^4, \\
&E_{03} = i(\frac{1}{2}e_1^2 - |b_{02}|^2).
\end{aligned}$$

[1] For the definition of F^{***}, see § 4.1. It means here that the coefficient of the z^2 term of ϕ is 1.

From the equation for b_{02} above, we obtain

$$e_1 = \frac{-(\frac{1}{2}+4b^2) \pm \sqrt{(\frac{1}{2}+4b^2)^2 - 4(12b_{02}^2 + 4b_{02}b^2)}}{2}$$

Since e_1 is a real number, we must have $(\frac{1}{2}+4b^2)^2 - 4(12b_{02}^2 + 4b_{02}b^2) \geq 0$, i.e.,

$$\left(\frac{1}{2}+4b^2\right)^2 + \frac{4}{3}b^4 \geq 48\left(b_{02}^2 + \frac{b^2}{6}\right)^2.$$

If we consider $F_p^{***} = (f_p^{***}, \phi_p^{***}, g_p^{***})$, it is of the same form

$$f_p^{***} = \frac{z - 2i\overline{b_{11}}z^2 + (ie_1 + i/2)zw - 4b_{02}z^3 + E_{11}z^2w + A_{12}zw^2 + A_{03}w^3}{1 - 2i\overline{b_{11}}z + ie_1w - 4b_{02}z^2 + E_{11}zw + E_{02}w^2 + E_{21}z^2w + E_{12}zw^2 + E_{03}w^3},$$
$$\phi_p^{***} = \frac{z^2 + b_{11}zw + b_{02}w^2 + B_{21}z^2w + B_{12}zw^2 + B_{03}w^3}{1 - 2i\overline{b_{11}}z + ie_1w - 4b_{02}z^2 + E_{11}zw + E_{02}w^2 + E_{21}z^2w + E_{12}zw^2 + E_{03}w^3},$$
$$g_p^{***} = \frac{w - 2i\overline{b_{11}}zw + ie_1w^2 - 4b_{02}z^2w + E_{11}zw^2 + C_{03}w^3}{1 - 2i\overline{b_{11}}z + ie_1w - 4b_{02}z^2 + E_{11}zw + E_{02}w^2 + E_{21}z^2w + E_{12}zw^2 + E_{03}w^3},$$

with $b_{02} > 0$ and $e_1 \in \mathbb{R}$. Here all coefficients, A_{12}, b_{11}, ..., are functions of $p \in \partial\mathbb{H}^2$. From above calculation, all of the coefficients (as functions of p) of F_p^{***} are bounded when $|b_{11}(p)|$ is bounded.

Similar conclusion holds for Case A_2 and Case B.

Then we take a sequence $p_m \in \partial\mathbb{H}^2$ so that the associated map $F_{p_m}^{***}$ satisfies

$$\lim_{m\to\infty} b_{11}(p_m) = \inf_p\{b_{11}(p)\}.$$

Then we show

$$F \textit{ is equivalent to } \widetilde{F} = \lim_{m\to\infty}(F_{p_m})^{***}.$$

Here we have to take care of the facts that p_m could go to ∞: $[0 : a : b] \in \partial\mathbb{H}^2$ and the equivalence is not obvious.

The limit map $\widetilde{F}$ has the minimum property for its parameter b_{11}, namely, if we denote by $b_{11}(p)$ the corresponding coefficient of the map $(\widetilde{F}_p)^{***}$ and $p = (z_0, w_0) = (z_0, u_0 + i|z_0|^2)$, we find

$$|b_{11}(p)|^2 = |b_{11}|^2 - i(\overline{b_{11}} + 2\overline{b_{11}}e_1 + 12b_{11}b_{02} + 4\overline{b_{11}}|b_{11}|^2)z_0$$
$$+i(b_{11} + 2b_{11}e_1 + 12\overline{b_{11}}b_{02} + 4b_{11}|b_{11}|^2)\overline{z_0} + 32b_{02}Re(b_{11})Im(b_{11})u_0 + o(1).$$

Since the critical point of the function $b_{11}(p)$ is zero by the minimum property, it gives the desired extra equation:

$$Im(b_{11})Re(b_{11}) = 0, \ and \ \overline{b_{11}} + 2e_1\overline{b_{11}} + 4\overline{b_{11}}|b_{11}|^2 + 12b_{02}b_{11} = 0. \tag{3.8}$$

It leads us consider Case(C): $b_{11} = 0$ and Case(D): $b_{11} \neq 0$.

Finally we consider all cases:

Case A1 C	cannot occur
Case A2 C	cannot occur
Case B C	$\exists$ a unique map
Case A1 D	cannot occur
Case A2 D	cannot occur
Case B D	cannot occur

The only map in $Rat(\mathbb{H}^2, \mathbb{H}^3)$ of degree 3 is of the normalized form $F = F^{***} = (f, \phi, g)$:

$$f = \frac{z + \frac{i}{2}zw - \frac{1}{16}zw^2}{1 + \frac{1}{16}w^2}, \ \phi = \frac{z^2 + \frac{i}{4}z^2w}{1 + \frac{1}{16}w^2}, \ g = \frac{w + \frac{1}{16}w^3}{1 + \frac{1}{16}w^2}. \tag{3.9}$$

We notice that it is too complicated to find (3.9) directly by the definition of F^{***}.

3.6 Classification of Maps from $\mathbb{B}^2$ With Degree Two

The classification problem for maps in $Rat(\mathbb{B}^2, \mathbb{B}^N)$ with degree 2 has been solved.

Theorem 3.6.1 *[JZ09] (i) Any nonlinear map in $Rat(\mathbb{B}^2, \mathbb{B}^N)$ with degree 2 is equivalent to a map $(F, 0)$ where $F \in Rat(\mathbb{B}^2, \mathbb{B}^5)$ is of one of the following forms:*
(I): $F = (G_t, 0)$ where $G_t \in Rat(\mathbb{B}^2, \mathbb{B}^4)$ is defined by

$$G_t(z, w) = (z^2, \sqrt{1 + \cos^2 t}\ zw, (\cos t)w^2, (\sin t)w), \quad 0 \le t < \pi/2. \tag{3.10}$$

(IIA): $F = (F_\theta, 0)$ where $F_\theta \in Rat(\mathbb{B}^2, \mathbb{B}^4)$ is defined by

$$F_\theta(z, w) = (z, (\cos\theta)w, (\sin\theta)zw, (\sin\theta)w^2), \quad 0 < \theta \le \frac{\pi}{2}. \tag{3.11}$$

(IIC): $F = F_{c_1,c_3,e_1,e_2} = \rho_5^{-1} \circ F \circ \rho_2 = (f, \phi_1, \phi_2, \phi_3, g) \in Rat(\mathbb{H}^2,\ \mathbb{H}^5)$ is of the form:

$$\begin{aligned} f &= \frac{z + (\frac{i}{2} + ie_1)zw}{1 + ie_1w + e_2w^2},\ \phi_1 = \frac{z^2}{1 + ie_1w + e_2w^2}, \\ \phi_2 &= \frac{c_1zw}{1 + ie_1w + e_2w^2},\ \phi_3 = \frac{c_3w^2}{1 + ie_1w + e_2w^2},\ g = \frac{w + ie_1w^2}{1 + ie_1w + e_2w^2}, \end{aligned}$$

where $c_1, c_3 > 0, -e_1, -e_2 \ge 0$, $e_1e_2 = c_3^2$, $-e_1 - e_2 = \frac{1}{4} + c_1^2$, satisfying one of the following conditions: either

$$\begin{cases} e_1 = \frac{-(\frac{1}{4}+c_1^2)-\sqrt{(\frac{1}{4}+c_1^2)^2-4c_3^2}}{2},\ e_2 = \frac{-(\frac{1}{4}+c_1^2)+\sqrt{(\frac{1}{4}+c_1^2)^2-4c_3^2}}{2}, \\ 0 < 4c_3^2 \le (\frac{1}{4} + c_1^2)^2, \end{cases} \tag{3.12}$$

or

$$\begin{cases} e_1 = \frac{-(\frac{1}{4}+c_1^2)+\sqrt{(\frac{1}{4}+c_1^2)^2-4c_3^2}}{2},\ e_2 = \frac{-(\frac{1}{4}+c_1^2)-\sqrt{(\frac{1}{4}+c_1^2)^2-4c_3^2}}{2}, \\ \frac{1}{2}c_1^2 + c_1^4 \le 4c_3^2 \le (\frac{1}{4} + c_1^2)^2. \end{cases} \tag{3.13}$$

(ii) Any two maps in $Rat(\mathbb{B}^2, \mathbb{B}^5)$ in the form of types (I), (IIA), and (IIC) above are equivalent if and only if they are identical.

In Faran's Theorem on $Rat(\mathbb{B}^2, \mathbb{B}^3)$, there are four maps, up to automorphisms, which are isolated. Nevertheless, for $Rat(\mathbb{B}^2, \mathbb{B}^N)$ with $N > 3$, there exists a continuous family of maps, up to automorphism. For example, D'Angelo constructed $F_t = (z, w\ cos\ t, (w\ sin\ t)z) \in Rat(\mathbb{B}^n, \mathbb{B}^{2n})$ with $t \in (0, \frac{\pi}{2})$ satisfies: F_t is equivalent to F_s if and only if $t = s$. To classify continuous family of maps, we have to use different technique.

3.7 Proof of Theorem 3.6.1 - Part 1

As a reduction in the proof of Theorem 3.6.1, Huang-Ji-Xu [HJX06] proved: Any map F in $Rat(\mathbb{H}^2, \mathbb{H}^N)$ with $deg(F) = 2$ is equivalent to a map $(G, 0)$ where $G = (f, \phi_1, \phi_2, \phi_3, g) \in Rat(\mathbb{H}^2, \mathbb{H}^5)$ is of the form (see also Lemma 2.3 below)

$$\begin{aligned} f(z,w) &= \frac{z - 2ibz^2 + (\frac{i}{2} + ie_1)zw}{1 + ie_1 w + e_2 w^2 - 2ibz}, \\ \phi_1(z,w) &= \frac{z^2 + bzw}{1 + ie_1 w + e_2 w^2 - 2ibz}, \\ \phi_2(z,w) &= \frac{c_2 w^2 + c_1 zw}{1 + ie_1 w + e_2 w^2 - 2ibz}, \\ \phi_3(z,w) &= \frac{c_3 w^2}{1 + ie_1 w + e_2 w^2 - 2ibz}, \\ g(z,w) &= \frac{w + ie_1 w^2 - 2ibzw}{1 + ie_1 w + e_2 w^2 - 2ibz}, \end{aligned}$$

where $b, -e_1, -e_2, c_1, c_2, c_3$ are real non-negative numbers satisfying $e_1 e_2 = c_2^2 + c_3^2$, $-e_1 - e_2 = \frac{1}{4} + b^2 + c_1^2$, $-be_2 = c_1 c_2$, and $c_3 = 0$ if $c_1 = 0$.

Since b and c_2 are determined by c_1, c_3, e_1 and e_2, a map in the above form is determined by c_1, c_3, e_1 and e_2. We denote a map of the above form, which is determined by c_1, c_3, e_1 and e_2, to be

$$F_{(c_1,c_3,e_1,e_2)} \in \mathcal{K}. \tag{3.14}$$

It was unclear which of the coefficients e_1, e_2, c_1 and c_3 of F are independent parameters.

Let us show why F is equivalent to another map $(G, 0)$ where $G \in Rat(\mathbb{B}^2, \mathbb{B}^5)$.

Let $F = (f, \phi_1, \phi_2, g)$ be a proper rational map of degree two from $\partial\mathbb{H}^2$ into $\partial\mathbb{H}^N$. Assume that $F(0) = 0$ and 0 is a generic point of F, namely, $\kappa_F(0) = 1$. Without loss of generality, we assume that $N \geq 4$. By Lemma 3.1 in [H03], we have $\sigma \in Aut_0(\partial\mathbb{H}^2)$ and $\tau \in \partial Aut_0(\partial\mathbb{H}^N)$ such that $\tau \circ F \circ \sigma$, still denoted by $F = (f, \phi, g)$, takes the following form:

$$\begin{aligned} f &= z + \frac{i}{2} zw + o_{wt}(3), \quad \frac{\partial^2 f}{\partial w^2}(0) = 0, \\ g &= w + o_{wt}(4), \\ \phi_1 &= z^2 + A_1 zw + B_1 w^2 + E_1 z^3 + \cdots, \\ \phi_j &= o_{wt}(2), \ j \geq 2. \end{aligned} \tag{3.15}$$

Replacing $(\phi_2, \cdots, \phi_{N-2})$ by $(\phi_2, \cdots, \phi_{N-2}) \cdot U$ with U a certain $(N-3) \times (N-3)$ unitary matrix, we can assume that $\phi_j = A_j zw + B_j w^2 + o(|(z,w)|^2)$ for $j \geq 2$ and $A_j = 0$ for $j \geq 3$.

In a similar manner, we can assume that $B_j = 0$ for $j \geq 4$ (if $N \geq 6$). Making use of the assumption that F has degree 2, we can thus assume in (3.15) that

$$\begin{aligned}\phi_2 =& A_2 zw + B_2 w^2 + o(|(z,w)|^2),\\ \phi_3 =& B_3 w^2 + o(|(z,w)|^2),\\ \phi_j =& 0,\ j \geq 4. \qquad \square\end{aligned} \tag{3.16}$$

3.8 Proof of Theorem 3.6.1 - Part 2

In[CJX06], by obtaining an extra equation, we got a more clearer picture on the maps as above.

Let us describe how to obtain this extra equation.

For any $F \in Rat(\mathbb{H}^2, \mathbb{H}^5)$ with $deg(F) = 2$, F is equivalent to another map $F^{***} \in Rat(\mathbb{H}^2, \mathbb{H}^5)$ of the above form. Also we can associate a family of maps $F_p \in Rat(\mathbb{H}^2, \mathbb{H}^5)$ for any $p \in \partial\mathbb{H}^2$, as well as the associated maps $(F_p)^{***}$ that is of the above form.

We define a real analytic function

$$\mathcal{W}(F_p^{***}) = c_1(p)^2 - e_1(p) - e_2(p)$$

where $c_1(p), e_1(p)$ and $e_2(p)$ are the coefficients of F_p^{***}:

$$f_p^{***}(z,w) = \frac{z - 2ib(p)z^2 + (\frac{i}{2} + ie_1(p))zw}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z}, \tag{3.17}$$

$$\phi_{1,p}^{***}(z,w) = \frac{z^2 + b(p)zw}{1 + ie_1(p)w + e_2 w^2 - 2ib(p)z}, \tag{3.18}$$

$$\phi_{2,p}^{***}(z,w) = \frac{c_2(p)w^2 + c_1(p)zw}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z}, \tag{3.19}$$

$$\phi_{3,p}^{***}(z,w) = \frac{c_3(p)w^2}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z}, \tag{3.20}$$

$$g_p^{***}(z,w) = \frac{w + ie_1(p)w^2 - 2ib(p)zw}{1 + ie_1(p)w + e_2 w^2 - 2ib(p)z}. \tag{3.21}$$

Here $b(p), e_1(p), e_2(p), c_1(p), c_2(p), c_3(p)$ satisfy

$$e_2(p)e_1(p) = c_2^2(p) + c_3^2(p),\ -e_2(p) = \frac{1}{4} + e_1(p) + b^2(p) + c_1^2(p),$$

and $-b(p)e_2(p) = c_1(p)c_2(p),\ c_3(p) = 0$ if $c_1(p) = 0$, with

$$c_1(p), c_2(p), b(p) \geq 0,\ e_2(p), e_1(p) \leq 0.$$

We observe that as long as $\mathcal{W}(F_p^{***})$ is bounded, all

$$e_1(p_m),\ e_2(p_m),\ c_1(p_m),\ c_2(p_m),\ c_3(p_m),\ b(p_m)$$

are uniformly bounded for all m. In fact, since $c_1(p_m), -e_1(p_m), -e_2(p_m)$ are non-negative, $c_1(p_m), e_1(p_m)$ and $e_2(p_m)$ are uniformly bounded for all m. From $-e_1(p_m) - e_2(p_m) = \frac{1}{4} + b^2(p_m) + c_1^2(p_m)$, $b(p_m)$ is uniformly bounded for any m. Finally, from $e_1(p_m)e_2(p_m) = c_2^2(p_m) + c_3^2(p_m)$, $c_2(p_m)$ and $c_3(p_m)$ are uniformly bounded.

The desired extra equation is obtained by moving up p to the extremal value as follows. We choose a sequence of $p_m \in \partial\mathbb{H}^2$ such that

$$p_m \to p_0 \in \overline{\partial\mathbb{H}^2} \quad and \quad \lim_m \mathcal{W}(F_{p_m}^{***}) = \inf_{p\in\partial\mathbb{H}^2-\Xi_F} \{\mathcal{W}(F_p^{***})\} \tag{3.22}$$

where Ξ_F is a proper real analytic variety such that $\forall p \in \partial\mathbb{H}^2 - \Xi_F$, F_p has geometric rank one at 0 so that $\mathcal{W}(F_p^{**})$ is well defined.

Then F is equivalent to $F_{p_0}^{***}$ which is of the above form and with the minimum property $\mathcal{W}(F_{p_0}^{***}) = \inf_{p\in\partial\mathbb{H}^2-\Xi_F} \mathcal{W}(F_p^{***})$.

A key lemma used to prove convergence of the limit map is the following result.

Lemma 3.8.1 *([CJX06] lemma 2.5) Let $F \in Rat(\partial\mathbb{H}^2, \partial\mathbb{H}^5)$ with $F(0) = 0$ and $deg(F) = 2$. Suppose that $p_m \in \partial\mathbb{H}^2$ is a sequence converging to 0, F_{p_m} is of rank 1 at 0 for any m and $F_{p_m}^{***}$ converges such that $\frac{\partial^2\phi_{1,m}^{***}}{\partial z\partial w}|_0$, $\frac{\partial^2\phi_{2,m}^{***}}{\partial w^2}|_0$, $\frac{\partial^2\phi_{2,m}^{***}}{\partial z\partial w}|_0$ and $\frac{\partial^2\phi_{3,m}^{***}}{\partial w^2}|_0$ are bounded for all m. Then*

(a) F is of geometric rank 1 at 0: $Rk_F(0) = 1$, and hence F^{***} is well-defined.

(b) $F_{p_m}^{***} \to F^{***}$.

(c) If we write $F_{p_m}^{***} = \widetilde{G}_{2,m} \circ \tau_{p_m} \circ F \circ \sigma_{p_m} \circ \widetilde{G}_{1,m}$ where σ_{p_m} and $\tau_{p_m} := \tau_{p_m}^F$ are as in [CJX06, (3)], $\widetilde{G}_{1,m}$ and $\widetilde{G}_{2,m}$ are as in [CJX06, (7)], then $\widetilde{G}_{1,m}$ and $\widetilde{G}_{2,m}$ are convergent to some $\widetilde{G}_1 \in Aut_0(\partial\mathbb{H}^2)$ and $\widetilde{G}_2 \in Aut_0(\partial\mathbb{H}^5)$ respectively.

Proof:(Sketch) (a) Suppose that F has rank 0 at 0. We'll seek a contradiction.

Denote $F^{**} = (f^{**}, \phi^{**}, g^{**})$. We only need to prove the following claim:

$$\frac{\partial^2 f^{**}}{\partial w^2}(0) = 0,\ \frac{\partial^2\phi^{**}}{\partial z^2}(0) = \frac{\partial^2\phi^{**}}{\partial z\partial w}(0) = (0,0,0). \tag{3.23}$$

In fact, by Lemma 2.4 [CJX06], F must be linear but this is a contradiction with $deg(F) = 2$.

Write

$$(F_{p_m})^{***} = (\widetilde{f}_m, \widetilde{\phi}_m, \widetilde{g}_m)$$

and also $(F_{p_m})^{***} = \tau^m \circ \left((F)^{**}_{q_m}\right)^{**} \circ \sigma_m$ where

$$\left((F)^{**}_{q_m}\right)^{**} = (\hat{f}_m, \hat{\phi}_m, \hat{g}_m),$$

$$\sigma_m(z,w) = \left(\frac{\lambda_m(z + a_m w)U_m}{1 - 2i\langle \overline{a_m}, z\rangle + (r_m - i|a_m|^2)w}, \frac{\lambda^2 w}{1 - 2i\langle \overline{a_m}, z\rangle + (r_m - i|a_m|^2)w}\right),$$

and

$$\tau^m(z^*, w^*) = \left(\frac{\lambda^*_m(z^* + a^*_m w^*)U^*_m}{1 - 2i\langle \overline{a^*_m}, z^*\rangle + (r^*_m - i|a^*_m|^2)w}, \frac{\lambda^{*2} w^*}{1 - 2i\langle \overline{a^*_m}, z^*\rangle + (r^*_m - i|a^*_m|^2)w^*}\right).$$

In order to prove Claim (3.23), it is enough to show that

$$\frac{\partial^2 \hat{f}_m}{\partial w^2}|_0 \to 0,\ \frac{\partial^2 \hat{\phi}_m}{\partial z^2}|_0 \to (0,0,0),\ \frac{\partial^2 \hat{\phi}_m}{\partial z \partial w}|_0 \to (0,0,0),\ as\ m \to \infty. \tag{3.24}$$

Then by the construction of F^{***} (see § 4.3), σ_m and τ_m satisfy the following properties.

$$(i)\quad \frac{\partial^2 \hat{f}_m}{\partial z \partial w}|_0 = \lambda^2_m \frac{\partial^2 \widetilde{f}_m}{\partial z \partial w}|_0,$$

$$(ii)\quad \frac{\partial^2 \hat{f}_m}{\partial w^2}|_0 = i\lambda^2_m a_m \frac{\partial^2 \widetilde{f}_m}{\partial z \partial w}|_0 U^{-1}_m + \lambda^3_m \frac{\partial^2 \widetilde{f}_m}{\partial w^2}|_0 U^{-1}_m,$$

$$(iii)\quad \frac{\partial^2 \hat{\phi}_m}{\partial z^2}|_0 = \lambda_m U^2_m \frac{\partial^2 \widetilde{\phi}_m}{\partial z^2}|_0 U^*_{22,m},$$

$$(iv)\quad \frac{\partial^2 \hat{\phi}_m}{\partial z \partial w}|_0 = \lambda_m \frac{\partial^2 \widetilde{\phi}_m}{\partial z^2}|_0 a_m U^2_m U^*_{22,m} + \lambda^2_m U_m \frac{\partial^2 \widetilde{\phi}_m}{\partial z \partial w}|_0 U^*_{22,m},$$

$$(v)\quad \frac{\partial^2 \hat{\phi}_m}{\partial w^2}|_0 = \lambda_m a^2_m \frac{\partial^2 \widetilde{\phi}_m}{\partial z^2}|_0 U^2_m U^*_{22,m} + 2\lambda^2_m a_m U_m \frac{\partial^2 \widetilde{\phi}_m}{\partial z \partial w}|_0 U^*_{22,m} + \lambda^3_m \frac{\partial^2 \widetilde{\phi}_m}{\partial w^2}|_0 U^*_{22,m}.$$

From (i), since F has rank 0 at 0, we see $\frac{\partial^2 \hat{f}_m}{\partial z \partial w}|_0 \to 0$. Recall that $\widetilde{F}_m$ has rank one at 0 and is of the form in § 3.7. Then $\frac{\partial^2 \widetilde{f}_m}{\partial z \partial w}|_0 = \frac{i}{2}$ so that $\lambda_m \to 0$ as m goes to ∞.

From (ii), since $\frac{\partial \widetilde{f}_m}{\partial w^2}|_0 = 0$, we know that $\lambda^2_m a_m$ is bounded.

From (iii), since $\lambda_m \to 0$ and $\frac{\partial^2 \widetilde{\phi}_m}{\partial z^2}|_0 = [1,0,0]$, we see $\frac{\partial^2 \hat{\phi}_m}{\partial z^2}|_0 \to \frac{\partial^2 \phi^{**}}{\partial z^2}|_0 = [0,0,0]$.

From (iv), the second term in the right hand side goes to zero for $\lambda_m \to 0$, and the first term in the right hand side is $\lambda_m \frac{\partial^2 \widetilde{\phi}_m}{\partial z^2}|_0 a_m U^2_m U^*_{22,m} = \frac{\lambda^2_m a_m}{\lambda_m}[1,0,0]U^2_m U^*_{22,m}$. Recall from (ii) that $\lambda^2_m a_m$ is bounded. On the other hand, $\frac{\partial^2 \hat{\phi}_m}{\partial z \partial w}|_0$ is bounded. All of these imply that $\lambda^2_m a_m$ must go to zero. Then from (ii), $\frac{\partial^2 \hat{f}_m}{\partial w^2}|_0 \to \frac{\partial^2 f^{**}}{\partial w^2}|_0 = 0$.

From (v), the second and the third terms on the right hand side converge to zero because of λ_m and $a_m\lambda_m^2 \to 0$. The first term on the right hand side is bounded and can be written as $\frac{\lambda_m^2 a_m^2}{\lambda_m}\frac{\partial^2\widetilde{\phi}_m}{\partial z^2}|_0 U_m^2 U_{22,m}^*$. This implies that $\lambda_m a_m \to 0$. Then from (iv), it proves $\frac{\partial^2\hat{\phi}_m}{\partial z\partial w}|_0 \to \frac{\partial^2\hat{\phi}}{\partial z\partial w} = [0,0,0]$. Our claim (3.24), as well as (3.23), is proved.

The part (b) is already included in the above proof. For the part (c), $\widetilde{G}_{1,m}$ is convergent because of the normalization procedure of F^{***} from F (cf. [Hu03]) and because of the part (a). □

The minimum property for $\mathcal{W}(F_p^{***})$ implies the vanishing of derivatives of the function $\mathcal{W}(F_p^{***})$ at p_0, which derives the extra equation.

In order to get this extra equation, we have to compute the first order derivatives of the function $\mathcal{W}(F_p^{***})$, which is done by the following lemma. The proof of this lemma used the differential formulas for F_p^* and F_p^{**} listed in Chapter 1. Although the computation is long, since every time it only counts for derivative at 0 so that lots of higher order terms can be dropped, the calculation is manageable.

Lemma 3.8.2 *([CJX06], lemma 3.1) Let $F = F_{c_1,c_3,e_1,e_2}$ and F_p^{***} be as above. Then for $p = (z_0, w_0) = (z_0, u_0 + i|z_0|^2) \in \partial\mathbb{H}^2$ near 0, we have real analytic functions*

$$b^2(p) = b^2 - 4b(2e_1 + c_1^2)\Im(z_0) + o(1), \quad c_1^2(p) = c_1^2 + 4c_1(bc_1 + 2c_2)\Im(z_0) + o(1),$$
$$e_2(p) + e_1(p) = e_2 + e_1 + 8b(e_1 + e_2)\Im(z_0) + o(1),$$
$$\mathcal{W}(F_p^{***}) = c_1^2(p) - e_1(p) - e_2(p) = c_1^2 - e_1 - e_2 + \Big(4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2)\Big)\Im(z_0) + o(1)$$

where we denote $o(k) = o(|(z_0, u_0)|^k)$.

If $c_1 = 0$, by the minimum property, it implies that the coefficient of $\Im(z_0)$ must be zero. Then we obtain

$$-8b(e_1 + e_2) = 0.$$

Since $-e_1 - e_2 = \frac{1}{4} + b^2 \neq 0$, it implies $b = 0$.

If $c_1 > 0$, by the minimum property of $F = F_0^{***}$, it implies that

$$4c_1(c_1 b + 2c_2) - 8b(e_1 + e_2) = 0.$$

Since $-e_1 - e_2 = \frac{1}{4} + b^2 + c_1^2 \neq 0$ and $c_1, b, c_2, -e_1, -e_2 \geq 0$, it implies $b = c_2 = 0$.

To study F, we distinguish two cases:

Case (I) $c_1 = b = 0$;
Case (II) $c_1 \neq 0$ and $b = c_2 = 0$.

It was proved in [CJX06] that F is equivalent to a new map $F_{c_1,c_3.e_1,e_2}$ that is of the form in one of the following types (from Case (I), we obtain (I); from Case (II), we obtain (IIA)(IIB) and (IIC)):

(I) $F_{0,0,e_1,e_2} = (f, \phi_1, \phi_2, \phi_3, g)$ is of the form

$$\begin{array}{l} f = \frac{z+(\frac{i}{2}+ie_1)zw}{1+ie_1w+e_2w^2},\ \phi_1 = \frac{z^2}{1+ie_1w+e_2w^2}, \\ \phi_2 = \frac{c_2w^2}{1+ie_1w+e_2w^2},\ \phi_3 = 0,\ g = \frac{w+ie_1w^2}{1+ie_1w+e_2w^2} \end{array} \tag{3.25}$$

where $e_1e_2 = c_2^2$ and $-e_1 - e_2 = \frac{1}{4}$. Here $e_2 \in [-\frac{1}{4}, 0)$ is a parameter. It then corresponds to the family $\{G_t\}_{0\leq t<\pi/2}$ in (3.10). When $e_2 = -\frac{1}{4}$, $F_{0,0,e_1,e_2}$ corresponds to G_0, i.e. $(z, w) \mapsto (z^2, \sqrt{2}zw, w^2, 0)$; when $e_2 \to 0$, $F_{0,0,e_1,e_2}$ goes to $G_{\pi/2} = F_{\pi/2}$, i.e., $(Z, w) \mapsto (z, zw, w^2)$.

(IIA) $F_{c_1,0,e_1,0} = (f, \phi_1, \phi_2, \phi_3, g)$ is of the form

$$f = \frac{z + (\frac{i}{2} + ie_1)zw}{1 + ie_1w},\ \phi_1 = \frac{z^2}{1 + ie_1w},\ \phi_2 = \frac{c_1zw}{1 + ie_1w},\ \phi_3 = 0,\ g = w \tag{3.26}$$

where $-e_1 = \frac{1}{4} + c_1^2$ and $c_1 \in [0, \infty)$ is a parameter. It corresponds to the family $\{F_\theta\}_{0<\theta\leq\pi/2}$ in (3.11). When $c_1 = 0$, $F_{c_1,0,e_1,0}$ corresponds to $F_{\pi/2}$; when $c_1 \to \infty$, $F_{c_1,0,e_1,0}$ goes to the linear map, i.e., $(z, w) \mapsto (z, w, 0)$.

(IIB) $F_{c_1,0,0,e_2} = (f, \phi_1, \phi_2, \phi_3, g)$ is of the form:

$$f = \frac{z + \frac{i}{2}zw}{1 + e_2w^2},\ \phi_1 = \frac{z^2}{1 + e_2w^2},\ \phi_2 = \frac{c_1zw}{1 + e_2w^2},\ \phi_3 = 0,\ g = \frac{w}{1 + e_2w^2}, \tag{3.27}$$

where $-e_2 = \frac{1}{4} + c_1^2$ and $c_1 \in (0, \infty)$ is a parameter. Notice that when $c_1 \to 0$, the map $F_{c_1,0,0,e_2}$ goes to the map G_0, i.e. the one in type (I) when $e_2 = -\frac{1}{4}$.

(IIC) $F_{c_1,c_3,e_1,e_2} = (f, \phi_1, \phi_2, \phi_3, g)$ is of the form:

$$\begin{array}{l} f = \frac{z+(\frac{i}{2}+ie_1)zw}{1+ie_1w+e_2w^2},\ \phi_1 = \frac{z^2}{1+ie_1w+e_2w^2}, \\ \phi_2 = \frac{c_1zw}{1+ie_1w+e_2w^2},\ \phi_3 = \frac{c_3w^2}{1+ie_1w+e_2w^2},\ g = \frac{w+ie_1w^2}{1+ie_1w+e_2w^2}, \end{array} \tag{3.28}$$

where $c_1, c_3 > 0, -e_1, -e_2 \geq 0, \quad e_1 e_2 = c_3^2, \quad -e_1 - e_2 = \frac{1}{4} + c_1^2$.

For any map F_{c_1,c_3,e_1,e_2} in one of these four types, we denote F_{c_1,c_3,e_1,e_2}, or (c_1, c_3, e_1, e_2), $\in \mathcal{K}_I$, $\mathcal{K}_{IIA}$, $\mathcal{K}_{IIB}$, and $\mathcal{K}_{IIC}$, respectively.

At this moment, it is not clear whether different such maps are not equivalent.

3.9 Proof of Theorem 3.6.1 - Part 3

It is proved by Ji-Zhang [JZ09] that the case (IIB) never occur.

We denote by $\mathcal{K}$ the collection of all such maps F_{c_1,c_3,e_1,e_2}. We may identify a map F_{c_1,c_3,e_1,e_2} with a point (c_1, c_3, e_1, e_2) in $\mathbb{R}^4$.

The set $\mathcal{K}$ is equal to a disjoint union

$$\mathcal{K} = \mathcal{K}_I \cup \mathcal{K}_{II}$$

where $\mathcal{K}_I = \{F_{c_1,c_3,e_1,e_2} \in \mathcal{K} \mid F_{c_1,c_2,e_1,e_2} \; is \; of \; form \; (I)\}$, etc. The set $\mathcal{K}$ is also equal to a disjoint union

$$\mathcal{K} = \mathcal{K}_{I,II,1+4e_2+2c_1^2>0} \cup \mathcal{K}_{I,II,1+4e_2+2c_1^2>0} \cup \mathcal{K}_{I,II,1+4e_2+2c_1^2>0},$$

where $\mathcal{K}_{I,II,1+4e_2+2c_1^2>0} = (\mathcal{K}_I \cup \mathcal{K}_{II}) \cap \{(c_1, c_3, e_1, e_2) \mid 1 + 4e_2 + 2c_1^2 > 0\}$, etc.

Lemma 3.9.1 *([JZ09], lemma 3.1)*

(a) If $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{I,II,1+4e_2+2c_1^2>0}$, then locally the function $\mathcal{W}((F_{c_1,c_3,e_1,e_2})_p^{***})$ is increasing as p moves along any ray from 0 in $\partial\mathbb{H}^2$.

(b) If $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{I,II,1+4e_2+2c_1^2=0}$, then locally the function $\mathcal{W}((F_{c_1,c_3,e_1,e_2})_p^{***})$ is constant as p moves along any ray from 0 in $\partial\mathbb{H}^2$.

(c) If $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{I,II,1+4e_2+2c_1^2<0}$, then locally the function $\mathcal{W}((F_{c_1,c_3,e_1,e_2})_p^{***})$ is decreasing as p moves along any ray from 0 in $\partial\mathbb{H}^2$.

Lemma 3.9.2 *([JZ09], lemma 3.2) (i) $\mathcal{K}_{II,e_1<e_2} \subseteq \mathcal{K}_{I,II,1+4e_2+2c_1^2>0}$, and $\mathcal{K}_{II,e_1=e_2} \subseteq \mathcal{K}_{I,II,1+4e_2+2c_1^2>0}$.*

(ii) Let $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{II,e_1>e_2}$. Then
(a) $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{I,II,1+4e_2+2c_1^2>0}$ if and only if $\frac{1}{2}c_1^2 + c_1^4 < 4c_3^2 < (\frac{1}{4} + c_1^2)^2$ holds.
(b) $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{I,II,1+4e_2+2c_1^2=0}$ if and only if $\frac{1}{2}c_1^2 + c_1^4 = 4c_3^2$ holds.
(c) $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{I,II,1+4e_2+2c_1^2<0}$ if and only if $0 \leq 4c_3^2 < \frac{1}{2}c_1^2 + c_1^4$ holds.

By last section, we can consider F_{c_1,c_3,e_1,e_2} satisfying the minimum property (3.22). Such map F_{c_1,c_3,e_1,e_2} will contradict with the statement in Lemma 3.9.1(c). Therefore, it follows:

Lemma 3.9.3 *([JZ09], lemma 3.4) Let* $(c_1, c_3, e_1, e_2) \in \mathcal{K}_I \cup \mathcal{K}_{II}$*. Then* F_{c_1,c_3,e_1,e_2} *satisfies (3.22) if and only if* $F_{c_1,c_3,e_1,e_2} \in \mathcal{K}^* := \mathcal{K}_I \cup \mathcal{K}_{II} - \mathcal{K}_{I,II,1+4e_2+2c_1^2<0}$.

This proves the part (i) of Theorem 3.6.1. From the definition of $\mathcal{K}$, e_1 and e_2 are determined by c_1 and c_3 through a quadratic equation. This show how we obtain the domain of the parameters c_1 and c_3 in Theorem 3.6.1.

We may outline the idea for the proof of Lemma 3.9.1 here. The monotonicity in Lemma 3.9.1 (a) means

$$\frac{d\mathcal{W}(F^{***}_{\Gamma(t)}))}{dt} = \lim_{\Delta t \to 0} \frac{\mathcal{W}(F^{***}_{\Gamma(t+\Delta t)}) - \mathcal{W}(F^{***}_{\Gamma(t)})}{\Delta t} \geq 0, \ \forall t \in [0, \delta]. \tag{3.29}$$

For any $0 < t < \delta$ and sufficiently small $\Delta t > 0$, if we can write

$$F^{***}_{\Gamma(t+\Delta t)} = \left(F^{***}_{\Gamma(t)} \right)^{***}_{q(t,\Delta t)} \tag{3.30}$$

for some differentiable map $q(t, \Delta t) \in \partial \mathbb{H}^2$, then from Lemma 3.8.2 we should have

$$\mathcal{W}(F^{***}_{\Gamma(t+\Delta t)}) = \mathcal{W}(F^{***}_{\Gamma(t)}) + \left[4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2) \right] (\Gamma(t)) \Im(q_1(t)) \Delta t + o(|\Delta t|), \tag{3.31}$$

where we write $q(t, \Delta t) := (q_1(t), q_2(t))\Delta t + o(|\Delta t|)$. Notice that $[4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2)](\Gamma(t)) \geq 0$ always holds because $c_1, c_2, -e_1 - e_2 \geq 0$. Then (3.29) follows if $\Im(q_1(t)) \geq 0$ holds. In particular, if $[4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2)](\Gamma(t)) \neq 0$ for any fixed $t \in [0, \delta)$, and if the following condition is satisfied:

$$\Im(q_1(t)) > 0, \quad \forall t \in [0, \delta], \tag{3.32}$$

then the strict inequality (3.29) holds. To prove (3.29), it suffices to prove (3.32). (3.32) is proved by local calculation of $\Im(q_1(t))$.

3.10 Proof of Theorem 3.6.1 - Part 4

As the final step to complete the proof of Theorem 3.6.1, it is proved by Ji-Zhang [JZ09] that the cases (I)(IIA) and (IIC) indeed give a complete classification for mappings in $Rat(\mathbb{B}^2, \mathbb{B}^N)$ with degree 2, up to equivalent classes.

To solve the classification problem, by Lemma 3.9.3, we need to show: for maps $F_{c_1',c_3',e_1',e_2'}$ and $F_{c_1'',c_3'',e_1'',e_2''}$ in $\mathcal{K}^*$, we have

$$F_{c_1',c_3',e_1',e_2'} \text{ is equivalent to } F_{c_1'',c_3'',e_1'',e_2''} \iff (c_1', c_3', e_1', e_2') = (c_1'', c_3'', e_1'', e_2''). \tag{3.33}$$

We first prove a local version of (3.33).

Lemma 3.10.1 *For any $P^{(0)} = (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) \in \mathcal{K}^*$, there is a neighborhood U of $P^{(0)}$ in $\mathcal{K}^*$ and a constant $c > 0$ such that for any point $(c_1', c_3', e_1', e_2'), (c_1'', c_3'', e_1'', e_2'') \in U$ with $F_{c_1'',c_3'',e_1'',e_2''} = (F_{c_1',c_3',e_1',e_2'})^{***}_p$ where $p = (a, b + i|a|^2) \in \partial\mathbb{H}^2$, $a \in \mathbb{C}$, $b \in \mathbb{R}$, $|p| := \max\{|a|, |b|\}$ $\leq c$, we have*

$$(c_1'', c_3'', e_1'', e_2'') = (c_1', c_3', e_1', e_2'). \tag{3.34}$$

To prove this, we use the monotone property in Lemma 3.9.1 to show:

$$\mathcal{W}(F_{c_1',c_3',e_1',e_2'}) = \mathcal{W}((F_{c_1',c_3',e_1',e_2'})^{***}_{\Gamma(0)}) \leq \mathcal{W}((F_{c_1',c_3',e_1',e_2'})^{***}_{\Gamma(t^*)}) = \mathcal{W}(F_{c_1'',c_3'',e_1'',e_2''}), \tag{3.35}$$

and

$$\mathcal{W}(F_{c_1'',c_3'',e_1'',e_2''}) = \mathcal{W}((F_{c_1'',c_3'',e_1'',e_2''})^{***}_{\widetilde{\Gamma}(0)}) \leq \mathcal{W}((F_{c_1'',c_3'',e_1'',e_2''})^{***}_{\widetilde{\Gamma}(\widetilde{t}^*)}) = \mathcal{W}((F_{c_1',c_3',e_1',e_2'}). \tag{3.36}$$

By (3.35) and (3.36), it follows that the function $\mathcal{W}((F_{c_1',c_3',e_1',e_2'})^{***}_{\Gamma}) = constant$. Then it implies that $(F_{c_1',c_3',e_1',e_2'})^{***}_{\Gamma(t)}$ is constant. Since $F_{c_1'',c_3'',e_1'',e_2''} = (F_{c_1',c_3',e_1',e_2'})^{***}_p$, Lemma 3.10.1 is proved.

Next, we prove the global version of (3.33). We need to show: if $F_{c_1^{(0)},c_3^{(0)},e_1^{(0)},e_2^{(0)}}$ and $F_{\widetilde{c}_1^{(0)},\widetilde{c}_3^{(0)},\widetilde{e}_1^{(0)},\widetilde{e}_2^{(0)}}$ in $\mathcal{K}^*$ are equivalent, then

$$(\widetilde{c}_1^{(0)}, \widetilde{c}_3^{(0)}, \widetilde{e}_1^{(0)}, \widetilde{e}_2^{(0)}) = (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}). \tag{3.37}$$

Let $\mathcal{E} := \{(c_1, c_3, e_1, e_2) \in \mathcal{K}_I \cup \mathcal{K}_{II} \mid (F_{c_1,c_3,e_1,e_2})^{***}_p \equiv F_{c_1,c_3,e_1,e_2}, \ \ \forall p \in \partial\mathbb{H}^2 \textit{ near } 0\}$. We assume that $(c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) \notin \mathcal{E}$; otherwise $F_{c_1^{(0)},c_3^{(0)},e_1^{(0)},e_2^{(0)}}$ and $F_{\widetilde{c}_1^{(0)},\widetilde{c}_3^{(0)},\widetilde{e}_1^{(0)},\widetilde{e}_2^{(0)}}$ cannot be equivalent.

Since $F_{c_1^{(0)},c_3^{(0)},e_1^{(0)},e_2^{(0)}}$ and $F_{\widetilde{c}_1^{(0)},\widetilde{c}_3^{(0)},\widetilde{e}_1^{(0)},\widetilde{e}_2^{(0)}}$ are equivalent,

$$F_{\widetilde{c}_1^{(0)},\widetilde{c}_3^{(0)},\widetilde{e}_1^{(0)},\widetilde{e}_2^{(0)}} = \Psi \circ F_{c_1^{(0)},c_3^{(0)},e_1^{(0)},e_2^{(0)}} \circ \Theta \tag{3.38}$$

where $\Theta \in Aut(\mathbb{H}^2)$ and $\Psi \in Aut(\mathbb{H}^5)$.

We take a real analytic curve $L = L(s) \in \mathcal{K}^* - \mathcal{E}$, $0 \leq s \leq 1$, where $\mathcal{E}$ is a such that $L(0) = (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)})$. In fact, since $(c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) \notin \mathcal{E}$ and $\mathcal{E}$ is closed, L could be taken in a neighborhood of $(c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)})$.

We shall use some deformation. By using automorphisms of balls, we can take a real analytic family of automorphisms $\Theta_s \in Aut(\partial\mathbb{H}^2)$, $\Psi_s \in Aut(\partial\mathbb{H}^5)$, $s \in [0,1]$, such that when $s = 0$, $\Theta_0 = \Theta$, $\Psi_0 = \Psi$; when $s \in (0,1)$, $\Theta_s(0) \neq \infty$, $\Psi_s \circ F_{L(s)} \circ \Theta_s(0) = 0$; when $s = 1$, $\Theta_1 = Id$, $\Psi_1 = Id$. Then we define

$$\hat{L}_0(s) := \Psi_s \circ F_{L(s)} \circ \Theta_s \in Rat(\mathbb{H}^2, \mathbb{H}^5), \quad 0 \leq s \leq 1,$$

such that $\hat{L}_0(s)(0) = 0$ for all s, $F_{\hat{L}_0(0)} = \Psi \circ F_{L(0)} \circ \Theta$ and $\hat{L}_0(1) = L(1)$. Our goal is to show: $\hat{L}_0(s) = L(s)$, $\forall s \in [0,1]$, so that $\hat{L}_0(0) = L(0)$, i.e., (3.37) holds.

Even though $(F_{\hat{L}_0(s)})^{***}$ is in $\mathcal{K}$ for any $s \in (0,1]$, it may not be in $\mathcal{K}^*$ because the minimum property (3.22) may not be satisfied. We claim that $(F_{\hat{L}_0(s)})^{***}$ is equivalent to another map $F_{\hat{L}(s)} \in \mathcal{K}^*$. More precisely, we want to find $q(s) \in \partial\mathbb{H}^2$ so that

$$F_{\hat{L}(s)} := (F_{\hat{L}_0(s)})^{***}_{q(s)} \in \mathcal{K}^*, \quad \forall s \in (0,1]. \tag{3.39}$$

As points in $\mathcal{K}$, we show

$$dist\bigg(F_{\hat{L}(s)}, \quad F_{\hat{L}_0(s)}\bigg) \to 0, \quad as\ s \to 1, \tag{3.40}$$

i.e.,

$$dist\bigg(F_{\hat{L}(s)}, \quad F_{L(s)}\bigg) \to 0, \quad as\ s \to 1.$$

Since both $F_{\hat{L}(s)} \in \mathcal{K}^*$ and $F_{L(s)} \in \mathcal{K}^* - \mathcal{E}$ where $s \in (s_0, 1]$ for some $s_0 > 0$ such that $0 \leq 1 - s_0$ is sufficiently small, by the local version of Theorem 3.6.1, we conclude

$$F_{\hat{L}(s)} = F_{L(s)}, \quad \forall s \in (s_0, 1].$$

Repeating this process. Finally by continuity $F_{\hat{L}(s)} = F_{L(s)}$, $\forall s \in [0,1]$. When restricted at 0, $F_{\hat{L}_0(0)} = F_{\hat{L}(0)} = F_{L(0)}$, so that (3.37) is proved.

Chapter 4

More Analytic Approaches

4.1 Five Facts in a Model Case

Theorem 4.1.1 *[HJ01] Let $F \in Prop_2(\mathbb{H}^n, \mathbb{H}^{2n-1})$. Then F is equivalent to a map that is either linear, or Whitney map: $W_{n,1}(z,w) = \big(z, w(z,w)\big)$ where $(z,w) \in \mathbb{C}^{n-1} \times \mathbb{C}$.*

Here is the main ingredient of the proof:

1. F^{**} can be further normalized into $F^{***} = (f, \phi, g)$:

$$
\begin{aligned}
&f_1 = z_1 + \frac{i}{2} z_1 w + o_{wt}(3),\\
&f_j = z_j + o_{wt}(3), \quad 2 \le j \le n-1,\\
&\phi_j = z_1 z_j + o_{wt}(2), 2 \le j \le n-1,\\
&g = w + o_{wt}(4),
\end{aligned}
$$

2. Show: The geometric rank $\kappa_0 = 1$.

3. Furthermore,

$$
\begin{aligned}
&f_1 = z_1 + \frac{i}{2} z_1 w + o_{wt}(3),\\
&f_j = z_j, \quad 2 \le j \le n-1,\\
&\phi_j = z_1 z_j + o_{wt}(2), 2 \le j \le n-1,\\
&g = w,
\end{aligned}
$$

4. F is equivalent to a map that satisfies

$$F = (z_1\widetilde{f}_1,\ z_2, ..., z_{n1}, z_1\widetilde{\phi}_1, ..., z_1\widetilde{\phi}_{n-1}, w).$$

   Here $\Phi = (\widetilde{f}_1, \widetilde{\phi}_1, ..., \widetilde{\phi}_{n-1})$ defines a biholomorphic map from $\mathbb{H}^n$ onto $\mathbb{B}^n$.

5. In particular, the restriction $F|_{\{z_1=0\}}$ is linear fractional.

4.2 Generalization of the Five Facts

The above five facts are generalized into the following results:

1. **Theorem 4.2.1** *([H03]) Let $F \in Prop_2(\mathbb{H}^n, \mathbb{H}^N)$. Then F is equivalent to a map $F_p^{***} = (f_p^{***}, \phi_p^{***}, g_p^{***})$ of the following form:*

$$\begin{cases} f_{l,p}^{***} = \sum_{j=1}^{\kappa_0} z_j f_{lj}^*(z,w), \quad f_{lj}^*(z,w) = \delta_l^j + \frac{i\delta_l^j \mu_l}{2} w + O(|(z,w)|^2),\ l \le \kappa_0; \\ f_{j,p}^{***} = z_j + o_{wt}(3), \quad \kappa_0 + 1 \le j \le n-1; \\ \phi_{lk,p}^{***} = \mu_{lk} z_l z_k + o_{wt}(2), \qquad \forall\ (l,k) \in \mathcal{S}; \\ g = w + o_{wt}(4), \end{cases}$$

   where

$$\mathcal{S}_0 = \{(j,l) : 1 \le j \le \kappa_0, j \le l, 1 \le l \le n-1\}$$

   is the index set for those $\phi_{lk,p}$ that have non-zero coefficients of the $z_l z_k$ terms,

$$\mathcal{S} := \mathcal{S}_0 \cup \left\{ (j,l) \quad | \quad j = \kappa_0 + 1, \kappa_0 + 1 \le l \le N - n - \frac{(2n - \kappa_0 - 1)\kappa_0}{2} \right\}$$

   is the index set for all $\phi_{lk,p}$, and

$$\mu_{jl} = \begin{cases} \sqrt{\mu_j + \mu_l}, & for\ j,l \le \kappa_0, j \ne l, \\ \sqrt{\mu_j}, & if\ j \le \kappa_0\ and\ l > \kappa_0\ or\ if\ j = l \le \kappa_0. \end{cases} \tag{4.1}$$

   (To see the outline of the proof, see Theorem 4.3.1 and its proof).

2. Due to the existence of the non-zero $z_l z_k$ terms of $\phi_{lk,p}^{***}$ above, which "occupy the room" in $\partial\mathbb{B}^N$, as application of Theorem 4.2.1, we immediately obtain the following result, which generalizes the second fact of the above five ones.

Corollary 4.2.2 *Let $F \in Prop_2(\mathbb{H}^n, \mathbb{H}^N)$ with geometric rank κ_0. Then*

$$N \geq n + \frac{\kappa_0(2n - \kappa_0 - 1)}{2}.$$

This inequality is sharp.

(Its proof will be found in § 4.3.)

[Example] If $F \in Prop_2(\mathbb{B}^n, \mathbb{B}^{2n-1})$ with $n \geq 3$, then $\kappa_0 \leq 1$. In fact, this follows from the inequality $2n - 1 \geq n + \frac{\kappa_0(2n-\kappa_0-1)}{2}$. □

3. **Theorem 4.2.3** *([HJX06], theorem 3.1) Let $F \in Prop_3(\mathbb{H}^n, \mathbb{H}^N)$ with geometric rank $\kappa_0 \leq n - 2$. Then F is equivalent to a map $F_p^{***} = (f_p^{***}, \phi_p^{***}, g_p^{***})$ of the following form:*

$$\begin{cases} f_{l,p}^{***} = \sum_{j=1}^{\kappa_0} z_j f_{lj}^*(z,w), \quad f_{lj}^*(z,w) = \delta_l^j + \frac{i\delta_l^j \mu_l}{2} w + O(|(z,w)|^2), \ l \leq \kappa_0; \\ f_{j,p}^{***} = z_j, \quad \kappa_0 + 1 \leq j \leq n-1; \\ \phi_{lk,p}^{***} = \mu_{lk} z_l z_k + \sum_{j=1}^{\kappa_0} z_j \phi_{lkj,p}^*, \quad \phi_{lkj,p}^*(z,w) = o_{wt}(2), \quad for \ (l,k) \in \mathcal{S}_0; \\ \phi_{lj,p}^{***} = \sum_{j=1}^{\kappa_0} z_j \phi_{lkj,p}^* = O(|(z,w)|^3) \quad for \ (l,k) \in \mathcal{S} - \mathcal{S}_0; \\ g_p^{***} = w; \end{cases}$$

Let us outline the idea to prove $g_p^{***} \equiv w$ and $f_{j,p}^{***} \equiv z_j$, $\forall \kappa_0 + 1 \leq j \leq n-1$.

First we consider to prove $g_p^{***} \equiv w$. It needs the following lemma:

Lemma 4.2.4 *Let $F = F^{**} \in Prop_2(\mathbb{H}^n, \mathbb{H}^N)$. If we further assume that $g(0,w) \equiv w$, then $g \equiv w$.*

Proof: Write $g = \sum_{m=1}^{\infty} g^{(m)}$ where $g^{(m)}$ is a weighted homogeneous polynomial of weighted degree m.

Considering the weighted $2k$ order terms in the basic equation $Im(g) = |\widetilde{f}|^2$ over $Im(w) = |z|^2$, we obtain

$$Im(g^{(2k)}) = \sum_{l=1}^{2k-1} \sum_{j=1}^{n-1} f_j^{(l)} \overline{f_j^{(2k-l)}} + \sum_{l=1}^{2k-1} \sum_{j=1}^{N-n} \phi_j^{(l)} \overline{\phi_j^{(2k-1)}} \tag{4.2}$$

whenever $Im(w) = |z|^2$. Since the right hand side doest not contain the z^I terms with $|I| = 2k$, $g^{(2k)}$ cannot contain the z^I terms with $|I| = 2k$. Since $g(0, w) = w$, $g^{(2k)}$ cannot contain the w^k terms. Hence

$$g^{(2k)} = \sum_{p=1}^{k-1} \sum_{|I|=2k-2p} c_I z^I w^p.$$

Since $g(0, w) = w$, from the basic equation $Im(g) = |\widetilde{f}|^2$ on $\partial\mathbb{H}_n$, it implies $\widetilde{f}(0, w) = 0$. Then $\widetilde{f}$ does not contain the w^p terms for any $p \geq 1$. By comparing the $z^I u^p$ terms in (4.2) where $|I| = 2k - 2p$, $c_I = 0$. Thus $g^{(2k)} \equiv 0$. Similarly, we obtain that $g^{(2k+1)} \equiv 0$ for $k \geq 1$. Therefore $g(z, w) \equiv w$. $\square$

We suppose that F, in addition, is C^3-smooth on $\partial\mathbb{H}^n$, and want to show that if the map F_p^{***} is as constructed in Theorem 4.2.1, then it satisfies $g_p^{***} \equiv w$. In fact, by Hopf lemma 1.7.3 and Lemma 4.2.4, it is sufficient to prove Lemma 4.2.5 below.

Lemma 4.2.5 *([H03]) Let F be a C^3-smooth map from $M \subset \partial\mathbb{H}_n$ into $\partial\mathbb{H}_N$ satisfying the condition for F_p^{***} in Theorem 4.2.1 with $1 \leq \kappa_0 \leq n - 2$. Then*

$$g(0, w) = w + o(|w|^3).$$

Next we show that $f_j = z_j$ for $\kappa_0 + 1 \leq j \leq n - 1$.

At this moment, we would like to assume the following "semi-linearity" property (see the fact five, or [H03]):

$$F(0, ..., 0, z_{\kappa_0+1}, ..., z_{n-1}, w) = (0, ..., 0, z_{\kappa_0+1}, ..., z_{n-1}, 0, ..., 0, w). \tag{4.3}$$

From (4.3), we can write $f_j = \sum_{l=1}^{\kappa_0} z_l f_{lj}^*$ and $\phi = \sum_{l=1}^{\kappa_0} z_l \phi_l^*$. Then from the above sections, we can write $F_p^{***} = (\widetilde{f}, g)$ as

$$\begin{cases} f_l = \sum_{j=1}^{\kappa_o} z_j f_{l,j}^*(z), \quad l \leq \kappa_0; \\ f_k = z_k + \sum_{j=1}^{\kappa_0} z_j f_{k,j}^*(z), \quad k \geq \kappa_0 + 1; \\ \phi_{lk} = z_l z_k + \sum_{j=1}^{\kappa_0} z_j \phi_{lk,j}^*(z), \quad (l, k) \in \mathcal{S}_0; \\ \phi_{st} = \sum_{j=1}^{\kappa_0} z_j \phi_{st,j}^*(z), \quad (s, t) \in \mathcal{S} - \mathcal{S}_0; \\ g \equiv w. \end{cases}$$

Substituting these into the equation $Im(g) = |\widetilde{f}|^2$. Fix $k \geq \kappa_0 + 1$. Considering the terms $\bar{z}_k z^I u^i$ (for arbitrary I and i) in $Im(g) = |\widetilde{f}|^2$, we have

$$0 = \bar{z}_k \sum_{j=1}^{\kappa_0} z_j f^*_{k,j}(z, u + i|z|^2).$$

Hence $\sum_{j=1}^{\kappa_0} f^*_{k,j}(z) \equiv 0$. This implies $f_k \equiv z_k$ for $\kappa_0 + 1 \leq k \leq n-1$.

4. **Theorem 4.2.6** *([HJX06], p.523) Let $F \in Prop_3(\mathbb{B}^n, \mathbb{B}^N)$ with $3 \leq n \leq N$ and geometric rank $\kappa_0 \leq n-2$. Then F is equivalent to a proper holomorphic map of the form*

$$H = (z_1, ..., z_{n-\kappa_0}, H_1, ..., H_{N-n+\kappa_0}),$$

where $H_j = \sum_{l=n-\kappa_0+1}^{n} z_l H_{j,l}$ with $H_{j,l}$ holomorphic over $\overline{\mathbb{B}^n}$. When $\kappa_0 = 1$, $F \in Prop_3(\mathbb{B}^n, \mathbb{B}^N)$ is equivalent to a new map (z, wh) where $h \in Rat(\mathbb{B}^n, \mathbb{B}^{N-n+1})$.

5. **Theorem 4.2.7** *[H03] Let $F \in Prop_3(\mathbb{H}^n, \mathbb{H}^N)$ with geometric rank $\kappa_0 \leq n-2$. The $\forall p \in \mathbb{B}^n$, $\exists$ affine $(n-\kappa_0)$-dimensional complex subspace S_p^a containing p such that*

$$F|_{S_p^a} \text{ is linear fractional.}$$

4.3 How to Construct F^{***}?

Recall for any $F \in Prop_2(\mathbb{H}^n, \mathbb{H}^N)$, F is equivalent to $F^{**} = (f^{**}, \phi^{**}, g^{**})$ such that

$$f^{**} = z + \frac{i}{2} a^{**(1)}(z) w + o_{wt}(3), \ \phi^{**} = \phi^{**(2)}(z) + o_{wt}(2), \ g^{**} = w + o_{wt}(4), \tag{4.4}$$

$$\langle \overline{z}, a^{**(1)}(z) \rangle |z|^2 = |\phi^{**(2)}(z)|^2.$$

We can further normalize this map to get more properties while it preserves the above properties of F^{**}.

How to define F^{***} in Theorem 4.2.1 from the map F^{**} preserving the property (4.4) ?

Consider $\sigma \in Aut_0(\mathbb{H}_n)$ and $\tau \in Aut_0(\mathbb{H}_N)$:

$$\sigma = \frac{(\lambda(z + aw) \cdot U, \ \lambda^2 w)}{1 - 2i\langle \overline{a}, z \rangle + (r - i|a|^2) w}, \tag{4.5}$$

where $\lambda > 0$, $r \in \mathbb{R}$, a is an $(n-1)$-tuple and U is an $(n-1)\times(n-1)$ unitary matrix. Let

$$\tau^*(z^*, w^*) = \frac{(\lambda^*(z^* + a^* w^*)\cdot U^*, \lambda^{*2} w^*)}{1 - 2i\langle \overline{a^*}, z^*\rangle + (r^* - i|a^*|^2)w^*} \tag{4.6}$$

where $\lambda^* > 0$, $r^* \in \mathbb{R}$, a^* is an $(N-1)$-tuple and U^* is an $(N-1)\times(N-1)$ unitary matrix.

Theorem 4.3.1 *[H03] (A) Let $F = (f, \phi, g)$ and $F^* = (f^*, \phi^*, g^*)$ be C^2-smooth CR map from a neighborhood of 0 in $\partial\mathbb{H}^n$ into $\partial\mathbb{H}^N$ $(N \geq n > 1)$, satisfies the condition (4.4). Suppose that $F^* = \tau^* \circ F \circ \sigma$ where σ and τ^* are as in (4.6) and (4.6). Then it holds that*

$$\lambda^* = \lambda^{-1},\ a_1^* = -\lambda^{-1} a \cdot U,\ a_2^* = 0,\ r^* = -\lambda^{-2} r,\ U^* = \begin{pmatrix} U^{-1} & 0 \\ 0 & U_{22}^* \end{pmatrix} \tag{4.7}$$

where $a^* = (a_1^*, a_2^*)$ with a_1^* its first $(n-1)$ components, U_{22}^* is an $(N-n)\times(N-n)$ unitary matrix. Conversely, suppose τ^* and σ, given as above, are related by (4.7). Suppose that F satisfies the condition (4.4). Then $F^* := \tau^* \circ F \circ \sigma$ also satisfies the (4.4).

(B) Let F and $F^* := \tau^* \circ F \circ \sigma$ both satisfy the condition (4.4). Let us denote

$$f(z,w) = z + \frac{i}{2} z\mathcal{A}w + \frac{1}{2}\frac{\partial^2 f}{\partial w^2}|_0 w^2 + o(|(z,w)|^2),$$
$$f^*(z,w) = z + \frac{i}{2} z\mathcal{A}^*w + \frac{1}{2}\frac{\partial^2 f^*}{\partial w^2}|_0 w^2 + o(|(z,w)|^2).$$

and

$$\phi(z,w) = \frac{1}{2} z(B^1, ..., , B^{N-n})z^t + z\mathcal{B}w + \frac{1}{2}\frac{\partial^2 \phi}{\partial w^2}|_0 w^2 + o(|(z,w)|^2),$$
$$\phi^*(z,w) = \frac{1}{2} z(B^{*1}, ..., , B^{*N-n})z^t + z\mathcal{B}^*w + \frac{1}{2}\frac{\partial^2 \phi^*}{\partial w^2}|_0 w^2 + o(|(z,w)|^2),$$

where

$$\mathcal{A} = -2i \begin{pmatrix} \frac{\partial^2 f_1}{\partial z_1 \partial w} & \cdots & \frac{\partial^2 f_{n-1}}{\partial z_1 \partial w} \\ \vdots & & \vdots \\ \frac{\partial^2 f_1}{\partial z_{n-1} \partial w} & \cdots & \frac{\partial^2 f_{n-1}}{\partial z_{n-1} \partial w} \end{pmatrix}\Bigg|_0$$

is the $(n-1)\times(n-1)$ matrix,

$$B^k = \begin{pmatrix} \frac{\partial^2 \phi_{(k)}}{\partial z_1^2} & \frac{\partial^2 \phi_{(k)}}{\partial z_1 \partial z_2} & \cdots & \frac{\partial^2 \phi_{(k)}}{\partial z_1 \partial z_{n-1}} \\ \frac{\partial^2 \phi_{(k)}}{\partial z_2 \partial z_1} & \frac{\partial^2 \phi_{(k)}}{\partial z_2^2} & \cdots & \frac{\partial^2 \phi_{(k)}}{\partial z_2 \partial z_{n-1}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 \phi_{(k)}}{\partial z_{n-1} \partial z_1} & \frac{\partial^2 \phi_{(k)}}{\partial z_{n-1} \partial z_2} & \cdots & \frac{\partial^2 \phi_{(k)}}{\partial z_{n-1}^2} \end{pmatrix}\Bigg|_0, \quad 1 \leq k \leq N-n,$$

are $(n-1)\times(n-1)$ matrices, and

$$\mathcal{B} = \left.\begin{pmatrix} \frac{\partial^2 \phi_{(1)}}{\partial z_1 \partial w} & \cdots & \frac{\partial^2 \phi_{(N-n)}}{\partial z_1 \partial w} \\ \vdots & & \vdots \\ \frac{\partial^2 \phi_{(1)}}{\partial z_{n-1} \partial w} & \cdots & \frac{\partial^2 \phi_{(N-n)}}{\partial z_{n-1} \partial w} \end{pmatrix}\right|_0$$

is an $(n-1)\times(N-n)$ matrix. $\mathcal{A}^*, B^{*k}, \mathcal{B}^*$ are defined similarly. Then

$$\mathcal{A}^* = \lambda^2 U \mathcal{A} U^{-1},$$

$$\frac{\partial^2 f^*}{\partial w^2}(0) = i\lambda^2 a U \mathcal{A} U^{-1} + \lambda^3 \frac{\partial^2 f}{\partial w^2}(0) U^{-1}.$$

$$\begin{array}{l} z(B^{*1}, ..., B^{*N-n})z^t = \lambda z U(B^1, ..., B^{N-n})U^t z^t U_{22}^*, \\ \mathcal{B}^* = \lambda U(B^1, ..., B^{N-n})U^t a^t U_{22}^* + \lambda^2 U \mathcal{B} U_{22}^*, \\ \frac{1}{2}\frac{\partial^2 \phi^*}{\partial w^2}|_0 = \frac{1}{2}\lambda a U(B^1, ..., B^{N-n})U^t a^t U_{22}^* + \lambda^2 a U \mathcal{B} U_{22}^* + \frac{1}{2}\lambda^3 \frac{\partial^2 \phi}{\partial w^2}|_0 U_{22}^*, \end{array} \tag{4.8}$$

(C) Let F_1 be a non-constant C^2 CR map from $M \subset \partial\mathbb{H}_n$ into $\partial\mathbb{H}_N$. Assume that $F_2 = \tau \circ F_1 \circ \sigma$ with $\sigma \in Aut(\mathbb{H}_n)$ and $\tau \in Aut(\mathbb{H}_N)$. Then

$$Rk_{F_2}(p) = Rk_{F_1(\sigma(p))}.$$

The normalization F^{***} in Theorem 4.2.1 is constructed by $\tau^* \circ F \circ \sigma$ for appropriate choice of τ^* and σ.

Proof of Theorem 4.2.1: (a) (b) By Theorem 4.3.1.

(c) Since f_j already are as in (2.8.1), from (2.72), we get $\sum_{j=1}^{\kappa_0} \mu_j |z_j|^2 |z|^2 = \sum_j |\phi_j^{(2)}(z)|^2$. Write $\phi_j^{(2)}(z) = \sum_{k\le l} a_{kl}^{(j)} z_k z_l$. Then (2.8.1) becomes

$$\sum_{j=1}^{\kappa_0} \mu_j |z_j|^2 |z|^2 = \sum_j a_{kl}^{(j)} \overline{a_{k'l'}^{(j)}} z_k z_l \overline{z_{k'} z_{l'}}.$$

Write $\alpha_{jl} := (a_{jl}^{(1)}, ..., a_{jl}^{(N-n)})$. We have

$$\langle a_{kl},\ \overline{a_{k'l'}}\rangle = \begin{cases} 0,\ if\ (k,l) \ne (k',l'), \\ \mu_k + \mu_l,\ \ if\ k,l \le \kappa_0, k \ne l,\ (k,l) = (k',l'), \\ \mu_k,\ \ if\ k \le \kappa_0,\ l > \kappa_0,\ (k,l) = (k',l'), \\ \mu_k,\ \ if\ k = l \le \kappa_0,\ (k,l) = (k',l'), \end{cases}$$

Hence $\{\alpha_{jl}\}_{(k,l)\in S_0}$ is a linearly independent system. This implies that $N-n \geq |\mathcal{S}_0|$. We extend $\{\frac{\alpha_{jl}}{|\alpha_{jl}|}\}$ to an $(N-n)\times(N-n)$ unitary matrix U_{22}^* and we replace ϕ by $\phi\cdot\overline{U_{22}^*}^t$. From the first identity of (4.8), we are done. $\square$

Proof of Corollary 4.2.2: It follows from $N-n\geq|\mathcal{S}_0|$. $\square$

4.4 Where is the Condition $\kappa_0 \leq n-2$ used ?

In Theorem 4.2.3 above, a very crucial condition is $\kappa_0 \leq n-2$. This condition indeed produces exact equations for the map F. In fact, by the normalization F^{**}, we have the curvature information:

$$\langle \overline{z}, a_p^{**(1)}(z)\rangle |z|^2 = |\phi_p^{**(2)}(z)|^2. \tag{4.9}$$

Write $a_p^{**(1)}(z) = z\mathcal{A}_p$ where

$$\mathcal{A}_p = -2i\left(\frac{\partial^2 f_{l,p}^{**}}{\partial z_j \partial w}\bigg|_0\right)$$

is an $(n-1)\times(n-1)$ Hermitian matrix.

Remarks

- The matrix $\mathcal{A}_p$ is semi-positive because of (4.9).
- (4.9) can be written as
$$z\mathcal{A}_p\overline{z}^t|z|^2 = |\phi_p^{**(2)}(z)|^2.$$
  Then for a non zero vector z, we have
$$\begin{aligned} |\phi_p^{**(2)}(z)|^2 = 0 \iff & z\mathcal{A}_p\overline{z}^t = 0 \\ \iff & z\mathcal{A}_p = 0 \qquad (\text{because } \mathcal{A}_p \geq 0) \\ \iff & \phi_p^{**(2)}(z) = 0 \end{aligned}$$
- We define a vector space $\mathcal{E}_p := \{\xi(p)\in\mathbb{C}^{n-1} \mid \xi(p)\cdot\mathcal{A}_p = 0\} \neq \emptyset$. Then
$$\xi(p)\in\mathcal{E}_p \iff \phi_p^{**(2)}(\xi(p)) = 0$$
  From these equations, it derives more equations by taking differentiation that make Theorem 4.2.3 possible.

4.5 Structure Theorem For Rank 1 Maps

As an application of Theorem 4.2.3, we have the following structure theorem on maps with geometric rank one. The key condition here is $\kappa_0 \le n-2$, which allows the maps have more rigidity property.

Theorem 4.5.1 *([HJX06], theorem 1.2) Let $F \in Prop_3(\mathbb{B}^n, \mathbb{B}^N)$ with $3 \le n \le N$ and geometric rank 1. Then F is equivalent to a proper holomorphic map of the form*

$$H := (z_1, \cdots, z_{n-1}, H_1, \cdots, H_{N-n+1}),$$

where $(H_1, \cdots, H_{N-n+1}) = w \cdot h$ with $h \in Rat(\mathbb{B}^n, \mathbb{B}^{N-n+1})$. Both H and h are affine linear maps along each hyperplane defined by $w = constant$.

In fact, from Theorem 4.2.3, when $\kappa_0 = 1$, we have

$$\begin{cases} f_{1,p}^{***} = z_1 f_1^*(z,w), \quad f_1^*(z,w) = 1 + \frac{i\mu_1}{2} w + O(|(z,w)|^2), \\ f_{j,p}^{***} = z_j, \quad 2 \le j \le n-1; \\ \phi_{1k,p}^{***} = \mu_{1k} z_1 z_k + z_1 \phi_{1k,p}^*, \quad \phi_{1k,p}^*(z,w) = o_{wt}(2), \quad for\ 1 \le k \le n-1; \\ \phi_{2\ell,p}^{***} = z_\ell \phi_{2\ell,p}^* = O(|(z,w)|^3) \quad for\ 2 \le \ell \le N-2n+1; \\ g = w. \end{cases}$$

By Cayley's transformation to obtain a new map $H : \mathbb{B}^n \to \mathbb{B}^N$:

$$H = (H_1, z_2, ..., z_{n-1}, H_n, ..., H_{N-n}, w).$$

We can make change on variables in the following way:

$$\begin{array}{ccc} z_1 & \leftrightarrow & z_n \\ \{z_2, ..., z_{n-1}\} & \leftrightarrow & \{z_1, ..., z_{n-2}\} \\ w & \leftrightarrow & z_{n-1} \end{array}$$

so that

$$H = (z_1, ..., z_{n-1}, H_1, H_2, ..., H_{N-n+1}).$$

As an application, we show the following result.

Theorem 4.5.2 *[HJX06] Let $F \in Rat(\mathbf{B}^n, \mathbf{B}^N)$ with geometric rank $\kappa_0 = 1$ and $n \ge 3$. Then $deg(F) \le \frac{N-1}{n-1}$.*

Proof:: For each $N \geq n \geq 3$, there is a unique positive integer k such that $k(n-1)+1 \leq N \leq (k+1)(n-1)$. We use induction on k. When $k=1$, $F \in Rat(\mathbb{B}^n, \mathbb{B}^{2n-2})$, by the first gap theorem, so that $deg(F) = 1 \leq \frac{N-1}{n-1}$ holds. Assume $deg(F) \leq \frac{N-1}{n-1}$ holds for any k. Consider $k+1$, by Theorem 4.2.6, F is equivalent to (z, wh) where $h \in Rat(\mathbb{B}^n, \mathbb{B}^{N-n+1})$. Then by the assumption, $deg(F) \leq 1 + deg(h) \leq 1 + \frac{(N-n+1)-1}{n-1} = \frac{N-1}{n-1}$. □

4.6 Proof of the Second Gap Theorem

The second gap theorem can be restated as

Theorem 4.6.1 *[HJX06] Let $F \in Prop_3(\mathbb{B}^n, \mathbb{B}^N)$ with $4 \leq n \leq N \leq 3n-4$. Then F is equivalent to $(F_\theta, 0)$ where*

$$F_\theta = (z, wcos\ \theta, z_1 wsin\ \theta, ..., z_{n-1} wsin\ \theta, w^2 sin\ \theta)$$

for some $\theta \in [0, \frac{\pi}{2}]$.

- In 2005, Hamada proved that any $F \in Prop_3(\mathbb{B}^n, \mathbb{B}^{2n})$ is equivalent to F_θ for some $\theta \in [0, \frac{\pi}{2}]$.

- By the inequality $N \geq n + \frac{\kappa_0(2n-\kappa_0-1)}{2}$, under the condition $N \leq 3n-4$, it implies that

  the geometric rank κ_0 of F is ≤ 1.

- Applying the structure theorem 4.5.1 for rank 1 maps, we can write

$$H := (z_1, \cdots, z_{n-1}, H_1, \cdots, H_{N-n+1}),$$

where $(H_1, \cdots, H_{N-n+1}) = w \cdot h$ with $h \in Rat(\mathbb{B}^n, \mathbb{B}^{N-n+1})$. Here

$$N - n + 1 \leq 3n - 4 - n + 1 = 2n - 3.$$

Then we can apply the first gap theorem to implies h is linear map.

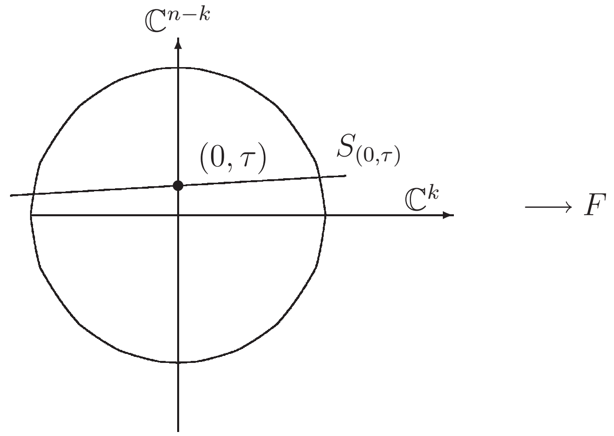
4.7 Rationality Problem

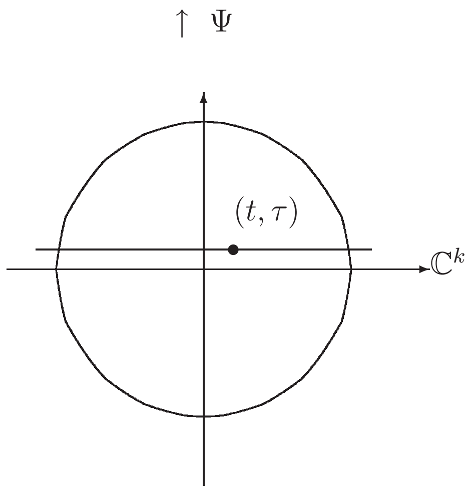
In 1989, Forstnerič proved [Fo89] that if $F \in Prop_{N-n+1}(\mathbb{B}^n, \mathbb{B}^N)$, then F must be a rational map with degree $deg(F) \leq N^2(N-n+1)$.

Theorem 4.7.1 *([HJX05], Corollary 1.3) If $F \in Prop_3(\mathbb{B}^n, \mathbb{B}^N)$ with either $\kappa_0 < n-1$ or $N \leq \frac{n(n+1)}{2}$, then F must be rational.*

- In order to prove that F is rational, by a theorem of Frostnerič, it suffices to prove that F is smooth on $\partial\mathbb{H}_n$.
- Under the hypothesis, F has partial k-linear property: for any point $Z \in \mathbb{B}^n - E$ where E is an affine subvariety, there is a unique k dimensional complex subspace S_Z on which F is linear fractional.
- Assume that $0 \in \mathbb{B}^n - E$ and $S_0 = \{z \mid z_{k+1} = ... = z_n = 0\}$.
- Construct a holomorphic map Ψ from a neighborhood of a rectangle $(-1-\epsilon, 1+\epsilon) \times (-\epsilon, \epsilon)$ in $\mathbb{C}^k \times \mathbb{C}^{n-k}$ to a neighborhood of $(-1-\epsilon, 1+\epsilon) \times \{0\}$ in $\mathbb{C}^k \times \mathbb{C}^{n-k}$ such that
  - $\Psi|_{S_0} \equiv Id$. ($\Longrightarrow \Psi$ is locally biholomorphic when ϵ is small)
  - For each line segment $L_{(t,\tau)}$ that (i) passes through the point (t, τ) and (ii) $L_{(t,\tau)}$ and S_0 are parallel, we have

$$\Psi(L_{(t,\tau)}) \subset S_{(0,\tau)}.$$





- For each fixed τ, since
$$F|_{S_{(0,\tau)}} = linear\ fractional,$$
we have
$$F \circ \Psi(t,\tau) = \frac{F(\tau) + \sum_{j=1}^{k} A_j(\tau)t_j}{1 + \sum_{j=1}^{k} b_j(\tau)t_j}.$$
On the other hand, we take a power series at the origin:
$$F \circ \Psi(t,\tau) = \sum_{\alpha} C_\alpha(\tau)t^\alpha \ \ \textit{is holomorphic near } (0,0).$$
$\overset{C_\alpha(\tau)\ is\ holomorphic}{\Longrightarrow}$ $A_j(\tau), b_j(\tau)$ and $F(\tau)$ are holomorphic of τ near 0.
- $F \circ \Psi(t,\tau)$ is holomorphic of (t,τ) whenever $\tau \sim 0$ and for any t.
- By the construction, $F \circ \Psi(t,\tau)$ is holomorphic is holomorphic of (t,τ) whenever (t,τ) in the rectangle $(-1-\epsilon, 1+\epsilon) \times (\epsilon, \epsilon)$.
- Choose Z_0 in the rectangle such that $F(Z_0) \in \partial\mathbb{B}^n$. Then
$$F = (F \circ \Psi) \circ (\Psi^{-1})$$
is holomorphic near $F(Z_0)$.

- F is C^∞ near $F(Z_0)$, so is on $\partial\mathbb{B}^n$.
- By Forstnerič Theorem, F is rational.

Chapter 5

More Geometric Approaches

5.1 Cartan's Moving Frame Theory

Invariants of a surface in $\mathbb{E}^3$ at a point[IL03] Let us consider (S, p) where S is a smooth surface in $\mathbb{E}^3$ and $p \in S$ is a point. To study (S, p), we could put (S, p) into a better position (normalized position). Namely, by taking a rotation and a translation, we can move S so that $p = (0, 0, 0)$ is the origin and the real surface S as a graph of a function f and that the tangent plane of S at 0 is the xy-plane:

$$z = f(x, y),\ f(0, 0) = 0,\ f_x(0, 0) = f_y(0, 0) = 0. \tag{5.1}$$

Geometrically, we moved the (S, p) into a "normalized position". Analytically, we have chosen a special coordinate system. Such normalization position for S is not unique; in fact, the above properties are preserved if we take any rotation in the xy-plane.

Suppose

$$z = f(x, y) = \sum_{j,k} a_{jk} x^j y^k,$$

where $a_{10} = f_x(0) = \frac{\partial f}{\partial x}|_0$, $a_{20} = \frac{1}{2} f_{xx}(0) = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}|_0$, etc.

If a function $h(a_{jk})$ is invariant under any rotation in the xy-plane, $h(a_{jk})$ is called a *differential invariant*.

For example, we consider Hessian

$$Hess(0, 0) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix} (0, 0)$$

to define

$$\begin{cases} K(0, 0) = det(Hess(0, 0)) = (f_{xx} f_{yy} - f_{xy} f_{yx})(0, 0), \\ H(0, 0) = \frac{1}{2} Trace(Hess(0, 0)) = \frac{1}{2}(f_{xx} + f_{yy})(0, 0). \end{cases} \tag{5.2}$$

We can verify that $K(0,0)$ and $H(0,0)$ are differential invariant. In fact, they are the value of the*Gaussian and mean curvatures* at the origin.

In the above, we fix a coordinate system (i.e., x-y-z) and the origin, which may be called a *frame*. Roughly speaking, a "frame" means: a choice of coordinate system, or a better position, or a normalized position, or an orthonormal basis of the tangent plane with the origin. In other words, we fix a frame at 0 of S.

Moving frames Consider a curve C in the space $\mathbb{E}^3$. Recall the Frénét-Serret frame: at any point at C, it has three vectors $\mathbf{T}, \mathbf{N}$ and $\mathbf{B}$, where $\mathbf{T}$ is the unit vector tangent to the curve, pointing in the direction of motion, $\mathbf{N} = \frac{d\mathbf{T}}{ds}/\|\frac{d\mathbf{T}}{ds}\|$ is the derivative of T with respect to the arclength parameter s of the curve, and $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ is the cross product of $\mathbf{T}$ and $\mathbf{N}$. It has

$$\begin{cases} \frac{d\mathbf{T}}{ds} = & \kappa\mathbf{N}, \\ \frac{d\mathbf{N}}{ds} = & -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \frac{d\mathbf{B}}{ds} = & -\tau\mathbf{N}, \end{cases}$$

where κ is the curvature of the curve and τ is the torsion of the curve.

We see that at every point p of the curve, there exists a frame $(\mathbf{T}, \mathbf{N}, \mathbf{B})_p$. These frames are continuous (or differentiable) of p. We call such frames *moving frames* along the curve. In this situation, every point is treated equally (no point is more special) and every frame is treated equally. κ and τ are invariants.

Cartan's moving frame theory will study submanifolds in which every point and every frame will be treated equally and that we should obtain some invariants.

Klein's Erlanger Programn Let G be a Lie group, and $H \subset G$ a closed Lie subgroup. Let $X := G/H$, the set of left cosets of H, is a homogeneous space with the induced differential structure from the quotient map. For material in this section, we refer [IL03].

By Klein's Erlanger Programn, we'll study geometry of submanifolds $M \subset X = G/H$, where two submanifolds $M, M' \subset X$ are *equivalent* if there is some $g \in G$ such that $g(M) = M'$.

$$\begin{array}{ccc} & & G \\ & s \nearrow & \downarrow \pi \\ M & \hookrightarrow & X = G/H \end{array}$$

[**Example**] Let us go back to real surfaces $S \hookrightarrow \mathbb{E}^3$:

$$\begin{array}{ccc} & & G = ASO(3) \\ & F \nearrow & \downarrow \pi \\ S & \hookrightarrow & \mathbb{E}^3 = G/H \end{array}$$

Here $i : S \hookrightarrow \mathbb{E}^3$ is the inclusion map,

$$\begin{aligned}
G &= ASO(3)\\
&= \text{the group of motions in } \mathbb{E}^3\\
&= \text{the space of orientated orthonormal frames of } \mathbb{E}^3\\
&= \text{the bundle of oriented orthonormal bases of } \mathbb{E}^3\\
&= \text{All adapted coordinates in } \mathbb{E}^3\\
&= \left\{ M = \begin{pmatrix} 1 & 0 \\ t & B \end{pmatrix}, t \in \mathbb{R}^3, B \in SO(3) \right\}.\\
H &= SO(3)\\
&= \text{all rotations.}\\
F &= \text{A first-order adapted lift (or a section)}\\
&= \text{A choice of adapted coordinates}\\
&= \text{A normalized position}
\end{aligned}$$

Write a lift $F(p) = (e_0(p), e_1(p), e_2(p), e_3(p))$ where $e_0(p) = [1, x, y, z]^t$ where $p = (x, y, z)^t \in S$, $(e_1, e_2, e_3)(p)$ are orthonormal, and $span(e_1(p), e_2(p)) = T_pS$. We said that F is a *first-order adapted lift.*

If we fix one lift $F(p) = \begin{bmatrix} 1 & 0 \\ p & Id \end{bmatrix}$, then any other first-order adapted lift $\widetilde{F}$ of S is of the form

$$\widetilde{F} = F \begin{bmatrix} 1 & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & 1 \end{bmatrix} = Fr \tag{5.3}$$

where $R : U \to SO(2)$ is a smooth function.

For each fixed point, we regard $F(p)$ is a frame (or normalized position) for S, regard another adapted lift $\widetilde{F}(p)$ as another frame (or normalized position) for S, and regard the matrix-valued map $r(p)$ is a rotations in the xy-plane. A section F is regarded as a family of frames (moving frames). □

In order to find differential invariants which are independent of choice of any such map r, we reformulate (5.3) in terms of Cartan's moving frame theory. One key idea to do this or to carry out the Klein's Erlanger Programm is the following theorem.

Theorem 5.1.1 *(Cartan's theorem, [IL03]) Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$ and Maurer-Cartan from ω over G. Let M be a manifold on which there exists a $\mathfrak{g}$-valued*

1-form ϕ such that $d\phi = -\phi \wedge \phi$. Then $\forall x \in M$, there exists a neighborhood U of x and a map $F : U \to G$ such that $F^*\omega = \phi$. Moreover, any two such maps F and $\widetilde{F}$ must satisfy $F = L_a \circ \widetilde{F}$ for some fixed $a \in G$, where L_a is a left translation of G.

Lie group and Lie algebra Let V be a real vector space of dimension n. Let $GL(V) \subset End(V)$ denote the group of all invertible linear maps. Let G be a Lie group. A *linear representation* of G is a group homomorphism $\rho : G \to GL(V)$. If V is endowed with a basis, we call the image $\rho(G)$ a *matrix Lie group*.

Let $\mathfrak{gl}(V) = End(V) = V \otimes V^*$. We identify $\mathfrak{gl}(V)$ with the set of $n \times n$ matrices where $n = \dim(V)$. We define a skew-symmetric multiplication [,] on $\mathfrak{gl}(V)$ by

$$[X, Y] = XY - YX,$$

where XY is the usual matrix multiplication. One can verify the *Jacobi identity*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0, \ \forall X, Y, Z \in \mathfrak{gl}(V).$$

A *Lie algebra* is a vector space $\mathfrak{g}$ equipped with a skew-symmetric bilinear operation $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called a *bracket*, that satisfies the Jacobi identity.

Let G be a Lie group. For each $g \in G$, we define the *left translation:*

$$L_g : G \to G, \ h \mapsto gh,$$

which derives

$$(L_g)_* : T_hG \to T_{gh}G.$$

A *left-invariant vector field* is a vector field X over G such that

$$(L_g)_* X = X, \quad \forall g \in G.$$

Since Lie bracket of two left-invariant vector fields is also left-invariant, the space of left-invariant vector fields $\Gamma^L(TG)$ is a Lie algebra of $\Gamma(TG)$.

A left-invariant vector field is determined by its value at just one point (say, at the identity element $e \in G$) because it is given at all other points by pushforward under left-translation. Thus we may identify $\Gamma^L(TG)$ with T_eG. We define $\mathfrak{g} = \Gamma^L(TG) \simeq T_eG$ to be the *Lie algebra* of G.

If $G \subseteq GL(V)$ is a matrix Lie group, then $\mathfrak{g} \simeq T_eG \subseteq \mathfrak{gl}(V) = End(V)$ is a matrix Lie algebra.

Maurer-Cartan form —— the intrinsic definition Maurer-Cartan form is defined over a Lie group G. It is not a standard one-form, but rather a $\mathfrak{g}$-valued one-form. If V is a vector space and M is a manifold, then a V-valued one-form is a collection of smooth maps: $T_xM \to V$. In other words, it is a smooth section of $T^*M \otimes V$. (If $V = \mathbb{R}$ or $\mathbb{C}$, it is the standard one-form. In our case, $V = T_eG$ where e is the identity element of G.)

The *Maurer-Cartan form* ω is a $\mathfrak{g}$-valued one-form on G defined by

$$\begin{array}{rccc} \omega: & T_gG & \to & T_eG \\ & v & \mapsto & \omega(v) = (dL_{g^{-1}})_* v \end{array}.$$

In other words, given an arbitrary Lie group G, we let $\mathfrak{g}$ denote its Lie algebra, which may be identified with T_eG (i.e., with the space of left-invariant vector fields). The *Maurer-Cartan form* ω of G is the unique left-invariant $\mathfrak{g}$-valued 1-form on G such that $\omega_e : T_eG \to \mathfrak{g}$ is the identity map.

Maurer-Cartan form —— the extrinsic definition If $G \subset GL(n)$ by a matrix valued inclusion $g = (g_{i,j})$, then one can write ω explicitly as

$$\omega = g^{-1}dg,$$

where $dg : T_gG \to \mathfrak{gl}(V)$ is the inclusion.

When G is a matrix Lie group, since $\omega = g^{-1}dg$ is a left-invariant $\mathfrak{g}$-valued 1-form such that $\omega_e : T_eG \to \mathfrak{g}$ is the identity map, then by the uniqueness, these two definitions of Maurer-Cartan form are the same.

We have the *Maurer-Cartan equation*:

$$d\omega = -\omega \wedge \omega. \tag{5.4}$$

In fact, $0 = d(Id) = d(g \cdot g^{-1}) = dg \cdot g^{-1} + gdg^{-1}$. Then $dg^{-1} = -g^{-1}dg \cdot g^{-1}$ so that

$$d\omega = d(g^{-1}dg) = dg^{-1} \wedge dg = -g^{-1}dg \cdot g^{-1} \wedge dg = -\omega \wedge \omega.$$

Transformation formula We consider the following diagram commutes:

$$\begin{array}{ccc} & & G \\ & F \nearrow & \downarrow \pi \\ M & \hookrightarrow & G/H \end{array}$$

Given a lift F of f, any other lift $\widetilde{F} : M \to G$ must be of the form

$$\widetilde{F}(x) = F(x)a(x) \tag{5.5}$$

for some map $a : M \to H$. It satisfies

$$\widetilde{F}^*(\omega) = a^{-1}F^*(\omega)a + a^{-1}da. \tag{5.6}$$

[Example] Going back to a surface $S \subset \mathbb{E}^3$.

$$\begin{array}{ccc} & & G = ASO(3) \\ & F \nearrow & \downarrow \pi \\ S & \hookrightarrow & \mathbb{E}^3 = G/H \end{array}$$

Here $G = ASO(3) = \left\{ M = \begin{pmatrix} 1 & 0 \\ t & B \end{pmatrix}, t \in \mathbb{R}^3, B \in SO(3) \right\}$, $H = SO(3)$ and F is a first-order adapted lift.

Write a lift

$$F(p) = (e_0(p), e_1(p), e_2(p), e_3(p)) = \begin{bmatrix} 1 & 0 \\ p & B(p) \end{bmatrix}$$

where $e_0(p) = [1, x, y, z]^t$, $p = (x, y, z)^t = (x, y, z(x, y))^t \in S$, $(e_1, e_2, e_3)(p)$ are orthonormal, and $span(e_1(p), e_2(p)) = T_pS$.

Since $\phi := F^{-1}dF$ is a $\mathfrak{g}$-valued one-form satisfying the equation $d\phi = -\phi \wedge \phi$, as in Theorem 5.1.1. Then $F^*\omega = \phi$, where ω is the Maurer-Cartan form over G.

We calculate $\phi = F^{-1}dF$ which equals to

$$\begin{bmatrix} 1 & 0 \\ p & B \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ dp & dB \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ B^{-1}dp & B^{-1}dB \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \phi^1 & \phi^1_1 & \phi^1_2 & \phi^1_3 \\ \phi^2 & \phi^2_1 & \phi^2_2 & \phi^2_3 \\ \phi^3 & \phi^3_1 & \phi^3_2 & \phi^3_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \phi^1 & 0 & -\phi^2_1 & -\phi^3_1 \\ \phi^2 & \phi^2_1 & 0 & -\phi^3_2 \\ \phi^3 & \phi^3_1 & \phi^3_2 & 0 \end{bmatrix}.$$

Then $dF = F\phi$, i.e.,

$$(de_0, de_1, de_2, e_3) = (e_0, e_1, e_2, e_3) \begin{bmatrix} 0 & 0 & 0 & 0 \\ \phi^1 & 0 & -\phi^1_2 & -\phi^3_1 \\ \phi^2 & \phi^2_1 & 0 & -\phi^3_2 \\ \phi^3 & \phi^3_1 & \phi^3_2 & 0 \end{bmatrix}$$

Hence $de_0 = e_1\phi^1 + e_2\phi^2 + e_3\phi^3$. On the other hand, $de_0 = (0, dx, dy, dz(x,y))^t$ is in the tangent space, i.e., in $span(e_1, e_2)$. Therefore, we have $de_0 = e_1\phi^1 + e_2\phi^2$, i.e., $de_0 = e_1F^*(\omega^1) + e_2F^*(\omega^2)$ so that

$$\phi^3 = F^*(\omega^3) = 0, \; and \; \phi^1 \wedge \phi^2 = F^*(\omega^1 \wedge \omega^2) \neq 0, \quad \forall p \in S. \tag{5.7}$$

By (5.7), $0 = F^*(\omega^3)$ implies

$$0 = F^*(d\omega^3).$$

By $d\omega = -\omega \wedge \omega$, we get $0 = -F^*(\omega^3_1 \wedge \omega^1 + \omega^3_2 \wedge \omega^2)$. By (5.7) $F^*\omega^1$ and $F^*\omega^2$ are linearly independent, we apply Cartan lemma [1] to obtain

$$F^* \begin{pmatrix} \omega^3_1 \\ \omega^3_2 \end{pmatrix} = F^* \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} F^* \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix},$$

where $h_{ij} = h_{ji}$ are some functions. We denote by $h_F = F^*(h_{ij})$ the matrix-valued function:

$$F^* \begin{pmatrix} \omega^3_1 \\ \omega^3_2 \end{pmatrix} = h_F F^* \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}.$$

If $\widetilde{F}$ is another adapted lift, we must have

$$\widetilde{F} = F \begin{pmatrix} 1 & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & 1 \end{pmatrix} = Fr,$$

where $R : U \to SO(2)$ is a smooth function. Then from $\widetilde{F}^*(\omega) = \widetilde{F}^{-1}d\widetilde{F}$ and $F^*(\omega) = F^{-1}dF$, we have

$$\begin{aligned} \widetilde{F}^{-1}d\widetilde{F} &= (Fr)^{-1}d(Fr) = r^{-1}(F^{1}dF)r + r^{-1}F^{-1}Fdr \\ &= \begin{pmatrix} 1 & & \\ & R^{-1} & \\ & & 1 \end{pmatrix} F^* \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & -\omega^2_1 & -\omega^3_1 \\ \omega^2 & \omega^2_1 & 0 & -\omega^3_2 \\ \omega^3 & \omega^3_1 & \omega^3_2 & 0 \end{pmatrix} \begin{pmatrix} 1 & & \\ & R & \\ & & 1 \end{pmatrix} + \begin{pmatrix} 0 & & \\ & R^{-1}dR & \\ & & 0 \end{pmatrix}. \end{aligned}$$

In particular,

$$\widetilde{F}^* \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = R^{-1}F^* \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}, \; \widetilde{F}^*(\omega^3_1, \omega^3_2) = F^*(\omega^3_1, \omega^3_2)R. \tag{5.8}$$

---

[1] Cartan's lemma: Let $v_1, ..., v_k$ are linearly independent vectors and let $w_1, ..., w_k$ are vectors such that $w_1 \wedge v_1 + ... + w_k \wedge v_k = 0$, then $w_i = \sum_j h_{ij}v_j$ where $h_{ij} = h_{ji}$, $1 \leq i, j \leq k$.

Since $R^{-1} = R^t$, we also have

$$h_{\widetilde{F}} = R^{-1} h_F R. \tag{5.9}$$

Then we obtain two invariants: the *mean curvature* $H := \frac{1}{2} trace(h_F)$ and *Gauss curvature* $K := det(h_F)$, which are well defined on U or M.

The case of n-dimensional submanifolds in $\mathbb{E}^{n+s}$ For high dimensional situation, we consider

$$G = ASO(n+s) = \left\{ M = \begin{pmatrix} 1 & 0 \\ t & B \end{pmatrix}, t \in \mathbb{R}^{n+s}, B \in SO(n+s) \right\}$$

which is the *group of Euclidean motions*,

$$H = SO(n+s),$$

which is the group of rotation and

$$X = \mathbb{E}^{n+s} = ASO(n+s)/SO(n+s).$$

Let $M \subset \mathbb{E}^{n+s}$ be an n-dimensional submanifold.

A map

$$s = (e_0, e_j, e_b) = \begin{bmatrix} 1 & 0 \\ t & B \end{bmatrix} : M \to G \tag{5.10}$$

is called a *first-order adapted lift* if $e_0 = (1, x)^t$, $x \in M$, (e_j, e_b) are orthonormal,

$$span\{e_j(x)\} = T_x M$$

and $e_b(x)$ are normal to M. Consequently,

$$s^* dx \equiv 0 \ mod\{x, e_j\}. \tag{5.11}$$

Let $\mathcal{F}^1$ denote the subbundle of $ASO(n+s)|_M$ of orientated first-ordered frames for M.

If $\widetilde{s}$ is another first-order adapted lift, then $\widetilde{s} = s \cdot g$ where

$$g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & g^i_j & 0 \\ 0 & 0 & u^a_b \end{bmatrix}$$

where $(g^i_j) \in SO(n)$ and $(u^a_b) \in SO(s)$. In other words, the motions in the fiber of $\mathcal{F}^1$ are given by such g.

As the same argument in Example above, the Maurer-Cartan form over $ASO(n+s)$ is of the form

$$\omega = \begin{pmatrix} 0 & 0 & 0 \\ \omega^i & \omega^i_j & \omega^i_b \\ \omega^a & \omega^a_j & \omega^a_b \end{pmatrix}. \tag{5.12}$$

$ds = s(s^*\omega)$. We have

$$dx = e_j\omega^j + e_a\omega^a.$$

Then pulling back by s, by (5.11), we obtain $s^*\omega^a = 0$ so that

$$s^*d\omega^a = 0. \tag{5.13}$$

From $d\omega = -\omega \wedge \omega$:

$$\begin{pmatrix} 0 & 0 & 0 \\ d\omega^i & d\omega^i_j & d\omega^i_b \\ d\omega^a & d\omega^a_j & d\omega^a_b \end{pmatrix} = -\begin{pmatrix} 0 & 0 & 0 \\ \omega^i & \omega^i_j & \omega^i_b \\ \omega^a & \omega^a_j & \omega^a_b \end{pmatrix} \wedge \begin{pmatrix} 0 & 0 & 0 \\ \omega^i & \omega^i_j & \omega^i_b \\ \omega^a & \omega^a_j & \omega^a_b \end{pmatrix}. \tag{5.14}$$

and by (5.13), we obtain

$$-s^*(\omega^a_j \wedge \omega^j) = 0.$$

By Cartan's lemma, we write

$$s^*\omega^a_j = h^a_{ij}s^*\omega^j$$

where $h^a_{ij} = h^a_{ji}$. It can be verified that $h^a_{ij}s^*\omega^i s^*\omega^j \otimes e_a$ is independent of choice of first order adapted lifts. Therefore it defines the *second fundamental form* of M

$$II_M := h^a_{ij}s^*\omega^i s^*\omega^j \otimes e_a \in \Gamma(M, S^2T^*M \otimes NM)$$

where NM denotes the normal bundle of M.

5.2 Flatness of CR Submanifolds

In Euclidean geometry, for a real submanifold $M^n \subset \mathbb{E}^{n+a}$, M is a piece of $\mathbb{E}^n$ if and only if its second fundamental form $II_M \equiv 0$.

In projective geometry, for a complex submanifold $M^n \subset \mathbb{CP}^{n+a}$, M is a piece of $\mathbb{CP}^n$ if and only if its projective second fundamental form $II_M \equiv 0$ (c.f. [IL03], p.81).

In CR geometry, we prove the CR analogue of this fact in this paper as follows:

Theorem 5.2.1 *(Ji-Yuan [JY09]) Let $H : M' \to \partial\mathbb{B}^{N+1}$ be a smooth CR-embedding of a strictly pseudoconvex CR real hypersurface $M' \subset \mathbb{C}^{n+1}$. Denote $M := H(M')$. If its CR second fundamental form $II_M \equiv 0$, then $M \subset F(\partial\mathbb{B}^{n+1}) \subset \partial\mathbb{B}^{N+1}$ where $F : \mathbb{B}^{n+1} \to \mathbb{B}^{N+1}$ is a certain linear fractional proper holomorphic map.*

It was proved by P. Ebenfelt, X. Huang and D. Zaitsev ([EHZ04], corollary 5.5), under the above same hypothese, that M' and hence M are locally CR-equivalent to the unit sphere $\partial\mathbb{B}^{n+1}$ in $\mathbb{C}^{n+1}$. This result allows us to consider

$$\begin{array}{ccccc} & & & G = SU(N+1,1) & \\ & & s \nearrow & \downarrow \pi & \\ F : \partial\mathbb{H}^{n+1} \to M = F(\partial\mathbb{H}^{n+1}) & \hookrightarrow & & \partial\mathbb{B}^{N+1} = G/H & \end{array}$$

There are several definitions of the CR second fundamental forms II_M of M. We have to prove that the above theorem is true for all of these definitions.

- Definition A, intrinsic one (Webster).
- Definition B, extrinsic one (cf. Ebenfelt-Huang-Zaitsev(2004)).
- Definition C, Cartan moving frame theory, with the group $G = GL^Q(CN+2)$.
- Definition D, Cartan moving frame theory, with the group $G = SU(N+1,1)$.

5.3 Definition A, the CR Second Fundamental Form

Let (M, θ) be a strictly pseudoconvex pseudohermitian manifold where θ is a contact form. Associated with a contact form θ one has the Reeb vector field R_θ, defined by the equations: (i) $d\theta(R_\theta, \cdot) \equiv 0$, (ii) $\theta(R_\theta) \equiv 1$.

If there are n complex 1-forms θ^α so that $\{\theta^1, ..., \theta^n\}$ forms a local basis for holomorphic cotangent bundle and

$$d\theta = i \sum_{\alpha,\beta=1}^{n} h_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}} \tag{5.15}$$

where $(h_{\alpha\overline{\beta}})$, called the *Levi form matrix*, is positive definite. Such θ^α may not be unique. Following Webster (1978), a coframe (θ, θ^α) is called *admissible* if (5.15) holds.

Theorem 5.3.1 *(Webster, 1978) Let (M^{2n+1}, θ) be a strictly pseudoconvex pseudohermitian manifold and let θ^j be as in (5.15). Then there are unique way to write*

$$d\theta^\alpha = \sum_{\gamma=1}^{n} \theta^\gamma \wedge \omega_\gamma^\alpha + \theta \wedge \tau^\alpha, \tag{5.16}$$

where τ^α are $(0,1)$-forms over M that are linear combination of $\theta^{\overline{\alpha}} = \overline{\theta^\alpha}$, and ω_α^β are 1-forms over M such that

$$0 = dh_{\alpha\overline{\beta}} - h_{\gamma\overline{\beta}}\omega_\alpha^\gamma - h_{\alpha\overline{\gamma}}\omega_{\overline{\beta}}^{\overline{\gamma}}. \tag{5.17}$$

We may denote $\omega_{\alpha\overline{\beta}} = h_{\gamma\overline{\beta}}\omega_\alpha^\gamma$ and $\overline{\omega_{\beta\overline{\alpha}}} = h_{\alpha\overline{\gamma}}\omega_{\overline{\beta}}^{\overline{\gamma}}$. In particular, if

$$h_{\alpha\beta} = \delta_{\alpha\beta}, \tag{5.18}$$

the identity in (5.17) becomes $0 = -\omega_{\alpha\overline{\beta}} - \overline{\omega_{\beta\overline{\alpha}}}$, i.e.,

$$0 = \omega_\alpha^\beta + \omega_{\overline{\beta}}^{\overline{\alpha}}. \tag{5.19}$$

Lemma 5.3.2 *([EHZ04], corollary 4.2) Let M and $\widetilde{M}$ be strictly pseudoconvex CR manifolds of dimensions $2n+1$ and $2\widetilde{n}+1$ respectively, and of CR dimensions n and $\widetilde{n}$ respectively. Let $F : M \to \widetilde{M}$ be a smooth CR-embedding. If (θ, θ^α) is a admissible coframe on M, then in a neighborhood of a point $\widetilde{p} \in F(M)$ in $\widetilde{M}$ there exists an admissible coframe $(\widetilde{\theta}, \widetilde{\theta}^A) = (\widetilde{\theta}, \widetilde{\theta}^\alpha, \widetilde{\theta}^\mu)$ on $\widetilde{M}$ with $F^*(\widetilde{\theta}, \widetilde{\theta}^\alpha, \widetilde{\theta}^\mu) = (\theta, \theta^\alpha, 0)$. In particular, the Reeb vector field $\widetilde{R}$ is tangent to $F(M)$. If we choose the Levi form matrix of M such that the functions $h_{\alpha\overline{\beta}}$ in (5.15) with respect to (θ, θ^α) to be $(\delta_{\alpha\overline{\beta}})$, then $(\widetilde{\theta}, \widetilde{\theta}^A)$ can be chosen such that the Levi form matrix of $\widetilde{M}$ relative to it is also $(\delta_{A\overline{B}})$. With this additional property, the coframe $(\widetilde{\theta}, \widetilde{\theta}^A)$ is uniquely determined along M up to unitary transformations in $U(n) \times U(\widetilde{n} - n)$.*

If (θ, θ^α) and $(\widetilde{\theta}, \widetilde{\theta}^A)$ are as above such that the condition on the Levi form matrices in Lemma 5.3.2 are satisfied, we say that the coframe $(\widetilde{\theta}, \widetilde{\theta}^A)$ is *adapted* to the coframe (θ, θ^α). In this case, by (5.19), we have $\theta = F^*\widetilde{\theta}$, $\theta^\alpha = F^*\widetilde{\theta}^\alpha$, and

$$d\theta^\alpha = \sum_{\gamma=1}^{n} \theta^\gamma \wedge \omega_\gamma^\alpha + \theta \wedge \tau^\alpha, \quad 0 = \ \omega_\alpha^\beta + \omega_{\overline{\beta}}^{\overline{\alpha}}, \quad \forall 1 \le \alpha, \beta \le n,$$

and

$$d\widetilde{\theta}^A = \sum_{B=1}^{\widetilde{n}} \widetilde{\theta}^C \wedge \widetilde{\omega}_C^A + \widetilde{\theta} \wedge \widetilde{\tau}^A, \quad 0 = \ \widetilde{\omega}_A^B + \widetilde{\omega}_{\overline{B}}^{\overline{A}}, \qquad \forall 1 \le A, B \le N.$$

For simplicity, we may denote $F^*\widetilde{\omega}_B^A$ by ω_B^A. We also denote $F^*\widetilde{\omega}_{A\overline{B}}$ by $\omega_{A\overline{B}}$ where $\omega_{A\overline{B}} = \omega_A^B$.

Write $\omega_\alpha{}^\mu = \omega_\alpha{}^\mu{}_\beta \theta^\beta$. The matrix of $(\omega_\alpha{}^\mu{}_\beta)$, $1 \le \alpha, \beta \le n$, $n+1 \le \mu \le \hat{n}$, defines the *CR second fundamental form* of M. It was used in [W79] and [Fa90].

5.4 Definition B, the CR Second Fundamental Form

Let $F : M \to \widetilde{M}$ be a smooth CR-embedding between $M \subset \mathbb{C}^{n+1}$ and $\widetilde{M} \subset \mathbb{C}^{N+1}$ where M and $\widetilde{M}$ are real strictly pseudoconvex hypersurfaces of dimensions $2n+1$ and $2\widetilde{n}+1$, and CR dimensions n and $\widetilde{n}$, respectively. Let $p \in M$ and $\widetilde{p} = F(p) \in \widetilde{M}$ be points. Let $\widetilde{\rho}$ be a local defining function for $\widetilde{M}$ near the point $\widetilde{p}$. Let

$$E_k(p) := span_{\mathbb{C}}\{L^{\bar{J}}(\widetilde{\rho}_{Z'} \circ F)(p) \mid J \in (Z_+)^n, 0 \le |J| \le k\} \subset T^{1,0}_{\widetilde{p}}\mathbb{C}^{N+1},$$

where $\widetilde{\rho}_{Z'} := \partial\widetilde{\rho}$ is the complex gradient (i.e., represented by vectors in $\mathbb{C}^{N+1}$ in some local coordinate system Z' near $\widetilde{p}$). $E_k(p)$ is independent of the choice of local defining function $\widetilde{\rho}$, coordinates Z' and the choice of basis of the CR vector fields $L_{\bar{1}}, ..., L_{\bar{n}}$.

The *CR second fundamental form* II_M of M is defined by (cf. [EHZ04], §2)

$$II_M(X_p, Y_p) := \overline{\pi\big(XY(\widetilde{\rho}_{\overline{Z}'} \circ f)(p)\big)} \in \overline{T'_{\widetilde{p}}\widetilde{M}/E_1(p)} \tag{5.20}$$

where $\widetilde{\rho}_{\overline{Z}'} = \overline{\partial}\widetilde{\rho}$ is represented by vectors in $\mathbb{C}^{N+1}$ in some local coordinate system Z' near $\widetilde{p}$, X, Y are any $(1,0)$ vector fields on M extending given vectors $X_p, Y_p \in T^{1,0}_p(M)$, and $\pi : T'_{\widetilde{p}}\widetilde{M} \to T'_{\widetilde{p}}\widetilde{M}/E_1(p)$ is the projection map.

5.5 Definition C, the CR Second Fundamental Form

Groups and geometry In Euclidean geometry, we consider

$$\begin{array}{ccc} & & G = ASO(n+s) \\ & s \nearrow & \downarrow \pi \\ M & \hookrightarrow & \mathbb{E}^{n+s} = ASO(n+s)/SO(n+s) \end{array}.$$

In projective geometry, we consider

$$\begin{array}{ccc} & & G = GL(\mathbb{C}^{N+1}) \\ & s \nearrow & \downarrow \pi \\ M & \hookrightarrow & \mathbb{CP}^N \end{array}.$$

In CR geometry, we will consider

$$\begin{array}{ccc} & & G = GL^Q(\mathbb{C}^{N+2}) \\ & s \nearrow & \downarrow \pi \\ M & \hookrightarrow & \partial\mathbb{H}^{N+1} \end{array} .$$

in this section and

$$\begin{array}{ccc} & & G = SU(N+1,1) \\ & s \nearrow & \downarrow \pi \\ M & \hookrightarrow & \partial\mathbb{H}^{N+1} \end{array} .$$

in the next section.

Construction of the group $GL^Q(\mathbb{C}^{N+2})$ We consider a real hypersurface Q in $\mathbb{C}^{N+2}$ defined by the homogeneous equation

$$\langle Z, Z\rangle := \sum_A Z^A\overline{Z^A} + \frac{i}{2}(\overline{Z^0}Z^{N+1} - Z^0\overline{Z^{N+1}}) = 0, \tag{5.21}$$

where $Z = (Z^0, Z^A, Z^{N+1})^t \in \mathbb{C}^{N+2}$. Let

$$\pi_0 : \mathbb{C}^{N+2} - \{0\} \to \mathbb{CP}^{N+1}, \quad (z_0, ..., z_{N+1}) \mapsto [z_0 : ... : z_{N+1}], \tag{5.22}$$

be the standard projection. For any point $x \in \mathbb{CP}^{N+1}$, $\pi_0^{-1}(x)$ is a complex line in $\mathbb{C}^{N+2}-\{0\}$. For any point $v \in \mathbb{C}^{N+2} - \{0\}$, $\pi_0(v) \in \mathbb{CP}^{N+1}$ is a point. The image $\pi_0(Q - \{0\})$ is the Heisenberg hypersurface $\partial\mathbb{H}^{N+1} \subset \mathbb{CP}^{N+1}$.

For any element $A \in GL(\mathbb{C}^{N+2})$:

$$A = (a_0, ..., a_{N+1}) = \begin{bmatrix} a_0^{(0)} & a_1^{(0)} & ... & a_{N+1}^{(0)} \\ a_0^{(1)} & a_1^{(1)} & ... & a_{N+1}^{(1)} \\ \vdots & \vdots & & \vdots \\ a_0^{(N+1)} & a_1^{(N+1)} & ... & a_{N+1}^{(N+1)} \end{bmatrix} \in GL(\mathbb{C}^{N+2}), \tag{5.23}$$

where each a_j is a column vector in $\mathbb{C}^{N+2}$, $0 \leq j \leq N+1$. This A is associated to an automorphism $A^\star \in Aut(\mathbb{CP}^{N+1})$ given by

$$A^\star\Big(\big[z_0 : z_1 : ... : z_{N+1}\big]\Big) = \Big[\sum_{j=0}^{N+1} a_j^{(0)} z_j : \sum_{j=0}^{N+1} a_j^{(1)} z_j : ... : \sum_{j=0}^{N+1} a_j^{(N+1)} z_j\Big]. \tag{5.24}$$

When $a_0^{(0)} \neq 0$, in terms of the non-homogeneous coordinates $(w_1, ..., w_n)$, $A^\star$ is a linear fractional from $\mathbb{C}^{N+1}$ which is holomorphic near $(0, ..., 0)$:

$$A^\star\big(w_1, ..., w_{N+1}\big) = \left(\frac{\sum_{j=0}^{N+1} a_j^{(1)} w_j}{\sum_{j=0}^{N+1} a_j^{(0)} w_j}, ..., \frac{\sum_{j=0}^{N+1} a_j^{(N+1)} w_j}{\sum_{j=0}^{N+1} a_j^{(0)} w_j}\right), \quad where\ w_j = \frac{z_j}{z_0}. \tag{5.25}$$

We denote $A \in GL^Q(\mathbb{C}^{N+2})$ if A satisfies $A(Q) \subseteq Q$ where we regard A as a linear transformation of $\mathbb{C}^{N+2}$. If $A \in GL^Q(\mathbb{C}^{N+2})$, we must have $A^\star(\partial\mathbb{H}^{N+1}) \subseteq \partial\mathbb{H}^{N+1}$, so that $A^\star \in Aut(\partial\mathbb{H}^{N+1})$. Conversely, if $A^\star \in Aut(\partial\mathbb{H}^{N+1})$, then $A \in GL^Q(\mathbb{C}^{N+2})$.

We define a bundle map:

$$\begin{array}{rrcl} \pi: & GL(\mathbb{C}^{N+2}) & \to & \mathbb{CP}^{N+1} \\ & A = (a_0, a_1, ..., a_{N+1}) & \mapsto & \pi_0(a_0). \end{array}$$

Then by (5.24), for any map $A \in GL(\mathbb{C}^{N+2})$, $A \in \pi^{-1}\big(\pi_0(a_0)\big) \iff A^\star([1:0:...:0]) = \pi_0(a_0)$. In particular, by the restriction, we consider a map

$$\begin{array}{rrcl} \pi: & GL^Q(\mathbb{C}^{N+2}) & \to & \partial\mathbb{H}^{N+1} \\ & A = (a_0, a_1, ..., a_{N+1}) & \mapsto & \pi_0(a_0). \end{array} \tag{5.26}$$

We get $\partial\mathbb{H}^{N+1} \simeq GL^Q(\mathbb{C}^{N+2})/P_1$ where P_1 is the isotropy subgroup of $GL^Q(\mathbb{C}^{N+2})$. Then by (5.24), for any map $A \in GL^Q(\mathbb{C}^{n+2})$,

$$A \in \pi^{-1}\big(\pi_0(a_0)\big) \iff A^\star([1:0:...:0]) = \pi_0(a_0). \tag{5.27}$$

CR submanifolds of $\partial\mathbb{H}^{N+1}$ Let $H : M' \to \partial\mathbb{H}^{N+1}$ be a CR smooth embedding where M' is a strictly pseudoconvex smooth real hypersurface in $\mathbb{C}^{n+1}$. We denote $M = H(M')$.

Let $R_{M'}$ be the Reeb vector field of M' with respect to a fixed contact form on M'. Then the real vector $R_{M'}$ generates a real line bundle over M', denoted by $\mathcal{R}_{M'}$. Since we can regard the rank n complex vector bundle $T^{1,0}M'$ as the rank $2n$ real vector bundle, over the real number field $\mathbb{R}$ we have:

$$TM' = T^cM' \oplus \mathcal{R}_{M'} \simeq T^{1,0}M' \oplus \mathcal{R}_{M'}. \tag{5.28}$$

given by

$$(a_j\frac{\partial}{\partial x_j}, b_j\frac{\partial}{\partial y_j}) + cR_{M'} \mapsto (a_j + ib_j)\frac{\partial}{\partial z_j} + cR_{M'}, \quad \forall a_j, b_j, c \in \mathbb{R}. \tag{5.29}$$

Since H is a CR embedding, we have

$$H_*(T^{1,0}M') = T^{1,0}M \subset T^{1,0}(\partial\mathbb{H}^{N+1}), TM \simeq H_*(T^{1,0}M') \oplus H_*(\mathcal{R}_{M'}) \subset T(\partial\mathbb{H}^{N+1}). \tag{5.30}$$

Lifts of the CR submanifolds Let $M = H(M') \subset \partial\mathbb{H}^{N+1}$ be as above. Consider the commutative diagram

$$\begin{array}{ccc} & & GL^Q(\mathbb{C}^{N+2}) \\ & e \nearrow & \downarrow \pi \\ M & \hookrightarrow & \partial\mathbb{H}^{N+1} \end{array}$$

Any map e satisfying $\pi \circ e = Id$ is called a *lift* of M to $GL^Q(\mathbb{C}^{N+2})$.

In order to define a more specific lifts, we need to give some relationship between geometry on $\partial\mathbb{H}^{N+1}$ and on $\mathbb{C}^{N+2}$ as follows. For any subset $X \in \partial\mathbb{H}^{N+1}$, we denote $\hat{X} := \pi_0^{-1}(X)$ where $\pi_0 : \mathbb{C}^{N+2} - \{0\} \to \mathbb{CP}^{N+1}$ is the standard projection map (5.22). In particular, for any $x \in M$, $\hat{x}$ is a complex line and for the real submanifold M^{2n+1}, the real submanifold $\hat{M}^{2n+3}$ is of dimension $2n+3$.

For any $x \in M$, we take $v \in \hat{x} = \pi_0^{-1}(x) \subset \mathbb{C}^{N+2} - \{0\}$, and we define

$$\hat{T}_x M = T_v \hat{M}, \quad \hat{T}_x^{1,0} M = T_v^{1,0} \hat{M}, \quad \hat{\mathcal{R}}_{M,x} := \mathcal{R}_{\hat{M},v}$$

where $\mathcal{R}_{\hat{M}} = \cup_{v \in \hat{M}} \mathcal{R}_{\hat{M},v}$. These definitions are independent of choice of v.

A lift $e = (e_0, e_\alpha, e_\mu, e_{N+1})$ of M into $GL^Q(\mathbb{C}^{N+2})$, where $1 \leq \alpha \leq n$ and $n+1 \leq \mu \leq N$, is called a *first-order adapted lift* if it satisfies the conditions:

$$e_0(x) \in \pi_0^{-1}(x), \quad span_{\mathbb{C}}(e_0, e_\alpha)(x) = \hat{T}_x^{1,0} M, \quad span(e_0, e_\alpha, e_{N+1})(x) = \hat{T}_x^{1,0} M \oplus \hat{\mathcal{R}}_{M,x} \tag{5.31}$$

where

$$span(e_0, e_\alpha, e_{N+1})(x) := \{c_0 e_0 + c_\alpha e_\alpha + c_{N+1} e_{N+1} \mid c_0, c_\alpha \in \mathbb{C},\ c_{N+1} \in \mathbb{R}\}. \tag{5.32}$$

Here we used (5.29) and the fact that the Reeb vector is real. Locally first-order adapted lifts always exist.

We have the restriction bundle $\mathcal{F}_M^0 := GL^Q(\mathbb{C}^{N+2})|_M$ over M. The subbundle $\pi : \mathcal{F}_M^1 \to M$ of $\mathcal{F}_M^0$ is defined by

$$\mathcal{F}_M^1 = \{(e_0, e_j, e_\mu, e_{N+1}) \in \mathcal{F}_M^0 \mid \ [e_0] \in M,\ (5.31)\ are\ satisfied\}.$$

Local sections of $\mathcal{F}_M^1$ are exactly all local first-order adapted lifts of M.

For two first-order adapted lifts $s = (e_0, e_j, e_\mu, e_{N+1})$ and $\widetilde{s} = (\widetilde{e}_0, \widetilde{e}_j, \widetilde{e}_\mu, \widetilde{e}_{N+1})$, by (5.31), we have

$$\begin{cases} \widetilde{e}_0 = g_0^0 e_0, \\ \widetilde{e}_j = g_j^0 e_0 + g_j^k e_k, \\ \widetilde{e}_\mu = g_\mu^0 e_0 + g_\mu^j e_j + g_\mu^\nu e_\nu + g_\mu^{N+1} e_{N+1}, \\ \widetilde{e}_{N+1} = g_{N+1}^0 e_0 + g_{N+1}^j e_j + g_{N+1}^{N+1} e_{N+1}, \end{cases} \tag{5.33}$$

Notice that by (5.29), g_{N+1}^{N+1} is some real-valued function, while other are complex-valued functions. In other words, $\widetilde{s} = s \cdot g$ where

$$g = (g_0, g_j, g_\mu, g_{N+1}) = \begin{pmatrix} g_0^0 & g_k^0 & g_\mu^0 & g_{N+1}^0 \\ 0 & g_k^j & g_\mu^j & g_{N+1}^j \\ 0 & 0 & g_\mu^\nu & 0 \\ 0 & 0 & g_\mu^{N+1} & g_{N+1}^{N+1} \end{pmatrix} \tag{5.34}$$

is a smooth map from M into $GL^Q(\mathbb{C}^{N+2})$. Then the fiber of $\pi : \mathcal{F}_M^1 \to M$ over a point is isomorphic to the group

$$G_1 = \left\{ g = \begin{pmatrix} g_0^0 & g_\beta^0 & g_\mu^0 & g_{N+1}^0 \\ 0 & g_\beta^\alpha & g_\mu^\alpha & g_{N+1}^\alpha \\ 0 & 0 & g_\mu^\nu & 0 \\ 0 & 0 & g_\mu^{N+1} & g_{N+1}^{N+1} \end{pmatrix} \in GL^Q(\mathbb{C}^{N+2}) \right\},$$

where we use the index ranges $1 \le \alpha, \beta \le n$ and $n+1 \le \mu, \nu \le N$.

We pull back the Maurer-Cartan form from $GL^Q(\mathbb{C}^{N+2})$ to $\mathcal{F}_M^1$ by a first-order adapted lift e of M as

$$\omega = \begin{pmatrix} \omega_0^0 & \omega_\beta^0 & \omega_\nu^0 & \omega_{N+1}^0 \\ \omega_0^\alpha & \omega_\beta^\alpha & \omega_\nu^\alpha & \omega_{N+1}^\alpha \\ \omega_0^\mu & \omega_\beta^\mu & \omega_\nu^\mu & \omega_{N+1}^\mu \\ \omega_0^{N+1} & \omega_\beta^{N+1} & \omega_\nu^{N+1} & \omega_{N+1}^{N+1} \end{pmatrix}.$$

Since $\omega = e^{-1}de$, i.e., $e\omega = de$. Then we have

$$de_0 = e_0\omega_0^0 + e_\alpha\omega_0^\alpha + e_\mu\omega_0^\mu + e_{N+1}\omega_0^{N+1}. \tag{5.35}$$

On the other hand, bu considering tangent vectors, we have

$$de_0 = e_0\omega_0^0 + e_\alpha\omega_0^\alpha + e_{N+1}\omega_0^{N+1}. \tag{5.36}$$

By (5.35) and (5.36), we conclude $\omega_0^\mu = 0$, $\forall \mu$. By the Maurer-Cartan equation $d\omega = -\omega \wedge \omega$, one gets $0 = d\omega_0^\nu = -\omega_\alpha^\nu \wedge \omega_0^\alpha - \omega_{N+1}^\nu \wedge \omega_0^{N+1}$, i.e., $0 = -\omega_\alpha^\nu \wedge \omega_0^\alpha,\ mod(\omega_0^{N+1})$. Then by Cartan's lemma,

$$\omega_\beta^\nu = q_{\alpha\beta}^\nu \omega_0^\alpha \quad mod(\omega_0^{N+1}),$$

for some functions $q_{\alpha\beta}^\nu = q_{\beta\alpha}^\nu$.

The CR second fundamental form In order to define the CR second fundamental form $II_M = II_M^s = q_{\alpha\beta}^\mu \omega_0^\alpha \omega_0^\beta \otimes \underline{e}_\mu$, mod$(\omega_0^{N+1})$, let us define $\underline{e}_\mu$ as follows.

For any first-order adapted lift $e = (e_0, e_\alpha, e_\nu, e_{N+1})$ with $\pi_0(e_0) = x$, we have $e_\alpha \in \hat{T}^{1,0}_x M$. Recall $T_E G(k,V) \simeq E^* \otimes (V/E)$ where $G(k,V)$ is the Grassmannian of k-planes that pass through the origin in a vector space V over $\mathbb{R}$ or $\mathbb{C}$ and $E \in G(k,V)$ ([IL03], p.73). Then $T_x M \simeq (\hat{x})^* \otimes (\hat{T}_x M/\hat{x})$ and hence the vector e_α induces $\underline{e_\alpha} \in T^{1,0}_x M$ by

$$\underline{e}_\alpha = e^0 \otimes \big(e_\alpha \ mod(e_0)\big),$$

where we denote by $(e^0, e^\alpha, e^\mu, e^{N+1})$ the dual basis of $(\mathbb{C}^{N+2})^*$. Similarly, we let

$$\underline{e}_\mu = e^0 \otimes \big(e_\mu \ mod \ \hat{T}^{(1,0)}_x M\big) \in N^{1,0}_x M, \tag{5.37}$$

where $N^{1,0}M$ is the CR normal bundle of M defined by $N^{1,0}_x M = T^{1,0}_x(\partial\mathbb{H}^{N+1})/T^{1,0}_x M$.

By direct computation, we obtain a tensor

$$II_M = II^e_M = q^\mu_{\alpha\beta}\omega^\alpha_0\omega^\beta_0 \otimes \underline{e}_\mu \in \Gamma\big(M, S^2 T^{1,0*}_{\pi_0(e_0)} M \otimes N^{1,0}_{\pi_0(e_0)} M\big) \ \ mod(\omega^{N+1}_0). \tag{5.38}$$

The tensor II_M is called the *CR second fundamental form* of M.

5.6 Definition D, the CR Second Fundamental Form

Q-frames We consider the real hypersurface Q in $\mathbb{C}^{N+2}$ defined by the homogeneous equation

$$\langle Z, Z\rangle := \sum_A Z^A\overline{Z^A} + \frac{i}{2}(Z^{N+1}\overline{Z^0} - Z^0\overline{Z^{N+1}}) = 0, \tag{5.39}$$

where $Z = (Z^0, Z^A, Z^{N+1})^t \in \mathbb{C}^{N+2}$. This can be extended to the scalar product

$$\langle Z, Z'\rangle := \sum_A Z^A\overline{Z'^A} + \frac{i}{2}(Z^{N+1}\overline{Z'}^0 - Z^0\overline{Z'^{N+1}}), \tag{5.40}$$

for any $Z = (Z^0, Z^A, Z^{N+1})^t, Z' = (Z'^0, Z'^A, Z'^{N+1})^t \in \mathbb{C}^{N+2}$. This product has the properties: $\langle Z, Z'\rangle$ is linear in Z and anti-linear in Z'; $\overline{\langle Z, Z'\rangle} = \langle Z', Z\rangle$; and Q is defined by $\langle Z, Z\rangle = 0$.

Let $SU(N+1,1)$ be the group of unimodular linear transformations of $\mathbb{C}^{N+2}$ that leave the form $\langle Z, Z\rangle$ invariant (cf. [CM74]).

By a *Q-frame* is meant an element $E = (E_0, E_A,\ E_{N+1}) \in GL(\mathbb{C}^{N+2})$ satisfying (cf. [CM74, (1.10)])

$$\begin{cases} det(E) = 1, \\ \langle E_A, E_B\rangle = \delta_{AB},\ \langle E_0, E_{N+1}\rangle = -\langle E_{N+1}, E_0\rangle = -\frac{i}{2}, \end{cases} \tag{5.41}$$

while all other products are zero.

There is exactly one transformation of $SU(N+1,1)$ which maps a given Q-frame into another. By fixing one Q-frame as reference, the group $SU(N+1,1)$ can be identified with the space of all Q-frames. Then $SU(N+1,1) \subset GL^Q(\mathbb{C}^{N+1})$ is a subgroup with the composition operation.

We define a bundle map:

$$\begin{array}{rccc} \pi: & GL(\mathbb{C}^{N+2}) & \to & \mathbb{CP}^{N+1} \\ & A = (a_0, a_1, ..., a_{N+1}) & \mapsto & \pi_0(a_0) \end{array}.$$

By taking restriction, we have the projection

$$\pi : SU(N+1,1) \to \partial\mathbb{H}^{N+1},\ (Z_0, Z_A, Z_{N+1}) \mapsto span(Z_0). \tag{5.42}$$

which is called a *Q-frames bundle*. We get $\partial\mathbb{H}^{N+1} \simeq SU(N+1,1)/P_2$ where P_2 is the isotropy subgroup of $SU(N+1,1)$. $SU(N+1,1)$ acts on $\partial\mathbb{H}^{N+1}$ effectively.

The Maurer-Cartan Form over $SU(N+1,1)$ Consider $E = (E_0, E_A, E_{N+1}) \in SU(N+1,1)$ as a local lift. Then the *Maurer-Cartan form* Θ on $SU(N+1,1)$ is defined by $dE = (dE_0, dE_A, dE_{N+1}) = E\Theta$, or $\Theta = E^{-1} \cdot dE$, i.e.,

$$d\begin{pmatrix} E_0 & E_A & E_{N+1}\end{pmatrix} = \begin{pmatrix} E_0 & E_B & E_{N+1}\end{pmatrix}\begin{pmatrix} \Theta_0^0 & \Theta_A^0 & \Theta_{N+1}^0 \\ \Theta_0^B & \Theta_A^B & \Theta_{N+1}^B \\ \Theta_0^{N+1} & \Theta_A^{N+1} & \Theta_{N+1}^{N+1}\end{pmatrix}, \tag{5.43}$$

where Θ_A^B are 1-forms on $SU(N+1,1)$. By (5.41) and (5.43), the Maurer-Cartan form (Θ) satisfies

$$\begin{array}{l} \Theta_0^0 + \overline{\Theta_{N+1}^{N+1}} = 0,\ \Theta_0^{N+1} = \overline{\Theta_0^{N+1}},\ \Theta_{N+1}^0 = \overline{\Theta_{N+1}^0}, \\ \Theta_A^{N+1} = 2i\overline{\Theta_0^A},\ \Theta_{N+1}^A = -\frac{i}{2}\overline{\Theta_A^0},\ \Theta_B^A + \overline{\Theta_A^B} = 0,\ \Theta_0^0 + \Theta_A^A + \Theta_{N+1}^{N+1} = 0, \end{array} \tag{5.44}$$

where $1 \le A \le N$. For example, from $\langle E_A, E_B\rangle = \delta_{AB}$, by taking differentiation, we obtain

$$\langle dE_A, E_B\rangle + \langle E_A, dE_B\rangle = 0.$$

By (5.43), we have

$$\begin{cases} dE_0 = E_0\Theta_0^0 + E_B\Theta_0^B + E_{N+1}\Theta_0^{N+1}, \\ dE_A = E_0\Theta_A^0 + E_B\Theta_A^B + E_{N+1}\Theta_A^{N+1}, \\ dE_{N+1} = E_0\Theta_{N+1}^0 + E_B\Theta_{N+1}^B + E_{N+1}\Theta_{N+1}^{N+1}. \end{cases}$$

Then

$$\langle E_0\Theta_A^0 + E_C\Theta_A^C + E_{N+1}\Theta_A^{N+1},\ E_B\rangle + \langle E_A, E_0\Theta_B^0 + E_D\Theta_B^D + E_{N+1}\Theta_B^{N+1}\rangle = 0,$$

which implies $\Theta_A^B + \overline{\Theta_B^A} = 0$. In particular, from (5.44), $\Theta_A^0 = -2i\overline{\Theta_{N+1}^A}$. Θ satisfies

$$d\Theta = -\Theta \wedge \Theta. \tag{5.45}$$

Let $M \hookrightarrow \partial\mathbb{H}^{N+1}$ be the image of $H : M' \to \partial\mathbb{H}^{N+1}$ where $M' \subset \mathbb{C}^{n+1}$ is a CR strictly pseudoconvex smooth hypersurface. Consider the inclusion map $M \hookrightarrow \partial\mathbb{H}^{N+1}$ and a lift $e = (e_0, e_1, ..., e_{N+1}) = (e_0, e_\alpha, e_\nu, e_{N+1})$ of M where $1 \le \alpha \le n$ and $n+1 \le \nu \le N$

$$\begin{array}{ccc} & & SU(N+1,1) \\ & e \nearrow & \downarrow \pi \\ M & \hookrightarrow & \partial\mathbb{H}^{N+1} \end{array}$$

We call e a *first-order adapted lift* if for any $x \in M$,

$$\pi_0\big(e_0(x)\big) = x,\ span_{\mathbb{C}}(e_0, e_\alpha)(x) = \hat{T}_x^{1,0}M,\ span(e_0, e_\alpha, e_{N+1})(x) = \hat{T}_x^{1,0}M \oplus \hat{\mathcal{R}}_{M,x}. \tag{5.46}$$

Locally first-order adapted lifts always exist. We have the restriction bundle $\mathcal{F}_M^0 := SU(N+1,1)|_M$ over M. The subbundle $\pi : \mathcal{F}_M^1 \to M$ of $\mathcal{F}_M^0$ is defined by

$$\mathcal{F}_M^1 = \{(e_0, e_j, e_\mu, e_{N+1}) \in \mathcal{F}_M^0 \mid\ [e_0] \in M,\ (5.46)\ are\ satisfied\}.$$

Local sections of $\mathcal{F}_M^1$ are exactly all local first-order adapted lifts of M. The fiber of $\pi : \mathcal{F}_M^1 \to M$ over a point is isomorphic to the group

$$G_1 = \left\{ g = \begin{pmatrix} g_0^0 & g_\beta^0 & g_\nu^0 & g_{N+1}^0 \\ 0 & g_\beta^\alpha & g_\nu^\alpha & g_{N+1}^\alpha \\ 0 & 0 & g_\nu^\mu & 0 \\ 0 & 0 & 0 & g_{N+1}^{N+1} \end{pmatrix} \in SU(N+1,1) \right\},$$

where we use the index ranges $1 \le \alpha, \beta \le n$ and $n+1 \le \mu, \nu \le N$.

By the remark below (5.33), g_{N+1}^{N+1} is real-valued. By (5.41), we have $\langle g_0, g_{N+1}\rangle = -\frac{i}{2}$, it implies $g_0^0 \cdot \overline{g_{N+1}^{N+1}} = 1$. In particular, both g_{N+1}^{N+1} and g_0^0 are real. Since $\langle g_0, g_\mu\rangle = 0$ and $g_0^0 \ne 0$, it implies $g_\mu^{N+1} = 0$. Since $\langle g_\alpha, g_\beta\rangle = \delta_{\alpha\beta}$, it implies that the matrix (g_α^β) is unitary. Since $deg(g) = 1$, it implies $g_0^0 \cdot det(g_\alpha^\beta) \cdot det(g_\mu^\nu) \cdot g_{N+1}^{N+1} = 1$, i.e., $det(g_\alpha^\beta) \cdot det(g_\mu^\nu) = 1$.

By considering all first-order adapted lifts from M into $SU(N+1,1)$, as the definition of II_M in Definition 3, we can defined CR second fundamental form II_M as in (5.38):

$$II_M = II^e_M = q^\mu_{\alpha\beta}\omega^\alpha_0\omega^\beta_0 \otimes \underline{e}_\mu \in \Gamma(M, S^2T^{1,0*}_{\pi_0(e_0)}M \otimes N^{1,0}_{\pi_0(e_0)}M), \quad mod(\omega^{N+1}_0), \tag{5.47}$$

which is a well-defined tensor, and is called the *CR second fundamental form* of M.

We remark that the notion of II_M in Definition 4 was introduced in a paper by S.H. Wang [Wa06].

5.7 Geometric Rank And The Second Fundamental Form

Geometric Rank and II_M

Lemma 5.7.1 *(i) ([JY09], theorem 7.1) Let $F \in Prop_k(\partial\mathbb{H}^{n+1}, \partial\mathbb{H}^{N+1})$ with $k \geq 2$ and $F(0) = 0$. Then there exists a neighborhood of 0 in $M := F(\partial\mathbb{H}^{n+1})$ and a C^{k-1}-smooth first-order adapted lift $e : U \to SU(N+1,1)$*

$$e = (e_0, e_j, e_b, e_{N+1}) \in SU(N+1,1), \quad 1 \leq j \leq n, \; n+1 \leq b \leq N-1. \tag{5.48}$$

(ii) ([JY09], Step 3 of the proof of Theorem 1.1) Let $F = F^{***} = (f, \phi, g)$, the induced first-order adapted lift s, and notation be as in Theorem 5.7.1. Then

$$h^\mu_{j,k}|_0 = \frac{\partial^2\phi_\mu}{\partial z_j \partial z_k}\Big|_0, \quad j,k \in \{1,2,...,n,N+1\} \tag{5.49}$$

where $II_M = h^\mu_{jk}\omega^j\omega^k \otimes \underline{e_\mu}$ is the CR second fundamental form.

Theorem 5.7.2 *[HJ09] Let $F \in Prop_2(\partial\mathbb{H}^{n+1}, \partial\mathbb{H}^{N+1})$. Then its geometric rank κ_0 equals to*

$$\kappa_0 = \sup_{p \in \partial\mathbb{H}^{n+1}} \left[n - \dim_{\mathbb{C}}\{\nu \mid II_{M,F(p)}(\nu,\nu) = 0\} \right]$$

where $II_{M,F(p)}$ is the CR second fundamental form of the submanifold M at the point $F(p)$. Here $\{\nu \mid II_{X,F(p)}(\nu,\nu) = 0\}$ is a vector space over $\mathbb{C}$.

Corollary 5.7.3 *Let $F \in Prop_2(\mathbb{H}^n, \mathbb{H}^N)$. Then*

$$\kappa_0 = 0 \iff II_M = 0.$$

Going back to Theorem 5.2.1. We have a lemma:

Lemma 5.7.4 *Let $H : M' \to \partial\mathbb{H}^{N+1}$ be a CR smooth embedding where M' is a strictly pseudoconvex smooth real hypersurface in $\mathbb{C}^{n+1}$. We denote $M = H(M')$. Then the following statements are equivalent:*
(i) The CR second fundamental form II_M by Definition A identically vanishes.
(ii) The CR second fundamental form II_M by Definition B identically vanishes.
(iii) The CR second fundamental form II_M by Definition C identically vanishes.
(iv) The CR second fundamental form II_M by Definition D identically vanishes.

Lemma 5.7.5 *(cf. [EHZ04], corollary 5.5) Let $H : M' \to M \hookrightarrow \partial\mathbb{H}^{N+1}$ be a smooth CR embedding of a strictly pseudoconvex smooth real hypersurface $M \subset \mathbb{C}^{n+1}$. Denote by $(\omega_{\alpha\ \beta}^{\ \mu})$ the CR second fundamental form matrix of H relative to an admissible coframe (θ, θ^A) on $\partial\mathbb{H}^{N+1}$ adapted to M. If $\omega_{\alpha\ \beta}^{\ \mu} \equiv 0$ for all α, β and μ, then M' is locally CR-equivalent to $\partial\mathbb{H}^{n+1}$.*

To prove Theorem 5.2.1, we apply Lemma 5.7.4 and Lemma 5.7.5 and the hypothesis that the CR second fundamental form identically vanishes to know that M is locally CR equivalent to $\partial\mathbb{H}^{n+1}$.

Then M is the image of a local smooth CR map $F : U \subset \partial\mathbb{H}^{n+1} \to M \subset \partial\mathbb{H}^{N+1}$ where U is a open set in $\partial\mathbb{H}^{n+1}$. By a result of Forstneric[Fo89], the map F must be a rational map. It suffices to prove that F is equivalent to a linear map. By the fact that F is linear if and only if its geometric rank is zero, it is sufficient to prove that the geometric rank of F is zero: $\kappa_0 = 0$. This can be done by applying Theorem 5.7.2.

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