Flatness of CR Submanifolds in a Sphere

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Dedicated to Professor Yang, Lo in the Occasion of his 70th Birthday

1 Introduction

The Cartan-Janet theorem asserted that for any analytic Riemannian manifold (M^n, g), there exist local isometric embeddings of M^n into Euclidean space $\mathbb{E}^N$ as N is sufficiently large. The CR analogue of Cartan-Janet theorem is not true in general. In fact, Forstneric [F086] and Faran [Fa88] proved the existence of real analytic strictly pseudoconvex hypersurfaces $M^{2n+1} \subset \mathbb{C}^{n+1}$ which do not admit any germ of holomorphic mapping taking M into sphere $\partial\mathbb{B}^{N+1}$ for any N.

There are recent progress on CR submanifolds in sphere $\partial\mathbb{B}^{N+1}$. Zaitsev [Za08] constructed explicit examples for the Forstneric and Faran phenomenon above. Ebenfelt, Huang and Zaitsev [EHZ04] proved rigidity of CR embeddings of general M^{2n+1} into spheres with CR co-dimension $< \frac{n}{2}$, which generalizes a result of Webster [We79] for the case of co-dimension one. S.-Y. Kim and J.-W. Oh [KO06] gave a necessary and sufficient condition for local embeddability into a sphere $\partial\mathbb{B}^{N+1}$ of a generic strictly pseudoconvex psuedohermitian CR manifold (M^{2n+1}, θ) in terms of its Chern-Moser curvature tensors and their derivatives.

In Euclidean geometry, for a real submanifold $M^n \subset \mathbb{E}^{n+a}$, M is a piece of $\mathbb{E}^n$ if and only if its second fundamental form $II_M \equiv 0$. In projective geometry, for a complex submanifold $M^n \subset \mathbb{CP}^{n+a}$, M is a piece of $\mathbb{CP}^n$ if and only if its projective second fundamental form $II_M \equiv 0$ (c.f. [IL03], p.81). In CR geometry, we prove the CR analogue of this fact in this paper as follows:

Theorem 1.1 *Let $H : M' \to \partial\mathbb{B}^{N+1}$ be a smooth CR-embedding of a strictly pseudoconvex CR real hypersurface $M' \subset \mathbb{C}^{n+1}$. Denote $M := H(M')$. If its CR second fundamental* *form $II_M \equiv 0$, then $M \subset F(\partial\mathbb{B}^{n+1}) \subset \partial\mathbb{B}^{N+1}$ where $F : \mathbb{B}^{n+1} \to \mathbb{B}^{N+1}$ is a certain linear fractional proper holomorphic map.*

Previously, it was proved by P. Ebenfelt, X. Huang and D. Zaitsev ([EHZ04], corollary 5.5), under the above same hypothese, that M' and hence M are locally CR-equivalent to the unit sphere $\partial\mathbb{B}^{n+1}$ in $\mathbb{C}^{n+1}$.

There are several definitions of the CR second fundamental forms II_M of M (see Section 3, 4, 5, and 6). The result in [EHZ04] used Definition 1 or 2. However, to prove Theorem 1.1, we need to use Definitions 3 and 4. We'll prove in Section 4 that $II_M \equiv 0$ by any one of the four definitions will imply $II_M \equiv 0$ for all other three definitions. One of the ingredients for our proof of Theorem 1.1 is the result of Ebenfelt-Huang-Zaitsev [EHZ04] so that M can be regarded as the image of a rational CR map $F : \partial\mathbb{H}^{n+1} \to M \subset \partial\mathbb{H}^{N+1}$. Another ingredient is a theorem of Huang ([Hu99]) that such a map F is linear if and only if its geometric rank κ_0 is zero. The idea about special lifts for maps between spheres was also used in [HJY09].

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2 Preliminaries

• **Maps between balls** We denote by $Prop(\mathbb{B}^n, \mathbb{B}^N)$ the space of all proper holomorphic maps from the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ to $\mathbb{B}^N$, denote by $Prop_k(\mathbb{B}^n, \mathbb{B}^N)$ the space $Prop(\mathbb{B}^n, \mathbb{B}^N) \cap C^k(\overline{\mathbb{B}^n})$, and denote by $Rat(\mathbb{B}^n, \mathbb{B}^N)$ the space $Prop(\mathbb{B}^n, \mathbb{B}^N) \cap \{rational\ maps\}$. We say that F and $G \in Prop(\mathbb{B}^n, \mathbb{B}^N)$ are *equivalent* if there are automorphisms $\sigma \in Aut(\mathbb{B}^n)$ and $\tau \in Aut(\mathbb{B}^N)$ such that $F = \tau \circ G \circ \sigma$.

Write $\mathbb{H}^n := \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \ \mathrm{Im}(w) > |z|^2\}$ for the Siegel upper-half space. Similarly, we can define the space $Prop(\mathbb{H}^n, \mathbb{H}^N)$, $Prop_k(\mathbb{H}^n, \mathbb{H}^N)$ and $Rat(\mathbb{H}^n, \mathbb{H}^N)$ similarly. By the Cayley transformation $\rho_n : \mathbb{H}^n \to \mathbb{B}^n$, $\rho_n(z, w) = (\frac{2z}{1-iw}, \ \frac{1+iw}{1-iw})$, we can identify a map $F \in Prop_k(\mathbb{B}^n, \mathbb{B}^N)$ or $Rat(\mathbb{B}^n, \mathbb{B}^N)$ with $\rho_N^{-1} \circ F \circ \rho_n$ in the space $Prop_k(\mathbb{H}^n, \mathbb{H}^N)$ or $Rat(\mathbb{H}^n, \mathbb{H}^N)$, respectively. We say that F and $G \in Prop(\mathbb{H}^n, \mathbb{H}^N)$ are *equivalent* if there are automorphisms $\sigma \in Aut(\mathbb{H}^n)$ and $\tau \in Aut(\mathbb{H}^N)$ such that $F = \tau \circ G \circ \sigma$.

We denote by $\partial\mathbb{H}^n = \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \ \mathrm{Im}(w) = |z|^2\}$ for the Heisenberg hypersurface. For any map $F \in Prop_2(\mathbb{H}^n, \mathbb{H}^N)$, by restricting on $\partial\mathbb{H}^n$, we can regard F as a C^2 CR map from $\partial\mathbb{H}^n$ to $\partial\mathbb{H}^N$.

We can parametrize $\partial\mathbb{H}^n$ by $(z, \overline{z}, u)$ through the map $(z, \overline{z}, u) \to (z, u + i|z|^2)$. In what follows, we will assign the weight of z and u to be 1 and 2, respectively. For a non-negative integer m, a function $h(z,\overline{z},u)$ defined over a small ball U of 0 in $\partial\mathbb{H}^n$ is said to be of quantity $o_{wt}(m)$ if $\frac{h(tz,t\overline{z},t^2u)}{|t|^m}\to 0$ uniformly for (z,u) on any compact subset of U as $t(\in\mathbb{R})\to 0$.

• **Partial normalization of** F Let $F=(f,\phi,g)=(\widetilde{f},g)=(f_1,\cdots,f_{n-1},\phi_1,\cdots,\phi_{N-n},g)$ be a non-constant map in $Prop_2(\mathbb{H}^n,\mathbb{H}^N)$ with $F(0)=0$. For each $p\in\partial\mathbb{H}^n$, we write $\sigma_p^0\in \mathrm{Aut}(\mathbb{H}^n)$ with $\sigma_p^0(0)=p$ and $\tau_p^F\in\mathrm{Aut}(\mathbb{H}^N)$ with $\tau_p^F(F(p))=0$ for the maps

$$\sigma_p^0(z,w)=(z+z_0,w+w_0+2i\langle z,\overline{z_0}\rangle), \tag{1}$$

$$\tau_p^F(z^*,w^*)=(z^*-\widetilde{f}(z_0,w_0),w^*-\overline{g(z_0,w_0)}-2i\langle z^*,\overline{\widetilde{f}(z_0,w_0)}\rangle). \tag{2}$$

F is equivalent to $F_p=\tau_p^F\circ F\circ\sigma_p^0=(f_p,\phi_p,g_p)$. Notice that $F_0=F$ and $F_p(0)=0$. The following is basic for the understanding of the geometric properties of F.

Lemma 2.1 *(\[§2, Lemma 5.3, Hu99\], \[Lemma 2.0, Hu03\]): Let F be a non-constant map in $Prop_2(\mathbb{H}^n,\mathbb{H}^N)$, $2\le n\le N$ with $F(0)=0$. For each $p\in\partial\mathbb{H}^n$, there is an automorphism $\tau_p^{**}\in Aut_0(\mathbb{H}^N)$ such that $F_p^{**}:=\tau_p^{**}\circ F_p$ satisfies the following normalization:*

$$f_p^{**}=z+\frac{i}{2}a_p^{**(1)}(z)w+o_{wt}(3),\ \ \phi_p^{**}=\phi_p^{**(2)}(z)+o_{wt}(2),\ \ g_p^{**}=w+o_{wt}(4), \tag{3}$$

$$\langle\overline{z},a_p^{**(1)}(z)\rangle|z|^2=|\phi_p^{**(2)}(z)|^2.$$

Let $\mathcal{A}(p)=-2i(\frac{\partial^2(f_p)_l^{**}}{\partial z_j\partial w}|_0)_{1\le j,l\le n-1}$. We call the rank of $\mathcal{A}(p)$, which we denote by $Rk_F(p)$, the *geometric rank* of F at p. $Rk_F(p)$ depends only on p and F, and is a lower semi-continuous function on p. We define the *geometric rank* of F to be $\kappa_0(F)=max_{p\in\partial\mathbb{H}^n}Rk_F(p)$. Notice that we always have $0\le\kappa_0\le n-1$. We define the geometric rank of $F\in\mathrm{Prop}_2(\mathbb{B}^n,\mathbb{B}^N)$ to be the one for the map $\rho_N^{-1}\circ F\circ\rho_n\in\mathrm{Prop}_2(\mathbb{H}^n,\mathbb{H}^N)$.

Lemma 2.2 *(ct. \[Hu99\], theorem 4.3) $F\in Prop_2(\mathbb{B}^n,\mathbb{B}^N)$ has geometric rank* 0 *if and only if F is equivalent to a linear map.*

Denote by $\mathcal{S}_0=\{(j,l):1\le j\le\kappa_0,1\le l\le(n-1),j\le l\}$ and write $\mathcal{S}:=\{(j,l):(j,l)\in\mathcal{S}_0,\ \text{or}\ \ j=\kappa_0+1,l\in\{\kappa_0+1,\cdots,\kappa_0+N-n-\frac{(2n-\kappa_0-1)\kappa_0}{2}\}\}$.

Lemma 2.3 *([Lemma 3.2, Hu03]): Let F be a C^2-smooth CR map from an open piece $M \subset \partial\mathbb{H}^n$ into $\partial\mathbb{H}^N$ with $F(0)=0$ and $Rk_F(0)=\kappa_0$. Let $P(n,\kappa_0)=\frac{\kappa_0(2n-\kappa_0-1)}{2}$. Then $N \geq n+P(n,\kappa_0)$ and there are $\sigma \in Aut_0(\partial\mathbb{H}^n)$ and $\tau \in Aut_0(\partial\mathbb{H}^N)$ such that $F_p^{***} = \tau \circ F \circ \sigma := (f,\phi,g)$ satisfies the following normalization conditions:*

$$\begin{cases} f_j = z_j + \dfrac{i\mu_j}{2} z_j w + o_{wt}(3), \quad \dfrac{\partial^2 f_j}{\partial w^2}(0)=0, \ j=1\cdots,\kappa_0, \ \mu_j>0, \\ f_j = z_j + o_{wt}(3), \quad j=\kappa_0+1,\cdots,n-1 \\ g = w + o_{wt}(4), \\ \phi_{jl} = \mu_{jl} z_j z_l + o_{wt}(2), \ \text{where} \ (j,l)\in\mathcal{S} \ \text{with} \ \mu_{jl}>0 \ \text{for} \ (j,l)\in\mathcal{S}_0 \\ \qquad \text{and} \ \mu_{jl}=0 \ \text{otherwise} \end{cases} \tag{4}$$

where $\mu_{jl}=\sqrt{\mu_j+\mu_l}$ for $j,l \leq \kappa_0$ $j \neq l$, $\mu_{jl}=\sqrt{\mu_j}$ if $j \leq \kappa_0$ and $l>\kappa_0$ or if $j=l\leq\kappa_0$.

• **Pseudohermitian metric and Webster connection** Let M be a C^2 smooth real hypersurface in $\mathbb{C}^{n+1}$. We denote by $T^cM = TM \cap iTM \subset TM$ its *maximal complex tangent bundle* with the complex structure $J: T^cM \to T^cM$. Here $J(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j}$ and $J(\frac{\partial}{\partial y_j}) = -\frac{\partial}{\partial x_j}$ in terms of holomorphic coordinates. We denote by $\mathcal{V} = T^{0,1}M = \{X + iJX \mid X \in T^cM\} \subset \mathbb{C}TM := TM \otimes \mathbb{C}$ the *CR bundle*. We also denote $T^{1,0}M = \overline{\mathcal{V}}$. All T^cM, $\mathcal{V}$ and $\overline{\mathcal{V}}$ are complex rank n vector bundles.

Write $T^0M := (T^{1,0}M \oplus T^{0,1}M)^\perp \subset \mathbb{C}T^*M$ for its rank one subbundle. Write $T'M := T^{0,1^\perp} \subset \mathbb{C}T^*M$ for its rank $n+1$ *holomorphic or (1,0) cotangent bundle* of M. Here $T^0 \subset T'M$.

A real nonvanishing 1-form θ over M is called a *contact form* if $\theta \wedge (d\theta)^n \neq 0$. Let M be as above given by a defining function r. Then the 1-form $\theta = i\partial r$ is a contact form of M.

We say that (M,θ) is *strictly pseudoconvex* if the Levi-form L_θ is positive definite for all $z \in M$. Here the *Levi-form* L_θ with respect to θ is defined by

$$L_\theta(\vec{u},\overline{\vec{v}}) := -id\theta(\vec{u} \wedge \overline{\vec{v}}), \quad \forall \vec{u},\vec{v} \in T_p^{1,0}(M), \ \forall p \in M.$$

Associated with a contact form θ one has the Reeb vector field R_θ, defined by the equations: (i) $d\theta(R_\theta,\cdot) \equiv 0$, (ii) $\theta(R_\theta) \equiv 1$. As a skew-symmetric form of maximal rank $2n$, the form $d\theta|_{T_pM}$ has a 1- dimensional kernel for each $p \in M^{2n+1}$. Hence equation (i) defines a unique line field $\langle R_\theta \rangle$ on M. The contact condition $\theta \wedge (d\theta)^n \neq 0$ implies that θ is non-trivial on that line field, so the unique real vector field is defined by the normalization condition (ii).

According Tanaka [T75] and Webester [We78], (M,θ) is called a *strictly pseudoconvex pseudohermitian manifold* if there are n complex 1-forms θ^α so that $\{\theta^1,...,\theta^n\}$ forms a local basis for holomorphic cotangent bundle $H^*(M)$ and

$$d\theta = i\sum_{\alpha,\beta=1}^{n} h_{\alpha\overline{\beta}}\theta^\alpha\wedge\theta^{\overline{\beta}} \tag{5}$$

where $(h_{\alpha\overline{\beta}})$, called the *Levi form matrix*, is positive definite. Such θ^α may not be unique. Following Webster [We78], a coframe (θ,θ^α) is called *admissible* if (5) holds. The admissible coframes are determined up to transformations $\widetilde{\theta}^\alpha = u^\alpha_\beta\theta^\beta$ where $(u^\alpha_\beta)\in GL(\mathbb{C}^n)$.

Theorem 2.4 *(Webster, [We78]) Let (M^{2n+1},θ) be a strictly pseudoconvex pseudohermitian manifold and let θ^j be as in (5). Then there are unique way to write*

$$d\theta^\alpha = \sum_{\gamma=1}^{n}\theta^\gamma\wedge\omega^\alpha_\gamma + \theta\wedge\tau^\alpha, \tag{6}$$

where τ^α are $(0,1)$-forms over M that are linear combination of $\theta^{\overline{\alpha}}=\overline{\theta^\alpha}$, and ω^β_α are 1-forms over M such that

$$0 = dh_{\alpha\overline{\beta}} - h_{\gamma\overline{\beta}}\omega^\gamma_\alpha - h_{\alpha\overline{\gamma}}\omega^{\overline{\gamma}}_{\overline{\beta}}. \tag{7}$$

We may denote $\omega_{\alpha\overline{\beta}} = h_{\gamma\overline{\beta}}\omega^\gamma_\alpha$ and $\overline{\omega_{\beta\overline{\alpha}}} = h_{\alpha\overline{\gamma}}\omega^{\overline{\gamma}}_{\overline{\beta}}$. In particular, if

$$h_{\alpha\beta} = \delta_{\alpha\beta}, \tag{8}$$

the identity in (7) becomes $0 = -\omega_{\alpha\overline{\beta}} - \overline{\omega_{\beta\overline{\alpha}}}$, i.e.,

$$0 = \omega^\beta_\alpha + \omega^{\overline{\alpha}}_{\overline{\beta}}. \tag{9}$$

The condition on τ^β means:

$$\tau^\beta = A^\beta_{\overline{\nu}}\theta^{\overline{\nu}},\quad A^{\alpha\beta} = A^{\beta\alpha}, \tag{10}$$

which holds automatically. The curvature is given by

$$d\omega_\alpha{}^\beta - \omega_\alpha{}^\gamma\wedge\omega_\gamma{}^\beta = R_\alpha{}^\beta{}_{\mu\overline{\nu}}\theta^\mu\wedge\theta^{\overline{\nu}} + W_\alpha{}^\beta{}_\mu\theta^\mu\wedge\theta - W^\beta{}_{\alpha\overline{\nu}}\theta^{\overline{\nu}}\wedge\theta + i\theta_\alpha\wedge\tau^\beta - i\tau_\alpha\wedge\theta^\beta \tag{11}$$

where the functions $R_\alpha{}^\beta{}_{\mu\overline{\nu}}$ and $W_\alpha{}^\beta{}_\mu$ represent the *pseudohermitian curvature* of (M,θ).

3 CR second fundamental forms —— Definition 1

We are going to survey four definitions of the CR second fundamental forms II_M of M in $\partial\mathbb{H}^{N+1}$. We start with Definition 1 which is the intrinsic one in terms of a coframe.

Lemma 3.1 *([EHZ04], corollary 4.2) Let M and $\widetilde{M}$ be strictly pseudoconvex CR-manifolds of dimensions $2n+1$ and $2\widetilde{n}+1$ respectively, and of CR dimensions n and $\widetilde{n}$ respectively. Let $F : M \to \widetilde{M}$ be a smooth CR-embedding. If (θ, θ^α) is a admissible coframe on M, then in a neighborhood of a point $\widetilde{p} \in F(M)$ in $\widetilde{M}$ there exists an admissible coframe $(\widetilde{\theta}, \widetilde{\theta}^A) = (\widetilde{\theta}, \widetilde{\theta}^\alpha, \widetilde{\theta}^\mu)$ on $\widetilde{M}$ with $F^*(\widetilde{\theta}, \widetilde{\theta}^\alpha, \widetilde{\theta}^\mu) = (\theta, \theta^\alpha, 0)$. In particular, the Reeb vector field $\widetilde{R}$ is tangent to $F(M)$. If we choose the Levi form matrix of M such that the functions $h_{\alpha\overline{\beta}}$ in (5) with respect to (θ, θ^α) to be $(\delta_{\alpha\overline{\beta}})$, then $(\widetilde{\theta}, \widetilde{\theta}^A)$ can be chosen such that the Levi form matrix of $\widetilde{M}$ relative to it is also $(\delta_{A\overline{B}})$. With this additional property, the coframe $(\widetilde{\theta}, \widetilde{\theta}^A)$ is uniquely determined along M up to unitary transformations in $U(n) \times U(\widetilde{n} - n)$.*

If (θ, θ^α) and $(\widetilde{\theta}, \widetilde{\theta}^A)$ are as above such that the condition on the Levi form matrices in Lemma 3.1 are satisfied, we say that the coframe $(\widetilde{\theta}, \widetilde{\theta}^A)$ is *adapted* to the coframe (θ, θ^α). In this case, by (9), we have $\theta = F^*\widetilde{\theta}$, $\theta^\alpha = F^*\widetilde{\theta}^\alpha$, and

$$d\theta^\alpha = \sum_{\gamma=1}^{n} \theta^\gamma \wedge \omega_\gamma^\alpha + \theta \wedge \tau^\alpha, \quad 0 = \ \omega_\alpha^\beta + \omega_{\overline{\beta}}^{\overline{\alpha}}, \quad \forall 1 \le \alpha, \beta \le n,$$

and

$$d\widetilde{\theta}^A = \sum_{B=1}^{\widetilde{n}} \widetilde{\theta}^C \wedge \widetilde{\omega}_C^A + \widetilde{\theta} \wedge \widetilde{\tau}^A, \quad 0 = \ \widetilde{\omega}_A^B + \widetilde{\omega}_{\overline{B}}^{\overline{A}}, \qquad \forall 1 \le A, B \le N.$$

For simplicity, we may denote $F^*\widetilde{\omega}_B^A$ by ω_B^A. We also denote $F^*\widetilde{\omega}_{A\overline{B}}$ by $\omega_{A\overline{B}}$ where $\omega_{A\overline{B}} = \omega_A^B$.

Write $\omega_\alpha{}^\mu = \omega_\alpha{}^\mu{}_\beta \theta^\beta$. The matrix of $(\omega_\alpha{}^\mu{}_\beta)$, $1 \le \alpha, \beta \le n$, $n+1 \le \mu \le \hat{n}$, defines the *CR second fundamental form* of M. It was used in [We79] and [Fa90].

4 CR second fundamental forms —— Definition 2

Definition 2 introduced in [EHZ04] is the extrinsic one in terms of defining function.

Let $F : M \to \widetilde{M}$ be a smooth CR-embedding between $M \subset \mathbb{C}^{n+1}$ and $\widetilde{M} \subset \mathbb{C}^{N+1}$ where M and $\widetilde{M}$ are real strictly pseudoconvex hypersurfaces of dimensions $2n+1$ and $2\widetilde{n}+1$, and

CR dimensions n and $\widetilde{n}$, respectively. Let $p \in M$ and $\widetilde{p} = F(p) \in \widetilde{M}$ be points. Let $\widetilde{\rho}$ be a local defining function for $\widetilde{M}$ near the point $\widetilde{p}$. Let

$$E_k(p) := span_{\mathbb{C}}\{L^{\bar{J}}(\widetilde{\rho}_{Z'} \circ F)(p) \mid J \in (Z_+)^n, 0 \leq |J| \leq k\} \subset T^{1,0}_{\widetilde{p}}\mathbb{C}^{N+1},$$

where $\widetilde{\rho}_{Z'} := \partial\widetilde{\rho}$ is the complex gradient (i.e., represented by vectors in $\mathbb{C}^{N+1}$ in some local coordinate system Z' near $\widetilde{p}$). Here we use multi-index notation $L^{\overline{J}} = L_{\overline{1}}^{J_1} \cdots L_{\overline{n}}^{J_n}$ and $|J| = J_1 + ... + J_n$. It was shown in [La01] that $E_k(p)$ is independent of the choice of local defining function $\widetilde{\rho}$, coordinates Z' and the choice of basis of the CR vector fields $L_{\overline{1}}, ..., L_{\overline{n}}$.

The *CR second fundamental form* II_M of M is defined by (cf. [EHZ04], §2)

$$II_M(X_p, Y_p) := \overline{\pi\big(XY(\widetilde{\rho}_{\overline{Z'}} \circ f)(p)\big)} \in \overline{T'_{\widetilde{p}}\widetilde{M}/E_1(p)} \tag{12}$$

where $\widetilde{\rho}_{\overline{Z'}} = \overline{\partial}\widetilde{\rho}$ is represented by vectors in $\mathbb{C}^{N+1}$ in some local coordinate system Z' near $\widetilde{p}$, X, Y are any $(1,0)$ vector fields on M extending given vectors $X_p, Y_p \in T_p^{1,0}(M)$, and $\pi : T'_{\widetilde{p}}\widetilde{M} \to T'_{\widetilde{p}}\widetilde{M}/E_1(p)$ is the projection map.

Since $\widetilde{M}$ and M are strictly pseudoconvex, the Levi form of $\widetilde{M}$ (at $\widetilde{p}$) with respect to $\widetilde{\rho}$ defines an isomorphism

$$\overline{T'_{\widetilde{p}}\widetilde{M}/E_1(p)} \cong T^{1,0}_{\widetilde{p}}\widetilde{M}/F_*(T^{1,0}_pM)$$

and the CR second fundamental form can be viewed as an $\mathbb{C}$-linear symmetric form

$$II_{M,p} : T^{1,0}_pM \times T^{1,0}_pM \to T^{1,0}_{\widetilde{p}}\widetilde{M}/F_*(T^{1,0}_pM) \tag{13}$$

that does not depend on the choice of $\widetilde{\rho}$ (cf.[EHZ04], §2).

The relation between Definition 1 and Definition 2 was discussed in [EHZ04]. Let $(M, \widetilde{M})$, $(\theta, \theta^\alpha), (\widetilde{\theta}, \widetilde{\theta}^A)$ be as in Lemma 3.1, and we abuse the structure bundle (θ, θ^α) on M with the structure bundle $(\widetilde{\theta}, \widetilde{\theta}^\alpha)$ on $\widetilde{M}$. We can choose a defining function $\widetilde{\rho}$ of $\widetilde{M}$ near a point $\widetilde{p} = F(p) \in \widetilde{M}$ where $p \in M$ such that $\theta = i\overline{\partial}\widetilde{\rho}$ on $\widetilde{M}$., i.e., in local coordinates Z' in $\mathbb{C}^{N+1}$, we have

$$\theta = i\sum_{k=1}^{N+1} \frac{\partial\widetilde{\rho}}{\partial\overline{Z'_k}} d\overline{Z'_k},$$

where we pull back the forms $d\overline{Z'_1}, ..., d\overline{Z'_{N+1}}$ to $\widetilde{M}$. Then we consider the coframe $(\theta, \theta^\alpha) = (F^*\widetilde{\theta}, F^*\widetilde{\theta}^\alpha)$ on M near p with $F(p) = \widetilde{p}$. We take its dual frame (T, L_A) of (θ, θ^A) and have

$$L_\beta(\widetilde{\rho}_{\overline{Z'}} \circ F) = -iL_\beta \lrcorner d\theta = g_{\beta\overline{C}}\theta^{\overline{C}} = g_{\beta\overline{\gamma}}\theta^{\overline{\gamma}}. \tag{14}$$

Here we used the definition of the construction, (5) and the dual relationship $\langle L_\beta, \theta^\alpha\rangle = \delta_\beta^\alpha$ and also notice that $g_{\beta\overline{\gamma}} = \delta_{\beta\gamma}$. Applying L_α to both sides of (14), we obtain

$$L_\alpha L_\beta(\widetilde{\rho}_{\overline{Z}'} \circ F) = g_{\beta\overline{\gamma}} L_\alpha \lrcorner d\theta^{\overline{\gamma}} = \omega_{\alpha\overline{\mu}\beta}\theta^{\overline{\mu}} \quad mod(\theta, \theta^{\overline{\alpha}})$$

which implies

$$II_M(L_\alpha, L_\beta) = \omega_{\alpha}{}^{\mu}{}_{\beta} L_\mu, \quad n+1 \leq \mu \leq N. \tag{15}$$

This identity gives the equivalent relation of the intrinsic and extrinsic definitions of II_M. Notice that we need a right choice of (θ, θ^α), (T, L_A) and $\widetilde{\rho}$.

By using $(\omega_{\alpha}{}^{b}{}_{\beta})$ and (15), as in (13), we can also define

$$II_{M,p} : T_p^{1,0}M \times T_p^{1,0}M \to T_{\widetilde{p}}^{1,0}\widetilde{M}/F_*(T_p^{1,0}M) \tag{16}$$

which is independent of the choice of the adapted coframe (θ, θ^A) in case $\widetilde{M}$ is locally CR embeddable in $\mathbb{C}^{N+1}$ (cf. [EHZ04], § 4).

5 CR second fundamental forms —— Definition 3

Definition 3 is the one as a tensor with respect to the group $GL^Q(\mathbb{C}^{N+2})$.

The bundle $GL^Q(\mathbb{C}^{N+2})$ over $\partial\mathbb{H}^{N+1}$ We consider a real hypersurface Q in $\mathbb{C}^{N+2}$ defined by the homogeneous equation

$$\langle Z, Z\rangle := \sum_A Z^A\overline{Z^A} + \frac{i}{2}(\overline{Z^0}Z^{N+1} - Z^0\overline{Z^{N+1}}) = 0, \tag{17}$$

where $Z = (Z^0, Z^A, Z^{N+1})^t \in \mathbb{C}^{N+2}$. Let

$$\pi_0 : \mathbb{C}^{N+2} - \{0\} \to \mathbb{CP}^{N+1}, \quad (z_0, ..., z_{N+1}) \mapsto [z_0 : ... : z_{N+1}], \tag{18}$$

be the standard projection. For any point $x \in \mathbb{CP}^{N+1}$, $\pi_0^{-1}(x)$ is a complex line in $\mathbb{C}^{N+2} - \{0\}$. For any point $v \in \mathbb{C}^{N+2} - \{0\}$, $\pi_0(v) \in \mathbb{CP}^{N+1}$ is a point. The image $\pi_0(Q - \{0\})$ is the Heisenberg hypersurface $\partial\mathbb{H}^{N+1} \subset \mathbb{CP}^{N+1}$.

For any element $A \in GL(\mathbb{C}^{N+2})$:

$$A = (a_0, ..., a_{N+1}) = \begin{bmatrix} a_0^{(0)} & a_1^{(0)} & ... & a_{N+1}^{(0)} \\ a_0^{(1)} & a_1^{(1)} & ... & a_{N+1}^{(1)} \\ \vdots & \vdots & & \vdots \\ a_0^{(N+1)} & a_1^{(N+1)} & ... & a_{N+1}^{(N+1)} \end{bmatrix} \in GL(\mathbb{C}^{N+2}), \tag{19}$$

where each a_j is a column vector in $\mathbb{C}^{N+2}$, $0 \leq j \leq N+1$. This A is associated to an automorphism $A^\star \in Aut(\mathbb{CP}^{N+1})$ given by

$$A^\star\Big(\big[z_0 : z_1 : ... : z_{N+1}\big]\Big) = \Big[\sum_{j=0}^{N+1} a_j^{(0)} z_j : \sum_{j=0}^{N+1} a_j^{(1)} z_j : ... : \sum_{j=0}^{N+1} a_j^{(N+1)} z_j\Big]. \tag{20}$$

When $a_0^{(0)} \neq 0$, in terms of the non-homogeneous coordinates $(w_1, ..., w_n)$, $A^\star$ is a linear fractional from $\mathbb{C}^{N+1}$ which is holomorphic near $(0, ..., 0)$:

$$A^\star\big(w_1, ..., w_{N+1}\big) = \bigg(\frac{\sum_{j=0}^{N+1} a_j^{(1)} w_j}{\sum_{j=0}^{N+1} a_j^{(0)} w_j}, ..., \frac{\sum_{j=0}^{N+1} a_j^{(N+1)} w_j}{\sum_{j=0}^{N+1} a_j^{(0)} w_j}\bigg), \quad \textit{where } w_j = \frac{z_j}{z_0}. \tag{21}$$

We denote $A \in GL^Q(\mathbb{C}^{N+2})$ if A satisfies $A(Q) \subseteq Q$ where we regard A as a linear transformation of $\mathbb{C}^{N+2}$. If $A \in GL^Q(\mathbb{C}^{N+2})$, we must have $A^\star(\partial\mathbb{H}^{N+1}) \subseteq \partial\mathbb{H}^{N+1}$, so that $A^\star \in Aut(\partial\mathbb{H}^{N+1})$. Conversely, if $A^\star \in Aut(\partial\mathbb{H}^{N+1})$, then $A \in GL^Q(\mathbb{C}^{N+2})$.

We define a bundle map:

$$\begin{array}{rccc} \pi: & GL(\mathbb{C}^{N+2}) & \rightarrow & \mathbb{CP}^{N+1} \\ & A = (a_0, a_1, ..., a_{N+1}) & \mapsto & \pi_0(a_0). \end{array}$$

Then by (20), for any map $A \in GL(\mathbb{C}^{N+2})$, $A \in \pi^{-1}\big(\pi_0(a_0)\big) \iff A^\star([1:0:...:0]) = \pi_0(a_0)$. In particular, by the restriction, we consider a map

$$\begin{array}{rccc} \pi: & GL^Q(\mathbb{C}^{N+2}) & \rightarrow & \partial\mathbb{H}^{N+1} \\ & A = (a_0, a_1, ..., a_{N+1}) & \mapsto & \pi_0(a_0). \end{array} \tag{22}$$

We get $\partial\mathbb{H}^{N+1} \simeq GL^Q(\mathbb{C}^{N+2})/P_1$ where P_1 is the isotropy subgroup of $GL^Q(\mathbb{C}^{N+2})$. Then by (20), for any map $A \in GL^Q(\mathbb{C}^{n+2})$,

$$A \in \pi^{-1}\big(\pi_0(a_0)\big) \iff A^\star([1:0:...:0]) = \pi_0(a_0). \tag{23}$$

CR submanifolds of $\partial\mathbb{H}^{N+1}$ Let $H: M' \rightarrow \partial\mathbb{H}^{N+1}$ be a CR smooth embedding where M' is a strictly pseudoconvex smooth real hypersurface in $\mathbb{C}^{n+1}$. We denote $M = H(M')$.

Let $R_{M'}$ be the Reeb vector field of M' with respect to a fixed contact form on M'. Then the real vector $R_{M'}$ generates a real line bundle over M', denoted by $\mathcal{R}_{M'}$. Since we can regard the rank n complex vector bundle $T^{1,0}M'$ as the rank $2n$ real vector bundle, over the real number field $\mathbb{R}$ we have:

$$TM' = T^cM' \oplus \mathcal{R}_{M'} \simeq T^{1,0}M' \oplus \mathcal{R}_{M'}. \tag{24}$$

given by

$$(a_j\frac{\partial}{\partial x_j}, b_j\frac{\partial}{\partial y_j}) + cR_{M'} \mapsto (a_j+ib_j)\frac{\partial}{\partial z_j} + cR_{M'}, \quad \forall a_j, b_j, c \in \mathbb{R}. \tag{25}$$

Since H is a CR embedding, we have

$$H_*(T^{1,0}M') = T^{1,0}M \subset T^{1,0}(\partial\mathbb{H}^{N+1}), TM \simeq H_*(T^{1,0}M') \oplus H_*(\mathcal{R}_{M'}) \subset T(\partial\mathbb{H}^{N+1}). \tag{26}$$

Lifts of the CR submanifolds Let $M = H(M') \subset \partial\mathbb{H}^{N+1}$ be as above. Consider the commutative diagram

$$\begin{array}{ccc} & & GL^Q(\mathbb{C}^{N+2}) \\ & e \nearrow & \downarrow \pi \\ M & \hookrightarrow & \partial\mathbb{H}^{N+1} \end{array}$$

Any map e satisfying $\pi \circ e = Id$ is called a *lift* of M to $GL^Q(\mathbb{C}^{N+2})$.

In order to define a more specific lifts, we need to give some relationship between geometry on $\partial\mathbb{H}^{N+1}$ and on $\mathbb{C}^{N+2}$ as follows. For any subset $X \subset \partial\mathbb{H}^{N+1}$, we denote $\hat{X} := \pi_0^{-1}(X)$ where $\pi_0 : \mathbb{C}^{N+2} - \{0\} \to \mathbb{CP}^{N+1}$ is the standard projection map (18). In particular, for any $x \in M$, $\hat{x}$ is a complex line and for the real submanifold M^{2n+1}, the real submanifold $\hat{M}^{2n+3}$ is of dimension $2n+3$.

For any $x \in M$, we take $v \in \hat{x} = \pi_0^{-1}(x) \subset \mathbb{C}^{N+2} - \{0\}$, and we define

$$\hat{T}_x M = T_v \hat{M}, \quad \hat{T}_x^{1,0} M = T_v^{1,0}\hat{M}, \quad \hat{\mathcal{R}}_{M,x} := \mathcal{R}_{\hat{M},v}$$

where $\mathcal{R}_{\hat{M}} = \cup_{v\in\hat{M}} \mathcal{R}_{\hat{M},v}$. These definitions are independent of choice of v.

A lift $e = (e_0, e_\alpha, e_\mu, e_{N+1})$ of M into $GL^Q(\mathbb{C}^{N+2})$, where $1 \le \alpha \le n$ and $n+1 \le \mu \le N$, is called a *first-order adapted lift* if it satisfies the conditions:

$$e_0(x) \in \pi_0^{-1}(x), \quad span(e_0, e_\alpha)(x) = \hat{T}_x^{1,0}M, \quad span(e_0, e_\alpha, e_{N+1})(x) = \hat{T}_x^{1,0}M \oplus \hat{\mathcal{R}}_{M,x} \tag{27}$$

where $span(e_0, e_\alpha)(x) = \mathbb{C} \otimes \{e_0 + a_\alpha e_\alpha + b e_{N+1} \mid a_\alpha \in \mathbb{C}, b \in \mathbb{R}\}|_x$, and

$$span(e_0, e_\alpha, e_{N+1})(x) := \mathbb{C} \otimes \{e_0 + a_\alpha e_\alpha + b e_{N+1} \,| a_\alpha \in \mathbb{C}, b \in \mathbb{R}\}|_x. \tag{28}$$

Here we used (25) and the fact that the Reeb vector is real. Locally first-order adapted lifts always exist (see Theorem 7.1 below).

We have the restriction bundle $\mathcal{F}_M^0 := GL^Q(\mathbb{C}^{N+2})|_M$ over M. The subbundle $\pi : \mathcal{F}_M^1 \to M$ of $\mathcal{F}_M^0$ is defined by

$$\mathcal{F}_M^1 = \{(e_0, e_j, e_\mu, e_{N+1}) \in \mathcal{F}_M^0 \mid \ [e_0] \in M, \ (27) \ are \ satisfied\}.$$

Local sections of $\mathcal{F}_M^1$ are exactly all local first-order adapted lifts of M.

For two first-order adapted lifts $s = (e_0, e_j, e_\mu, e_{N+1})$ and $\widetilde{s} = (\widetilde{e}_0, \widetilde{e}_j, \widetilde{e}_\mu, \widetilde{e}_{N+1})$, by (27), we have

$$\begin{cases} \widetilde{e}_0 = g_0^0 e_0, \\ \widetilde{e}_j = g_j^0 e_0 + g_j^k e_k, \\ \widetilde{e}_\mu = g_\mu^0 e_0 + g_\mu^j e_j + g_\mu^\nu e_\nu + g_\mu^{N+1} e_{N+1}, \\ \widetilde{e}_{N+1} = g_{N+1}^0 e_0 + g_{N+1}^j e_j + g_{N+1}^{N+1} e_{N+1}, \end{cases} \tag{29}$$

In other words, $\widetilde{s} = s \cdot g$ where

$$g = (g_0, g_j, g_\mu, g_{N+1}) = \begin{pmatrix} g_0^0 & g_k^0 & g_\mu^0 & g_{N+1}^0 \\ 0 & g_k^j & g_\mu^j & g_{N+1}^j \\ 0 & 0 & g_\mu^\nu & 0 \\ 0 & 0 & g_\mu^{N+1} & g_{N+1}^{N+1} \end{pmatrix} \tag{30}$$

is a smooth map from M into $GL^Q(\mathbb{C}^{N+2})$. Then the fiber of $\pi : \mathcal{F}_M^1 \to M$ over a point is isomorphic to the group

$$G_1 = \left\{ g = \begin{pmatrix} g_0^0 & g_\beta^0 & g_\mu^0 & g_{N+1}^0 \\ 0 & g_\beta^\alpha & g_\mu^\alpha & g_{N+1}^\alpha \\ 0 & 0 & g_\mu^\nu & 0 \\ 0 & 0 & g_\mu^{N+1} & g_{N+1}^{N+1} \end{pmatrix} \in GL^Q(\mathbb{C}^{N+2}) \right\},$$

where we use the index ranges $1 \leq \alpha, \beta \leq n$ and $n+1 \leq \mu, \nu \leq N$.

We pull back the Maurer-Cartan form from $GL^Q(\mathbb{C}^{N+2})$ to $\mathcal{F}_M^1$ by a first-order adapted lift e of M as

$$\omega = \begin{pmatrix} \omega_0^0 & \omega_\beta^0 & \omega_\nu^0 & \omega_{N+1}^0 \\ \omega_0^\alpha & \omega_\beta^\alpha & \omega_\nu^\alpha & \omega_{N+1}^\alpha \\ \omega_0^\mu & \omega_\beta^\mu & \omega_\nu^\mu & \omega_{N+1}^\mu \\ \omega_0^{N+1} & \omega_\beta^{N+1} & \omega_\nu^{N+1} & \omega_{N+1}^{N+1} \end{pmatrix}.$$

Since $\omega = e^{-1}de$, i.e., $e\omega = de$. Then we have

$$de_0 = e_0\omega_0^0 + e_\alpha\omega_0^\alpha + e_\mu\omega_0^\mu + e_{N+1}\omega_0^{N+1}. \tag{31}$$

On the other hand, we have $de_0 \equiv 0 \text{ mod}\{e_0, e_\alpha, e_{N+1}\}$ when pullback to $\mathcal{F}_M^1$. Then we conclude $\omega_0^\mu = 0$, $\forall \mu$. By the Maurer-Cartan equation $d\omega = -\omega \wedge \omega$, one gets $0 = d\omega_0^\nu = -\omega_\alpha^\nu \wedge \omega_0^\alpha - \omega_{N+1}^\nu \wedge \omega_0^{N+1}$, i.e., $0 = -\omega_\alpha^\nu \wedge \omega_0^\alpha,\ mod(\omega_0^{N+1})$. Then by Cartan's lemma,

$$\omega_\beta^\nu = q_{\alpha\beta}^\nu \omega_0^\alpha \ \ mod(\omega_0^{N+1}),$$

for some functions $q^\nu_{\alpha\beta} = q^\nu_{\beta\alpha}$.

The CR second fundamental form In order to define the CR second fundamental form $II_M = II^s_M = q^\mu_{\alpha\beta}\omega^\alpha_0\omega^\beta_0 \otimes \underline{e}_\mu$, mod$(\omega^{N+1}_0)$, let us define $\underline{e}_\mu$ as follows.

For any first-order adapted lift $e = (e_0, e_\alpha, e_\nu, e_{N+1})$ with $\pi_0(e_0) = x$, we have $e_\alpha \in \hat{T}^{1,0}_x M$. Recall $T_E G(k, V) \simeq E^* \otimes (V/E)$ where $G(k, V)$ is the Grassmannian of k-planes that pass through the origin in a vector space V over $\mathbb{R}$ or $\mathbb{C}$ and $E \in G(k, V)$ ([IL03], p.73). Then $T_x M \simeq (\hat{x})^* \otimes (\hat{T}_x M/\hat{x})$ and hence the vector e_α induces $\underline{e_\alpha} \in T^{1,0}_x M$ by

$$\underline{e}_\alpha = e^0 \otimes \big(e_\alpha \; mod(e_0)\big),$$

where we denote by $(e^0, e^\alpha, e^\mu, e^{N+1})$ the dual basis of $(\mathbb{C}^{N+2})^*$. Similarly, we let

$$\underline{e}_\mu = e^0 \otimes \big(e_\mu \; mod \; \hat{T}_x M\big) \in N_x M, \tag{32}$$

where $N_x M$ is the CR normal bundle of M defined by $N_x M = T_x(\partial\mathbb{H}^{N+1})/T_x M$.

By direct computation, we obtain a tensor

$$II_M = II^e_M = q^\mu_{\alpha\beta}\omega^\alpha_0\omega^\beta_0 \otimes \underline{e}_\mu \in \Gamma\big(M, S^2 T^{1,0*}_{\pi_0(e_0)} M \otimes N_{\pi_0(e_0)} M\big) \;\; mod(\omega^{N+1}_0). \tag{33}$$

The tensor II_M is called the *CR second fundamental form* of M.

Pulling back a lift Let $M \subset \partial\mathbb{H}^{N+1}$ be as above with a point $Q_0 \in M$. Let $A \in GL^Q(\mathbb{C}^{N+2})$, $A^\star \in Aut(\partial\mathbb{H}^{N+1})$ with $A^\star(Q_0) = P_0$ and $\widetilde{M} = A^\star(M)$. Let $\widetilde{s} : \widetilde{M} \to GL^Q(\mathbb{C}^{N+2})$ be a lift. We claim:

$$s := A^{-1} \cdot \widetilde{s} \circ A^\star, \tag{34}$$

is also a lift from M into $GL^Q(\mathbb{C}^{N+2})$. In fact, in order to prove that s is a lift, it suffices to prove: $\pi s = Id$, i.e., for any point $Q \in M$ near Q_0, $\pi s(Q) = Q$. In fact,

$$\pi s(Q) = \pi(A^{-1} \cdot \widetilde{s} \circ A^\star)(Q) = \pi(A^{-1} \cdot \widetilde{s}(P)) = (A^\star)^{-1}(\pi\widetilde{s}(P)) = (A^\star)^{-1}(P) = Q.$$

so that our claim is proved.

If, in addition, $\widetilde{s}$ is a first-order adapted lift of $\widetilde{M}$ into $GL^Q(\mathbb{C}^{N+2})$, s is also a first-order adapted lift of M into $GL^Q(\mathbb{C}^{N+2})$.

Let Ω be the Maurer-Cartan form over $GL^Q(\mathbb{C}^{N+2})$. Then by the invariant property $A^*\Omega = \Omega$, we have $s^*\Omega = (A^{-1} \cdot \widetilde{s} \circ A^\star)^*\Omega = (A^\star)^*(\widetilde{s})^*(A^{-1})^*\Omega = (A^\star)^*(\widetilde{s})^*\Omega$, i.e., it holds on M that

$$\omega = (A^\star)^*\widetilde{\omega} \tag{35}$$

where $\omega = s^*\Omega$ and $\widetilde{\omega} = \widetilde{s}^*\Omega$ so that $\omega^\alpha_0 = (A^\star)^*\widetilde{\omega}^\alpha_0$ and $\omega^\mu_\beta = (A^\star)^*\widetilde{\omega}^\mu_\beta$. The last equality yields

$$q^\mu_{\alpha\beta} = \widetilde{q}^\mu_{\alpha\beta} \circ A^\star. \tag{36}$$

6 CR second fundamental forms —— Definition 4

Definition 4 will be the one as a tensor with respect to the group $SU(N+1,1)$.

As for Definition 3, we consider the real hypersurface Q in $\mathbb{C}^{N+2}$ defined by the homogeneous equation

$$\langle Z, Z\rangle := \sum_A Z^A\overline{Z^A} + \frac{i}{2}(Z^{N+1}\overline{Z^0} - Z^0\overline{Z^{N+1}}) = 0, \tag{37}$$

where $Z = (Z^0, Z^A, Z^{N+1})^t \in \mathbb{C}^{N+2}$. This can be extended to the scalar product

$$\langle Z, Z'\rangle := \sum_A Z^A\overline{Z'^A} + \frac{i}{2}(Z^{N+1}\overline{Z'}^0 - Z^0\overline{Z'^{N+1}}), \tag{38}$$

for any $Z = (Z^0, Z^A, Z^{N+1})^t, Z' = (Z'^0, Z'^A, Z'^{N+1})^t \in \mathbb{C}^{N+2}$. This product has the properties: $\langle Z, Z'\rangle$ is linear in Z and anti-linear in Z'; $\overline{\langle Z, Z'\rangle} = \langle Z', Z\rangle$; and Q is defined by $\langle Z, Z\rangle = 0$.

Let $SU(N+1,1)$ be the group of unimodular linear transformations of $\mathbb{C}^{N+2}$ that leave the form $\langle Z, Z\rangle$ invariant (cf. [CM74]).

By a *Q-frame* is meant an element $E = (E_0, E_A,\ E_{N+1}) \in GL(\mathbb{C}^{N+2})$ satisfying (cf. [CM74, (1.10)])

$$\begin{cases} det(E) = 1, \\ \langle E_A, E_B\rangle = \delta_{AB},\ \langle E_0, E_{N+1}\rangle = -\langle E_{N+1}, E_0\rangle = -\frac{i}{2}, \end{cases} \tag{39}$$

while all other products are zero.

There is exactly one transformation of $SU(N+1,1)$ which maps a given Q-frame into another. By fixing one Q-frame as reference, the group $SU(N+1,1)$ can be identified with the space of all Q-frames. Then $SU(N+1,1) \subset GL^Q(\mathbb{C}^{N+1})$ is a subgroup with the composition operation. By (22) and the restriction, we have the projection

$$\pi : SU(N+1,1) \to \partial\mathbb{H}^{N+1},\ (Z_0, Z_A, Z_{N+1}) \mapsto span(Z_0). \tag{40}$$

which is called a *Q-frames bundle*. We get $\partial\mathbb{H}^{N+1} \simeq SU(N+1,1)/P_2$ where P_2 is the isotropy subgroup of $SU(N+1,1)$. $SU(N+1,1)$ acts on $\partial\mathbb{H}^{N+1}$ effectively.

Consider $E = (E_0, E_A, E_{N+1}) \in SU(N+1,1)$ as a local lift. Then the *Maurer-Cartan form* Θ on $SU(N+1,1)$ is defined by $dE = (dE_0, dE_A, dE_{N+1}) = E\Theta$, or $\Theta = E^{-1}\cdot dE$, i.e.,

$$d\begin{pmatrix} E_0 & E_A & E_{N+1}\end{pmatrix} = \begin{pmatrix} E_0 & E_B & E_{N+1}\end{pmatrix} \begin{pmatrix} \Theta_0^0 & \Theta_A^0 & \Theta_{N+1}^0 \\ \Theta_0^B & \Theta_A^B & \Theta_{N+1}^B \\ \Theta_0^{N+1} & \Theta_A^{N+1} & \Theta_{N+1}^{N+1} \end{pmatrix}, \tag{41}$$

where Θ_A^B are 1-forms on $SU(N+1,1)$. By (39) and (41), the Maurer-Cartan form (Θ) satisfies

$$\begin{aligned}&\Theta_0^0+\overline{\Theta_{N+1}^{N+1}}=0,\ \Theta_0^{N+1}=\overline{\Theta_0^{N+1}},\ \Theta_{N+1}^0=\overline{\Theta_{N+1}^0},\\ &\Theta_A^{N+1}=2i\overline{\Theta_0^A},\ \Theta_{N+1}^A=-\frac{i}{2}\overline{\Theta_A^0},\ \Theta_B^A+\overline{\Theta_A^B}=0,\ \Theta_0^0+\Theta_A^A+\Theta_{N+1}^{N+1}=0,\end{aligned} \tag{42}$$

where $1\le A\le N$. For example, from $\langle E_A,E_B\rangle=\delta_{AB}$, by taking differentiation, we obtain

$$\langle dE_A,E_B\rangle+\langle E_A,dE_B\rangle=0.$$

By (41), we have

$$\begin{cases}dE_0=E_0\Theta_0^0+E_B\Theta_0^B+E_{N+1}\Theta_0^{N+1},\\ dE_A=E_0\Theta_A^0+E_B\Theta_A^B+E_{N+1}\Theta_A^{N+1},\\ dE_{N+1}=E_0\Theta_{N+1}^0+E_B\Theta_{N+1}^B+E_{N+1}\Theta_{N+1}^{N+1}.\end{cases}$$

Then

$$\langle E_0\Theta_A^0+E_C\Theta_A^C+E_{N+1}\Theta_A^{N+1},\ E_B\rangle+\langle E_A,E_0\Theta_B^0+E_D\Theta_B^D+E_{N+1}\Theta_B^{N+1}\rangle=0,$$

which implies $\Theta_A^B+\overline{\Theta_B^A}=0$. In particular, from (42), $\Theta_A^0=-2i\overline{\Theta_{N+1}^A}$. Θ satisfies

$$d\Theta=-\Theta\wedge\Theta. \tag{43}$$

Let $M\hookrightarrow\partial\mathbb{H}^{N+1}$ be the image of $H:M'\to\partial\mathbb{H}^{N+1}$ where $M'\subset\mathbb{C}^{n+1}$ is a CR strictly pseudoconvex smooth hypersurface. Consider the inclusion map $M\hookrightarrow\partial\mathbb{H}^{N+1}$ and a lift $e=(e_0,e_1,...,e_{N+1})=(e_0,e_\alpha,e_\nu,\ e_{N+1})$ of M where $1\le\alpha\le n$ and $n+1\le\nu\le N$

$$\begin{array}{ccc} & & SU(N+1,1)\\ & e\nearrow & \downarrow\pi\\ M & \hookrightarrow & \partial\mathbb{H}^{N+1}\end{array}$$

We call e a *first-order adapted lift* if for any $x\in M$, (27) is satisfied:

$$\pi_0\big(e_0(x)\big)=x,\ span(e_0,e_\alpha)(x)=\hat{T}_x^{1,0}M,\ span(e_0,e_\alpha,e_{N+1})(x)=\hat{T}_x^{1,0}M\oplus\hat{\mathcal{R}}_{M,x}. \tag{44}$$

Locally first-order adapted lifts always exist (see Theorem 7.1 below). We have the restriction bundle $\mathcal{F}_M^0:=SU(N+1,1)|_M$ over M. The subbundle $\pi:\mathcal{F}_M^1\to M$ of $\mathcal{F}_M^0$ is defined by

$$\mathcal{F}_M^1=\{(e_0,e_j,e_\mu,e_{N+1})\in\mathcal{F}_M^0\ |\ \ [e_0]\in M,\ (44)\ are\ satisfied\}.$$

Local sections of $\mathcal{F}^1_M$ are exactly all local first-order adapted lifts of M. The fiber of $\pi : \mathcal{F}^1_M \to M$ over a point is isomorphic to the group

$$G_1 = \left\{ g = \begin{pmatrix} g_0^0 & g_\beta^0 & g_\nu^0 & g_{N+1}^0 \\ 0 & g_\beta^\alpha & g_\nu^\alpha & g_{N+1}^\alpha \\ 0 & 0 & g_\nu^\mu & 0 \\ 0 & 0 & 0 & g_{N+1}^{N+1} \end{pmatrix} \in SU(N+1,1) \right\},$$

where we use the index ranges $1 \le \alpha, \beta \le n$ and $n+1 \le \mu, \nu \le N$.

By (39), we have $\langle g_0, g_{N+1} \rangle = -\frac{i}{2}$, it implies $g_0^0 \cdot \overline{g_{N+1}^{N+1}} = 1$ so that $g_{N+1}^{N+1} = \frac{1}{\overline{g_0^0}}$. Since $\langle g_0, g_\mu \rangle = 0$ and $g_0^0 \neq 0$, it implies $g_\mu^{N+1} = 0$. Since $\langle g_\alpha, g_\beta \rangle = \delta_{\alpha\beta}$, it implies that the matrix (g_α^β) is unitary. Since $deg(g) = 1$, it implies $g_0^0 \cdot det(g_\alpha^\beta) \cdot det(g_\mu^\nu) \cdot g_{N+1}^{N+1} = 1$. By (25) and (44), g_{N+1}^{N+1} is a real if $g_{N+1}^0 = 0$; g_{N+1}^{N+1}/g_{N+1}^0 is real if $g_{N+1}^0 \neq 0$.

By considering all first-order adapted lifts from M into $SU(N+1,1)$, as the definition of II_M in Definition 3, we can defined CR second fundamental form II_M as in (33):

$$II_M = II_M^e = q^\mu_{\alpha\beta} \omega_0^\alpha \omega_0^\beta \otimes \underline{e}_\mu \in \Gamma(M, S^2 T^{1,0*}_{\pi_0(e_0)} M \otimes N_{\pi_0(e_0)} M), \quad mod(\omega_0^{N+1}), \tag{45}$$

which is a well-defined tensor, and is called the *CR second fundamental form* of M.

We remark that II_M in Definition 4 was studied in [Wa09].

Pulling back a lift Let $M \subset \partial\mathbb{H}^{N+1}$ be as above with a point $Q_0 \in M$. Let $A \in SU(N+1,1)$, $A^\star \in Aut(\partial\mathbb{H}^{N+1})$ with $A^\star(Q_0) = P_0$ and $\widetilde{M} = A^\star(M)$. Let $\widetilde{s} : \widetilde{M} \to SU(N+1,1)$ be a lift. We claim:

$$s := A^{-1} \cdot \widetilde{s} \circ A^\star, \tag{46}$$

is also a lift from M into $SU(N+1,1)$. Similarly as in (35) and (36), we have

$$\omega = (A^\star)^* \widetilde{\omega} \tag{47}$$

and

$$q^\mu_{\alpha\beta} = \widetilde{q}^\mu_{\alpha\beta} \circ A^\star. \tag{48}$$

where $\omega = s^*\Omega$, $\widetilde{\omega} = \widetilde{s}^*\Omega$ and Ω is the Maurer-Cartan form over $SU(N+1,1)$.

[Example] Consider the maps in (1) and (2):

$$\sigma_p^0(z,w) = (z + z_0, w + w_0 + 2i\langle z, \overline{z_0} \rangle),$$
$$\tau_p^F(z^*, w^*) = (z^* - \widetilde{f}(z_0,w_0), w^* - \overline{g(z_0,w_0)} - 2i\langle z^*, \overline{\widetilde{f}(z_0,w_0)} \rangle)$$

where $p = (z_0, w_0)$, $z = \mathbb{C}^n$, $w = z_{n+1}$, $\sigma_p^0 \in Aut(\partial\mathbb{H}^{n+1})$, and $\tau_p^F \in Aut(\partial\mathbb{H}^{N+1})$.

By (19) and (21), these two maps correspond to two matrices:

$$A_{\sigma_p^0} = \begin{bmatrix} 1 & 0 & ... & 0 & 0 \\ z_{01} & 1 & ... & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_{0n} & 0 & ... & 1 & 0 \\ w_0 & 2i\overline{z_{01}} & ... & 2i\overline{z_{0n}} & 1 \end{bmatrix} \in SU(n+1,1) \tag{49}$$

and

$$A_{\sigma_p^F} = \begin{bmatrix} 1 & 0 & ... & 0 & 0 \\ -\widetilde{f}_{01} & 1 & ... & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\widetilde{f}_{0N-n} & 0 & ... & 1 & 0 \\ -\overline{g(z_0,w)} & -2i\overline{\widetilde{f}_1(z_0,w_0)} & ... & -2i\overline{\widetilde{f}_{N-n}(z_0,w_0)} & 1 \end{bmatrix} \in SU(N+1,1) \tag{50}$$

where $z_0 = (z_{01}, ..., z_{0n})$ and $w_0 = z_{0n+1}$. $\square$

[Example] Consider the map $F_{\lambda,r,\vec{a},U} = (f,g) \in Aut_0(\partial\mathbb{H}^{n+1})$

$$f(z) = \frac{\lambda(z+\vec{a}w)U}{1-2i\langle z, \overline{\vec{a}}\rangle - (r+i\|\vec{a}\|^2)w}, \quad g(z) = \frac{\lambda^2 w}{1-2i\langle z, \overline{\vec{a}}\rangle - (r+i\|\vec{a}\|^2)w}$$

where $\lambda > 0, r \in \mathbb{R}, \vec{a} \in \mathbb{C}^n$ and $U = (u_{\alpha\beta})$ is an $(n-1)\times(n-1)$ unitary matrix. By (19) and (21), its corresponding matrix,

$$A_{F_{\lambda,r,\vec{a},U}} = \begin{bmatrix} 1 & -2i\overline{a_1} & ... & -2i\overline{a_n} & -(r+i\|\vec{a}\|^2) \\ 0 & \lambda u_{11} & ... & \lambda u_{1n} & \lambda a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \lambda u_{n1} & ... & \lambda u_{nn} & \lambda a_n \\ 0 & 0 & ... & 0 & \lambda^2 \end{bmatrix}, \tag{51}$$

is not in $SU(n+1,1)$ in general. In fact, we can write

$$F_{\lambda,r,\vec{a},U} = F_{\lambda,0,0,Id} \circ F_{1,0,0,U} \circ F_{1,r,\vec{a},Id}. \tag{52}$$

or $A_{F_{\lambda,r,\vec{a},U}} = A_{F_{\lambda,0,0,Id}} \cdot A_{F_{1,0,0,U}} \cdot A_{F_{1,r,\vec{a},Id}}$. Here $A_{F_{1,0,0,U}}$ and $A_{F_{1,r,\vec{a},Id}}$ are in $SU(N+1,1)$; while $A_{F_{\lambda,0,0,Id}}$ is in $SU(N+1,1)$ if and only if $\lambda = 1$. Therefore

$$A_{F_{\lambda,r,\vec{a},U}} \textit{ is in } SU(n+1,1) \textit{ if and only if } \lambda = 1. \tag{53}$$

7 Existence of First-order Adapted Lifts from M into $SU(N+1,1)$ or into $GL^Q(\mathbb{C}^{N+2})$

Existence of first-order adapted lifts. Let $(M',0)$ be a germ of smooth real hypersurface in $\mathbb{C}^{n+1}$ defined by the defining function

$$r = \sum_{j=1}^{n} z_j \overline{z}_j + \frac{i}{2}(w - \overline{w}) + o(2). \tag{54}$$

We take

$$\theta = i\partial r = i\bigg(\sum_{j=1}^{n} \overline{z_j} dz_j - \frac{1}{2}dw)\bigg) + o(1).$$

as a contact form of M'.

Write $w = u + iv$. Here $v = \sum_{j=1}^{n} |z_j|^2 + o(2)$. Take (z_j, u) as a coordinates system of M'. By considering the coordinate map: $h : \mathbb{C}^n \times \mathbb{R} \to M',\ (z_j, u) \mapsto (z_j, u + i|z|^2 + o(2))$, we get the pushforward

$$h_*(\frac{\partial}{\partial z_j}) = L_j := \frac{\partial}{\partial z_j} + i\big(\overline{z_j} + o(1)\big)\frac{\partial}{\partial u}, \quad h_*(\frac{\partial}{\partial u}) = R_{M'} := (1 + o(1))\frac{\partial}{\partial u}$$

for $j = 1, 2, ..., n$. Then $\{L_j\}_{1 \le j \le n}$ form a basis of the complex tangent bundle $T^{1,0}M'$ of M'. Since $d\alpha = -i \sum_{j=1}^{n} dz_j \wedge d\overline{z_j}$, we see that R is the Reeb vector field of M'. In particular, as the restriction at 0, we have

$$L_j|_0 = \frac{\partial}{\partial z_j}|_0, \quad R_{M'}|_0 = \frac{\partial}{\partial u}|_0. \tag{55}$$

Theorem 7.1 *Let $M \hookrightarrow \partial\mathbb{H}^{N+1}$ be the image of $H : M' \to \partial\mathbb{H}^{N+1}$ where $M' \subset \mathbb{C}^{n+1}$ is a smooth strictly pseudoconvex CR-hypersurface. Then for any point in M, the first-order adapted lift $E = (E_0, E_\alpha, E_\mu, E_{N+1})$ of M into $SU(N+1,1)$ (hence into $GL^Q(\mathbb{C}^{N+2})$) exists in some neighborhood of the point in M.*

Proof: **Step 1.** Without of loss of generality, we assume that $0 \in M$ so that it suffices to construct a lift $E = (E_0, E_\alpha, E_\mu, E_{N+1})$ in a neighborhood of the point 0. Here we denote $[1 : 0 : ... : 0]$ by 0.

Assume that M' is defined by the equation $Im\ w = |z|^2 + o(|z|^2)$ in $(z,w) \in \mathbb{C}^n \times \mathbb{C}$ where $w = u + iv$. Assume that $H = (1, f_\alpha, \phi_\mu, g)$ is the smooth CR embedding of M' into $\partial\mathbb{H}^{N+1}$ with $H(0) = 0$ and

$$f = z + O(|(z,w)|^2), \phi = O(|(z,w)|^2),\ g = w + O(|(z,w)|^2). \tag{56}$$

Let $L_\alpha, \alpha = 1, 2, ..., n$ be a basis of the CR vector fields and R is the Reeb vector field on M'. Then as in (55) with (56), we have

$$L_\alpha|_0 = \frac{\partial}{\partial z_j}|_0, \quad and \quad R|_0 = \frac{\partial}{\partial u}|_0. \tag{57}$$

It follows that $\bar{L}_\alpha H = 0$ as H is a CR map. By the Lewy extension theorem, H extends holomorphically to one side of M', denoted by D, where D is obtained by attaching the holomorphic discs. By applying the maximum principle and the Hopf lemma to the subharmonic function $\sum |f_\alpha|^2 + \sum |\phi_\mu|^2 + \frac{i}{2}(g - \bar{g})$ on D, it follows that $\frac{\partial Im\ g}{\partial v}(0) \neq 0$. Since $\frac{\partial g}{\partial \bar{w}} = 0$ and $\frac{\partial Im\ g}{\partial u}(0) = 0$, we have $Rg(0) = \frac{\partial g}{\partial u}(0) = \frac{\partial Im\ g}{\partial v}(0) \neq 0$.

Step 2. Direct construction of E_0, E_α and E_{N+1} We define

$$E_0 := \begin{bmatrix} 1 \\ f_\alpha(z,w) \\ \phi_\mu(z,w) \\ g(z,w) \end{bmatrix} \tag{58}$$

which can be regarded as a point in $\partial\mathbb{H}^{N+1}$. Then $\langle E_0, E_0 \rangle = 0$ holds:

$$\sum f_\alpha \bar{f}_\alpha + \sum \phi_\mu \bar{\phi}_\mu + \frac{i}{2}(g - \bar{g}) = 0, \quad on\ M. \tag{59}$$

Apply the CR vector field L_β to E_0, we define

$$\widetilde{E}_\beta = (0, L_\beta f_\alpha, L_\beta \phi_\mu, L_\beta g)^t,$$

which form the basis of the complex tangent bundle $T^{1,0}_{\pi_0(E_0)}(M)$. Then in a neighborhood of 0 in M, we have as in (27)

$$span(E_0, \widetilde{E}_\alpha) = \hat{T}^{(1,0)}_{\pi_0(E_0)} M.$$

Now, we have $\langle E_0, \widetilde{E}_\alpha \rangle = 0$ by applying L_β to (59):

$$\sum \bar{f}_\alpha L_\beta f_\alpha + \sum \bar{\phi}_\mu L_\beta \phi_\mu + \frac{i}{2} L_\beta g = 0. \tag{60}$$

By the Gram-Schmid orthonormalization procedure, we can obtain, from $\{\widetilde{E}_\beta\}$, an orthonormal set with respect to the usual Hermitian inner product $\langle\ ,\ \rangle_0$; we denote it by $\{E_\beta\}$. By the definition (38), we notice that for any $Z = (Z^0, Z^A, Z^{N+1})$ and $Z' = (Z'^0, Z'^A, Z'^{N+1})$,

$$\langle Z,\ Z'\rangle = \left\langle (\frac{i}{2}Z^{N+1}, Z^A, -\frac{i}{2}Z^0),\ (Z'_0, Z'^A, Z'^{N+1})\right\rangle_0 = \langle \hat{Z}, Z'\rangle_0, \tag{61}$$

where $\langle\ ,\ \rangle_0$ is the usual Hermitian inner product and $\hat{Z} := (\frac{i}{2}Z^{N+1}, Z^A, -\frac{i}{2}Z^0)$. Then we see from (60) that

$$\langle E_0,\ E_\beta\rangle = \left\langle (\frac{i}{2}g, f_\alpha, \phi_\mu, -\frac{i}{2}),\ (0, L_\beta f_\alpha, L_\beta\phi_\mu, L_\beta g)\right\rangle_0 = 0.$$

Also we observe $\langle E_\alpha,\ E_\beta\rangle = \langle E_\alpha,\ E_\beta\rangle_0 = \delta_{\alpha\beta}$. Then $\langle E_0, E_0\rangle = 0, \langle E_0, E_\beta\rangle = 0$ and $\langle E_\alpha, E_\beta\rangle = \delta_{\alpha\beta}$ hold.

Applying the Reeb vector field R, we define another vector

$$\widetilde{E}_{N+1} := (0, R\ f_\alpha, R\ \phi_\mu, R\ g)^t$$

over a neighborhood of 0 in M such that

$$span(E_0, E_\alpha, \widetilde{E}_{N+1}) = \hat{T}_{\pi_0(E_0)}M$$

as in (27). We want to construct

$$E_{N+1} = AE_0 + B_\alpha E_\alpha + C\widetilde{E}_{N+1}$$

such that

$$\langle E_{N+1},\ E_0\rangle = \frac{i}{2},\ \langle E_\alpha, E_{N+1}\rangle = 0,\ \textit{and}\ \langle E_{N+1},\ E_{N+1}\rangle = 0.$$

From $\langle E_{N+1}, E_0\rangle = \frac{i}{2}$, we get $\langle AE_0 + B_\alpha E_\alpha + C\widetilde{E}_{N+1},\ E_0\rangle = \frac{i}{2}$ so that

$$C = \frac{i}{2\langle \widetilde{E}_{N+1}, E_0\rangle}. \tag{62}$$

By (57), we notice that

$$\langle \widetilde{E}_{N+1}, E_0\rangle|_0 = \sum \frac{\partial f_\alpha}{\partial u}|_0 \bar{f}_\alpha(0) + \sum \frac{\partial \phi_\mu}{\partial u}|_0 \bar{\phi}_\mu(0) + \frac{i}{2}\frac{\partial g}{\partial u}|_0$$

and therefore $\langle \widetilde{E}_{N+1}, E_0\rangle(0) = \frac{i}{2}R\ g(0) \neq 0$.

From $\langle E_{N+1}, E_\alpha\rangle = 0$, we get $\langle AE_0 + B_\beta E_\beta + C\widetilde{E}_{N+1},\ E_\alpha\rangle = 0$ so that

$$B_\alpha = -C\delta_{\beta\alpha}\langle \widetilde{E}_{N+1}, E_\beta\rangle = -C\langle \widetilde{E}_{N+1}, E_\alpha\rangle. \tag{63}$$

From $\langle E_{N+1}, E_{N+1}\rangle = 0$, we get $\langle AE_0 + B_\beta E_\beta + C\widetilde{E}_{N+1},\ AE_0 + B_\beta E_\beta + C\widetilde{E}_{N+1}\rangle = 0$. Since $C\langle \widetilde{E}_{N+1}, E_0\rangle = \frac{i}{2}$, $\overline{C}\langle E_0, \widetilde{E}_{N+1}\rangle = -\frac{i}{2}$, $B_\alpha = -C\langle \widetilde{E}_{N+1}, E_\alpha\rangle$ and $\overline{B_\alpha} = -\overline{C}\langle E_\alpha, \widetilde{E}_{N+1}\rangle$ by (62) and (63), we obtain

$$-\frac{i}{2}A + \frac{i}{2}\overline{A} - \sum_\alpha |B_\alpha|^2 + |C|^2\langle E_{N+1}, E_{N+1}\rangle = 0,$$

so that

$$Im(A) = \sum_\alpha |B_\alpha|^2 - |C|^2\langle E_{N+1}, E_{N+1}\rangle. \tag{64}$$

Therefore E_{N+1} is determined.

So far we have $\langle E_0, E_0\rangle = \langle E_{N+1}, E_{N+1}\rangle = \langle E_0, E_\beta\rangle = \langle E_{N+1}, E_\beta\rangle = 0$, $\langle E_\alpha, E_\beta\rangle = \delta_{\alpha\beta}$ and $\langle E_0, E_{N+1}\rangle = -\frac{i}{2}$ hold.

Step 3. Construction of E From Step 2, at the point 0, we have vectors

$$E_0|_0 = [1:0:...:0],\ E_1|_0 = [0:1:0:...:0], ..., E_n|_0 = [0:0:...:1:0:...:0], \tag{65}$$

and

$$E_{N+1}|_0 = [0:0:...:0:1]. \tag{66}$$

Therefore we can define E at the point 0 by

$$E(0) := Id \in SU(N+1,1). \tag{67}$$

For any other point P in a small neighborhood of 0 in M, we are going to define $E(P) \in SU(N+1,1)$ as follows.

Write $H(p) = P$ for some $p \in M'$. Then we take a map $\Psi_P \in SU(N+1,1)$ such that

$$\Psi_P^\star(P) = 0,\ \ T_0^{1,0}\Psi(M) = span(E_0|_0, E_\alpha|_0),\ \ and\ \ T_0\Psi(M) = span(E_0|_0, E_\alpha|_0, E_{N+1}|_0)$$

as in (27), where $E_0|_0, E_\alpha|_0$ and $E_{N+1}|_0$ are defined in (65) and (66). The map Ψ_P can be defined as $A_{F_{1,r,\vec{a},U}} \circ A_{\sigma_p^F}$ where $A_{\sigma_p^F} \in SU(N+1,1)$ as in (50) and $A_{F_{1,r,\vec{a},U}} \in SU(N+1,1)$ as in (51). Notice in the construction of the normalization F^{**} and F^{***}, we can always choose $\lambda = 1$ so that (52) can be used. Ψ_P is smooth as P varies. Then we define

$$E(P) := (\Psi_P^\star)^* E(0) = (\Psi_P)^{-1}E(0). \tag{68}$$

This definition is the same as in (46). Since Ψ_P is invariant for the Hermitian scalar product $\langle\ ,\ \rangle$ defined in (38) and $E(0)$ satisfies the identities (39), it implies that $E(P)$ satisfies the identities (39), i.e., $E(p) \in SU(N+1,1)$.

As a matrix, we denote $E(P) = (\hat{E}_0, \hat{E}_\alpha, \hat{E}_\mu, \hat{E}_{N+1})$. Since the map Ψ_P preserves the CR structures and the tangent vector spaces of M and $\Psi_P(M)$, we have as in (27)

$$span(\hat{E}_0, \hat{E}_\alpha) = span(E_0, E_\alpha)|_P, \quad span(\hat{E}_0, \hat{E}_\alpha, \hat{E}_{N+1}) = span(E_0, E_\alpha, E_{N+1})|_P.$$

where E_0, E_α and E_{N+1} are constructed in Step 2. We remark that we can replace $(\hat{E}_0, \hat{E}_\alpha, \hat{E}_{N+1})$ by (E_0, E_α, E_{N+1}). $\square$

Existence of a more special first-order adapted lifts when M is spherical When $M = F(\partial\mathbb{H}^{n+1})$ where $F \in Prop_2(\mathbb{H}^{n+1}, \mathbb{H}^{N+1})$, we can construct a more special first-order adapted lift of M into $SU(N+1,1)$ as follows (cf. [HJY09]).

Let $F = (f, \phi, g) \in Prop_2(\partial\mathbb{H}^{n+1}, \partial\mathbb{H}^{N+1})$ be any map with $F = F_p^{***}$. Then $F(0) = 0$. We introduce a local biholomorphic map near the origin

$$F_{fg} := (f, g) : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1},\ (z, z_{N+1}) \mapsto (f, g) = (\hat{z}, \hat{z}_{N+1})$$

with its inverse

$$F_{fg}^{-1} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1},\ (\hat{z}, \hat{z}_{N+1}) \mapsto ((F_{fg}^{-1})^{(1)}, ..., (F_{fg}^{-1})^{(n)}, (F_{fg}^{-1})^{(N+1)}) = (z, z_{N+1}).$$

Here we use $(\hat{z}, \hat{z}_{N+1})$ as a coordinates system of $M = F(\partial\mathbb{H}^{n+1})$ near $F(0) = 0$. Denote $Proj_{fg} : \mathbb{C}^{N+1} \to \mathbb{C}^{n+1}, (\hat{z}, \hat{z}_\mu, \hat{z}_{N+1}) \mapsto (\hat{z}, \hat{z}_{N+1})$. Then we have $Proj_{fg} \circ F = F_{fg}$:

$$\begin{array}{ccc} F : \partial\mathbb{H}^{n+1} & \to & M \\ & \searrow F_{fg} & \downarrow Proj_{fg} \\ & & \mathbb{C}^{n+1} \end{array}$$

We also have a pair of inverse maps $F : \partial\mathbb{H}^{n+1} \to M$ *and* $(F_{fg}^{-1}) \circ Proj_{fg} : M \to \partial\mathbb{H}^{n+1}$.

Locally we can regard M as a graph: $F \circ F_{fg}^{-1} : \mathbb{C}^{n+1} \to M \subset \mathbb{C}^{N+2}$:

$$(\hat{z}, \hat{z}_{N+1}) \mapsto \big(\hat{z},\ \phi\big((F_{fg})^{-1}(\hat{z}, \hat{z}_{N+1})\big),\ \ \hat{z}_{N+1}\big)$$

Now let us define a lift of M into $SU(N+1,1)$

$$e = (e_0, e_\alpha, e_\mu, e_{N+1}) \in SU(N+1,1), \quad 1 \le \alpha \le n,\ n+1 \le \mu \le N \tag{69}$$

as follows.

We define $e_0 : M \hookrightarrow \mathbb{C}^{N+2}$ be the inclusion:

$$e_0(\hat{z}, \hat{z}_{N+1}) = F \circ F_{fg}^{-1}(\hat{z}, \hat{z}_{N+1}) = \Big[1 : \ \hat{z} : \ \phi\big((F_{fg})^{-1}(\hat{z}, \hat{z}_{N+1})\big) : \ \hat{z}_{N+1}\Big]^t \tag{70}$$

$\forall(\hat{z}, \hat{z}_{N+1}) \in \mathbb{C}^{n+1}$. We define $e_\alpha : M \to \mathbb{C}^{N+2}$ for $1 \le \alpha \le n$:

$$e_\alpha := \frac{1}{\sqrt{|L_\alpha f|^2 + |L_\alpha \phi|^2}}\big[0 : L_\alpha f : L_\alpha \phi : L_\alpha g\big]^t \circ F_{fg}^{-1}. \tag{71}$$

where $L_\alpha = \frac{\partial}{\partial z^\alpha} + 2i\bar{z}^\alpha \frac{\partial}{\partial z^{N+1}}$. By the definition (38), we have $\langle e_0, e_0\rangle = 0$ because $f \cdot \overline{f} + \phi \cdot \overline{\phi} - \frac{1}{2i}(g - \overline{g}) = \hat{z} \cdot \overline{\hat{z}} + \phi\big((F_{fg})^{-1}(\hat{z}, \hat{z}_{N+1})\big)\overline{\phi\big((F_{fg})^{-1}(\hat{z}, \hat{z}_{N+1})\big)} + \frac{i}{2}(\hat{z}_{N+1} - \overline{\hat{z}_{N+1}}) = 0$ holds on $\partial\mathbb{H}^{n+1}$, and $\langle e_0, e_\alpha\rangle = 0$ because $L_\alpha f \cdot \overline{f} + L_\alpha \phi \cdot \overline{\phi} + \frac{i}{2} L_\alpha g = 0$ holds on $\partial\mathbb{H}^{n+1}$, and $\langle e_\alpha, e_\beta\rangle = \delta_{\alpha\beta}$ because $L_\alpha f \cdot \overline{L_\beta f} + L_\alpha \phi \cdot \overline{L_\beta \phi} = 0$ holds on $\partial\mathbb{H}^{n+1}$ for $\alpha \ne \beta$.

If we define $\widetilde{e}_{N+1} := (0, Tf, T\phi, Tg)^t \circ F_{fg}^{-1}$, where $T = \frac{\partial}{\partial u}$ with $z^{N+1} = u + iv$, then $span(e_0, e_\alpha, \widetilde{e}_{N+1}) = \hat{T}_{\pi_0(e_0)} M$. We then find coefficient functions A, B_α and C such that $e_{N+1} = Ae_0 + \sum B_\alpha e_\alpha + C\widetilde{e}_{N+1}$ satisfies

$$\langle e_0, e_{N+1}\rangle = -\frac{i}{2}, \ \langle e_\alpha, e_{N+1}\rangle = 0, \ \langle e_{N+1}, e_{N+1}\rangle = 0. \tag{72}$$

8 Relationship among four definitions of II_M

Lemma 8.1 *Let $H : M' \to \partial\mathbb{H}^{N+1}$ be a CR smooth embedding where M' is a strictly pseudoconvex smooth real hypersurface in $\mathbb{C}^{n+1}$. We denote $M = H(M')$. Then the following statements are equivalent:*

(i) The CR second fundamental form II_M by Definition 1 identically vanishes.

(ii) The CR second fundamental form II_M by Definition 2 identically vanishes.

(iii) The CR second fundamental form II_M by Definition 3 identically vanishes.

(iv) The CR second fundamental form II_M by Definition 4 identically vanishes.

Proof (i) $\Longleftrightarrow$ (ii) by (15).

(iii) $\Longleftrightarrow$ (iv) The equivalence follows by the facts that, for Definition 3 and 4, $II_M^e \equiv 0$ for one first-order adapted lift e if and only if $II_M^s \equiv 0$ for any first-order adapted lift s, that a first-order adapted lift from M to $SU(N+1,1)$ must be a first-order adapted lift from M to $GL^Q(\mathbb{C}^{N+2})$.

(iv) $\Longrightarrow$ (i): Let $M \subset \partial\mathbb{H}^{N+1}$ be a $(2n+1)$ dimensional CR submanifold with CR dimension n that admits a first-order adapted lift e into $SU(N+1,1)$. Consider the pull-backed Maurer-Cartan form over M by e

$$\omega = \begin{pmatrix} \omega_0^0 & \omega_\beta^0 & \omega_\nu^0 & \omega_{N+1}^0 \\ \omega_0^\alpha & \omega_\beta^\alpha & \omega_\nu^\alpha & \omega_{N+1}^\alpha \\ 0 & \omega_\beta^\mu & \omega_\nu^\mu & \omega_{N+1}^\mu \\ \omega_0^{N+1} & \omega_\beta^{N+1} & 0 & \omega_{N+1}^{N+1} \end{pmatrix}.$$

with

$$\begin{aligned} &\omega_0^0 + \overline{\omega_{N+1}^{N+1}} = 0,\ \omega_0^{N+1} = \overline{\omega_0^{N+1}},\ \omega_{N+1}^0 = \overline{\omega_{N+1}^0}, \\ &\omega_A^{N+1} = 2i\overline{\omega_0^A},\ \omega_{N+1}^A = -\tfrac{i}{2}\overline{\omega_A^0},\ \omega_B^A + \overline{\omega_A^B} = 0,\ \omega_0^0 + \omega_A^A + \omega_{N+1}^{N+1} = 0, \end{aligned} \tag{73}$$

where $1 \le A \le N$.

Let $\theta = \omega_0^{N+1}$ which is a real 1-form by (73). By $d\omega = -\omega \wedge \omega$ and (73), we obtain

$$d\theta = -\omega_0^{N+1} \wedge \omega_0^0 - \omega_\alpha^{N+1} \wedge \omega_0^\alpha - \omega_{N+1}^{N+1} \wedge \omega_0^{N+1} = 2i\omega_0^\alpha \wedge \overline{\omega_0^\alpha} - \theta \wedge (\omega_0^0 + \overline{\omega_0^0}) = i\theta^\alpha \wedge \overline{\theta^\alpha},$$

where we denote

$$\theta^\alpha = \sqrt{2}\omega_0^\alpha + c_\alpha \theta \tag{74}$$

for some functions c_α. Therefore, (8) holds and hence M is a strictly pseudoconvex pseudohermitian manifold with an admissible coframe (θ, θ^α). Hence Definition 4 of $II_M \equiv 0$ implies Definition 1 of $II_M \equiv 0$.

(i) $\Longrightarrow$ (iv): Definition 1 of II_M gives a coframe (θ, θ^α) which corresponds to Definition 2 of II_M with respect to a defining function ρ of M in $\partial\mathbb{H}^{N+1}$.

Now take a first-order adapted lift e from M into $SU(N+1,1)$. By (74), it corresponds to a coframe (θ, θ^α) on M and by (16), it corresponds Definition 2 of II_M by some choice of the defining function $\hat\rho$ of M in $\partial\mathbb{H}^{N+1}$.

The above ρ and $\hat\rho$ may not be the same. But Definition 2 of $II_M \equiv 0$ is independent of choice of defining functions, which gives (i) $\Longrightarrow$ (iv). $\square$

9 Proof of Theorem 1.1

Lemma 9.1 *(cf. [EHZ04], corollary 5.5) Let $H : M' \to M \hookrightarrow \partial\mathbb{H}^{N+1}$ be a smooth CR embedding of a strictly pseudoconvex smooth real hypersurface $M \subset \mathbb{C}^{n+1}$. Denote by $(\omega_\alpha{}^\mu{}_\beta)$ the CR second fundamental form matrix of H relative to an admissible coframe (θ, θ^A) on $\partial\mathbb{H}^{N+1}$ adapted to M. If $\omega_\alpha{}^\mu{}_\beta \equiv 0$ for all α, β and μ, then M' is locally CR-equivalent to $\partial\mathbb{H}^{n+1}$.*

Proof of Theorem 1.1 **Step 1. Reduction to a problem for geometric rank** By Lemma 8.1 and Lemma 9.1 and the hypothesis that the CR second fundamental form identically vanishes, we know that M is locally CR equivalent to $\partial\mathbb{H}^{n+1}$.

Then M is the image of a local smooth CR map $F : U \subset \partial\mathbb{H}^{n+1} \to M \subset \partial\mathbb{H}^{N+1}$ where U is a open set in $\partial\mathbb{H}^{n+1}$. By a result of Forstneric[Fo89], the map F must be a rational map. It suffices to prove that F is equivalent to a linear map. By Lemma 2.2, it is sufficient to prove that the geometric rank of F is zero: $\kappa_0 = 0$.

Suppose $\kappa_0 > 0$ and we seek a contradiction.

Step 2. Reduction to a lift of $\big((H\circ\tau_p^F)(M), 0\big)$ Take any point $p \in U \subset \partial\mathbb{H}^{n+1}$ with $\kappa_0 = \kappa_0(p) > 0$, and consider the associated map (see Lemma 2.1)

$$F_p^{***} = H\circ\tau_p^F\circ F\circ\sigma_p^0\circ G : \partial\mathbb{H}^{n+1} \to \partial\mathbb{H}^{N+1}, \quad F_p^{***}(0) = 0, \tag{75}$$

where σ_p^0 is defined in (1), τ_p^F is defined in (2), $G \in Aut_0(\mathbb{H}^{n+1})$ and $H \in Aut_0(\mathbb{H}^{N+1})$ are automorphisms. By Theorem 2.3, $F_p^{***} = (f, \phi, g)$ satisfies the following normalization conditions:

$$\begin{cases} f_j = z_j + \dfrac{i\mu_j}{2} z_j w + o_{wt}(3), \quad \dfrac{\partial^2 f_j}{\partial w^2}(0) = 0, \ j = 1\cdots,\kappa_0, \ \mu_j > 0, \\ f_j = z_j + o_{wt}(3), \quad j = \kappa_0+1,\cdots,n-1 \\ g = w + o_{wt}(4), \\ \phi_{jl} = \mu_{jl} z_j z_l + o_{wt}(2), \ \text{where } \ (j,l)\in\mathcal{S} \text{ with } \ \mu_{jl} > 0 \text{ for } \ (j,l)\in\mathcal{S}_0 \\ \qquad \text{and } \mu_{jl} = 0 \text{ otherwise} \end{cases} \tag{76}$$

where $\mu_{jl} = \sqrt{\mu_j + \mu_l}$ for $j,l \le \kappa_0$ $j \ne l$, $\mu_{jl} = \sqrt{\mu_j}$ if $j \le \kappa_0$ and $l > \kappa_0$ or if $j = l \le \kappa_0$. Here the assumption that $\kappa_0 > 0$ is used.

From (75) we obtain

$$\begin{array}{ccc} (M, F(p)) & \xrightarrow{H\circ\tau_p^F} & \big(H\circ\tau_p^F(M), 0\big) \\ \uparrow F & & \uparrow F_p^{***} \\ (\partial\mathbb{H}^{n+1}, p) & \xleftarrow{\sigma_p^0\circ G} & (\partial\mathbb{H}^{n+1}, 0) \end{array}$$

If we can show that there exists a first-order adapted lift e from the submanifold $H\circ\tau_p^F(M)$ near 0 into $SU(N+1,1)$ such that the corresponding CR second fundamental form

$$II^e_{H\circ\tau_p^F(M)} \ne 0 \ at\ 0, \tag{77}$$

then we obtain a first-order adapted lift $\widetilde{e} := \left(H\circ\tau_p^F\right)^{-1}\circ e\circ H\circ\tau_p^F$ from the submanifold M near $F(p)$ into $GL^Q(\mathbb{C}^{N+1})$ such that the corresponding CR second fundamental form

$$II_M^{\widetilde{e}} \neq 0 \ at \ F(p). \tag{78}$$

Notice that the map $H\circ\tau_p^F \in GL^Q(\mathbb{C}^{N+2})$ but $H\circ\tau_p^F \notin SU(N+1,1)$, so that the lift $\widetilde{e}$ is not from M into $SU(N+1,1)$. This is why we have to introduce Definition 3.

Since we take arbitrary $p\in\partial\mathbb{H}^{n+1}$, from (78) it concludes that $II_M \not\equiv 0$, but this is a desired contradiction.

Step 3. Calculation of the second fundamental form It remains to prove existence of the lift e such that (77) holds.

The lift e constructed in the second half of Section 7 is a first-order adapted lift from $H\circ\tau_p^F(M)$ near 0 into $SU(N+1,1)$ which defines a CR second fundamental form as a tensor $II^e_{H\circ\tau_p^F(M)} = q^\mu_{\alpha\beta}\omega^\alpha\omega^\beta\otimes(\underline{e_\mu})$ in (45). If we can show

$$q^\mu_{\alpha\beta}(0) = \frac{\partial^2\phi_\mu}{\partial z_\alpha\partial z_\beta}\bigg|_0, \tag{79}$$

where $F_p^{***} = (f,\phi,g) = (f_\alpha,\phi_\mu,g)$. Since we assume that $\kappa_0 > 0$, by (76) and (79), it implies $q^\mu_{\alpha\beta}(0)\neq 0, \forall\alpha,\beta$ and μ, i.e., $II^e_{H\circ\tau_p^F(M)}\neq 0$. This proves (77).

Let $E = (e_0, e_\alpha, \hat{E}_\mu, e_{N+1})$ be the lift constructed in Theorem 7.1 (see the remark at the end of the proof of Theorem 7.1) and in (70) (71) and (72). Since $E|_0 = Id$, we have

$$\omega|_0 = (E^{-1}|_0)(dE)|_0 = dE|_0$$

so that

$$\omega|_0 = \begin{bmatrix} 0 & * & \dots & * \\ dz_1 & * & \dots & * \\ \vdots & \vdots & & \vdots \\ dz_n & * & \dots & * \\ * & * & \dots & * \\ \vdots & \vdots & & \vdots \\ * & * & \dots & * \\ dw & * & \dots & * \end{bmatrix}\Bigg|_0 .$$

Hence $\omega_0^1|_0 = dz_1,\ \dots,\ \omega_0^n|_0 = dz_n,\ \omega_0^{N+1}|_0 = dz_{N+1}$. Then by applying the chain rule, we obtain

$$\omega^\mu_j|_0 = dE^\mu_j|_0 = d\big((L_j\phi_\mu)\circ(F_{fg})^{-1}\big)\big|_0 = \frac{\partial}{\partial z_k}\big((L_j\phi_\mu)\circ(F_{fg})^{-1}\big)|_0 dz_k = \frac{\partial^2\phi_\mu}{\partial z_k\partial z_j}|_0\omega_0^k|_0,$$

for any $j, k \in \{1, 2, ..., n, N+1\}$, $n+1 \leq \mu \leq N$. Hence (79) is proved. The proof of Theorem 1.1 is complete. $\square$

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