Science in China Series A: Mathematics
2009, Vol. 52, No. , 1–17
www.SciChina.com
www.springerlink.com

Classification of Rational Holomorphic Maps from $\mathbb{B}^2$ into $\mathbb{B}^N$ with Degree 2

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Abstract Rational proper holomorphic maps from the unit ball in $\mathbb{C}^2$ into the unit ball $\mathbb{C}^N$ with degree 2 are classified, up to automorphisms of balls.

Keywords: rational holomorphic maps between balls, classification, maps with degree 2

MSC(2000): 32H02

1 Introduction

Denote by $Prop(\mathbb{B}^n, \mathbb{B}^N)$ the space of proper holomorphic maps from the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ into the unit ball $\mathbb{B}^N \subset \mathbb{C}^N$, $Prop_k(\mathbb{B}^n, \mathbb{B}^N) := Prop(\mathbb{B}^n, \mathbb{B}^N) \cap C^k(\overline{\mathbb{B}^n})$ and $Rat(\mathbb{B}^n, \mathbb{B}^N) := Prop(\mathbb{B}^n, \mathbb{B}^N) \cap \{rational\ maps\}$. We recall that $F, G \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ are said to be *equivalent* if there are automorphisms $\sigma \in Aut(\mathbb{B}^n)$ and $\tau \in Aut(\mathbb{B}^N)$ such that $F = \tau \circ G \circ \sigma$. In this paper, we study the classification problem for elements in $Rat(\mathbb{B}^2, \mathbb{B}^N)$ with degree two. For an element F in $Rat(\mathbb{B}^2, \mathbb{B}^N)$, there is a naturally associated invariant $Rk_F \leqslant 1$, called the geometric rank of the map (for the definition, see §2). Since F is linear if and only if its geometric rank $Rk_F = 0$, we only need to consider maps with geometric rank $Rk_F = 1$. By using Cayley transformation $\rho_k : \mathbb{H}^k \to \mathbb{B}^k$ where $\mathbb{H}^k$ is the Siegel upper-half space (see § 2), studying $Rat(\mathbb{B}^2, \mathbb{B}^N)$ is equivalent to studying $Rat(\mathbb{H}^2, \mathbb{H}^N)$.

Making use of results obtained in the previous work [8] [1], we give a complete description for the modular space for maps in $Rat(\mathbb{B}^2, \mathbb{B}^N)$ with degree $\leqslant 2$ under the above mentioned equivalence relation. Our main result is the following Theorem 1.1. Notice that when $N = 3$, $Rat(\mathbb{B}^2, \mathbb{B}^3)$ has been classified by Faran [4]; and when $N = 4$, a complete list of monomial maps in $Rat(\mathbb{B}^2, \mathbb{B}^4)$ has been given by D'Angelo [3].

Theorem 1.1. *(i) Any nonlinear map in $Rat(\mathbb{B}^2, \mathbb{B}^N)$ with degree 2 is equivalent to a map $(F, 0)$ where $F \in Rat(\mathbb{B}^2, \mathbb{B}^5)$ is of one of the following forms:*
(I): $F = (G_t, 0)$ where $G_t \in Rat(\mathbb{B}^2, \mathbb{B}^4)$ is defined by

$$G_t(z, w) = (z^2, \sqrt{1 + \cos^2 t}\ zw, (\cos t)w^2, (\sin t)w), \quad 0 \leqslant t < \pi/2. \tag{1}$$

(IIA): $F = (F_\theta, 0)$ where $F_\theta \in Rat(\mathbb{B}^2, \mathbb{B}^4)$ is defined by

$$F_\theta(z, w) = (z, (\cos\theta)w, (\sin\theta)zw, (\sin\theta)w^2), \quad 0 < \theta \leqslant \frac{\pi}{2}. \tag{2}$$

(IIC): $F = F_{c_1,c_3,e_1,e_2} = \rho_5^{-1} \circ F \circ \rho_2 = (f, \phi_1, \phi_2, \phi_3, g) \in Rat(\mathbb{H}^2, \mathbb{H}^5)$ is of the form:

$$\begin{aligned} f &= \frac{z + (\frac{i}{2} + ie_1)zw}{1 + ie_1 w + e_2 w^2},\ \phi_1 = \frac{z^2}{1 + ie_1 w + e_2 w^2}, \\ \phi_2 &= \frac{c_1 zw}{1 + ie_1 w + e_2 w^2},\ \phi_3 = \frac{c_3 w^2}{1 + ie_1 w + e_2 w^2},\ g = \frac{w + ie_1 w^2}{1 + ie_1 w + e_2 w^2}, \end{aligned}$$

where $c_1, c_3 > 0, -e_1, -e_2 \geqslant 0$, $e_1 e_2 = c_3^2$, $-e_1 - e_2 = \frac{1}{4} + c_1^2$, satisfying one of the following conditions: either

$$\begin{cases} e_1 = \frac{-(\frac{1}{4}+c_1^2) - \sqrt{(\frac{1}{4}+c_1^2)^2 - 4c_3^2}}{2},\ e_2 = \frac{-(\frac{1}{4}+c_1^2) + \sqrt{(\frac{1}{4}+c_1^2)^2 - 4c_3^2}}{2}, \\ 0 < 4c_3^2 \leqslant (\frac{1}{4} + c_1^2)^2, \end{cases} \tag{3}$$

Received March 7, 2008; accepted June 6, 2008
DOI:
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or

$$\left\{\begin{array}{l} e_1 = \frac{-(\frac{1}{4}+c_1^2)+\sqrt{(\frac{1}{4}+c_1^2)^2-4c_3^2}}{2},\ e_2 = \frac{-(\frac{1}{4}+c_1^2)-\sqrt{(\frac{1}{4}+c_1^2)^2-4c_3^2}}{2}, \\ \frac{1}{2}c_1^2 + c_1^4 \leqslant 4c_3^2 \leqslant (\frac{1}{4}+c_1^2)^2. \end{array}\right. \tag{4}$$

(ii) Any two maps in $Rat(\mathbb{B}^2, \mathbb{B}^5)$ *in the form of types (I), (IIA), and (IIC) above are equivalent if and only if they are identical.*

Next, we give a review on the development of this problem and outline the proof for Theorem 1.1 as follows. For some notations to be used, we refer the reader to §2.

• **A result obtained in [8]** A classification result was proved in the last section of [8] under the action of the isotropic automorphism groups of the Heisenberg hypersurfaces, which gives in particular the following: Any map F in $Rat(\mathbb{H}^2, \mathbb{H}^N)$ with $deg(F) = 2$ is equivalent to a map $(G, 0)$ where $G = (f, \phi_1, \phi_2, \phi_3, g) \in Rat(\mathbb{H}^2, \mathbb{H}^5)$ is of the form (see also Lemma 2.3 below)

$$\begin{array}{l} f(z,w) = \frac{z-2ibz^2+(\frac{i}{2}+ie_1)zw}{1+ie_1w+e_2w^2-2ibz}, \\ \phi_1(z,w) = \frac{z^2+bzw}{1+ie_1w+e_2w^2-2ibz},\ \phi_2(z,w) = \frac{c_2w^2+c_1zw}{1+ie_1w+e_2w^2-2ibz}, \\ \phi_3(z,w) = \frac{c_3w^2}{1+ie_1w+e_2w^2-2ibz},\ g(z,w) = \frac{w+ie_1w^2-2ibzw}{1+ie_1w+e_2w^2-2ibz}, \end{array} \tag{5}$$

where $b, -e_1, -e_2, c_1, c_2, c_3$ are real non-negative numbers satisfying $e_1e_2 = c_2^2 + c_3^2$, $-e_1 - e_2 = \frac{1}{4} + b^2 + c_1^2$, $-be_2 = c_1c_2$, and $c_3 = 0$ if $c_1 = 0$.

Since b and c_2 are determined by c_1, c_3, e_1 and e_2, a map in the form of (5) is determined by c_1, c_3, e_1 and e_2. We denote a map of the form (5) determined by c_1, c_3, e_1 and e_2 to be

$$F_{(c_1,c_3,e_1,e_2)} \in \mathcal{K}. \tag{6}$$

Sometimes we regard a such map $F_{(c_1,c_3,e_1,e_2)}$ as a point: $(c_1, c_3, e_1, e_2) \in \mathcal{K}$. It was unclear in [8] which of the coefficients e_1, e_2, c_1 and c_3 of F are independent parameters.

• **Review of the result in [1]** In [1], by obtaining an extra equation, we got a clearer picture on the maps in (5).

For any $F \in Rat(\mathbb{H}^2, \mathbb{H}^5)$ with $deg(F) = 2$, if the geometric rank of F at the origin is one: $Rk_F(0) = 1$, then by a normalization procedure (see Lemma 2.2 and 2.3 below, or [7][8]), F is equivalent to another map $F^{***} \in Rat(\mathbb{H}^2, \mathbb{H}^5)$ of the form (5). Also we can associate a family of maps $F_p \in Rat(\mathbb{H}^2, \mathbb{H}^5)$ for any $p \in \partial\mathbb{H}^2$. Let us define $\Xi_F := \{p \in \partial\mathbb{H}^2 \mid Rk_{F_p}(0) = 0\}$ to be the set of p at which the geometric rank of F_p at the origin is zero. If $p \notin \Xi_F$, we obtain a normalized map $(F_p)^{***}$ that is of the form (5), and we define a real analytic function $\mathcal{W}(F_p^{***}) = c_1(p)^2 - e_1(p) - e_2(p)$ where $c_1(p), e_1(p)$ and $e_2(p)$ are the coefficients of F_p^{***} as in (5).

The desired extra equation is obtained by moving up p to the extremal value as follows. We choose a sequence of $p_m \in \partial\mathbb{H}^2 - \Xi_F$ such that $Rk_{F_{p_m}}(0) = 1$, $p_m \to p_0 \in \overline{\partial\mathbb{H}^2}$ and $\lim_m \mathcal{W}(F_{p_m}^{***}) = \inf_{p\in\partial\mathbb{H}^2-\Xi_F}\{\mathcal{W}(F_p^{***})\}$.

If $p_0 \in \partial\mathbb{H}^2$, by [1, § 4], we can write

$$F_{p_m}^{***} = (F_{p_0})_{q_m}^{***} \tag{7}$$

where $q_m \in \partial\mathbb{H}^2$ and $q_m \to 0$. Then it implies by [1, Lemma 2.5] that $Rk_{F_{p_0}}(0) = 1$, and that F is equivalent to $F_{p_0}^{***}$ which is of the form (5) and with the minimum property $\mathcal{W}(F_{p_0}^{***}) = \inf_{p\in\partial\mathbb{H}^2-\Xi_F}\mathcal{W}(F_p^{***})$. The minimum property implies the vanishing of derivatives of the function $\mathcal{W}(F_p^{***})$ at p_0, which derives the extra equation.

If $p_0 = \infty$, by [1, § 4] we can similarly write

$$F_{p_m}^{***} = (\tau_\infty \circ F \circ \sigma_\infty)_{q_m}^{***} \tag{8}$$

where $\sigma_\infty \in Aut(\partial\mathbb{B}^2)$, $\tau_\infty \in Aut(\partial\mathbb{B}^5)$, $q_m \in \partial\mathbb{H}^2$ and $q_m \to 0$ so that, by the same argument above, $Rk_{\tau_\infty\circ F\circ\sigma_\infty}(0) = 1$ and that F is equivalent to $(\tau_\infty \circ F \circ \sigma_\infty)^{***}$ which is of the form (5). The minimum property also derives the extra equation.

With the extra equation described above, it was proved in [1] that F is equivalent to another map $F_{c_1,c_3.e_1,e_2}$ $\in \mathcal{K}$ satisfying the property

$$\mathcal{W}\big((F_{c_1,c_3.e_1,e_2})_p^{***}\big) \geqslant \mathcal{W}\big((F_{c_1,c_3.e_1,e_2})_0^{***}\big),\ \forall p \in \partial\mathbb{H}^2 \ near\ 0. \tag{9}$$

and that the new map $F_{c_1,c_3.e_1,e_2}$ is of the form in one of the following types:

(I) $F_{0,0,e_1,e_2} = (f, \phi_1, \phi_2, \phi_3, g)$ is of the form

$$\begin{array}{l} f = \frac{z+(\frac{i}{2}+ie_1)zw}{1+ie_1w+e_2w^2},\ \phi_1 = \frac{z^2}{1+ie_1w+e_2w^2}, \\ \phi_2 = \frac{c_2w^2}{1+ie_1w+e_2w^2},\ \phi_3 = 0,\ g = \frac{w+ie_1w^2}{1+ie_1w+e_2w^2} \end{array} \tag{10}$$

where $e_1e_2 = c_2^2$ and $-e_1 - e_2 = \frac{1}{4}$. Here $e_2 \in [-\frac{1}{4}, 0)$ is a parameter. It then corresponds to the family $\{G_t\}_{0\leqslant t<\pi/2}$ in (1). When $e_2 = -\frac{1}{4}$, $F_{0,0,e_1,e_2}$ corresponds to G_0, i.e. $(z,w) \mapsto (z^2, \sqrt{2}zw, w^2, 0)$; when $e_2 \to 0$, $F_{0,0,e_1,e_2}$ goes to $G_{\pi/2} = F_{\pi/2}$, i.e., $(Z,w) \mapsto (z, zw, w^2)$.

(IIA) $F_{c_1,0,e_1,0} = (f, \phi_1, \phi_2, \phi_3, g)$ is of the form

$$f = \frac{z + (\frac{i}{2} + ie_1)zw}{1 + ie_1 w},\ \phi_1 = \frac{z^2}{1 + ie_1 w},\ \phi_2 = \frac{c_1 zw}{1 + ie_1 w},\ \phi_3 = 0,\ g = w \tag{11}$$

where $-e_1 = \frac{1}{4} + c_1^2$ and $c_1 \in [0, \infty)$ is a parameter. It corresponds to the family $\{F_\theta\}_{0<\theta\leqslant\pi/2}$ in (2). When $c_1 = 0$, $F_{c_1,0,e_1,0}$ corresponds to $F_{\pi/2}$; when $c_1 \to \infty$, $F_{c_1,0,e_1,0}$ goes to the linear map, i.e., $(z,w) \mapsto (z, w, 0)$.

(IIB) $F_{c_1,0,0,e_2} = (f, \phi_1, \phi_2, \phi_3, g)$ is of the form:

$$f = \frac{z + \frac{i}{2}zw}{1 + e_2 w^2},\ \phi_1 = \frac{z^2}{1 + e_2 w^2},\ \phi_2 = \frac{c_1 zw}{1 + e_2 w^2},\ \phi_3 = 0,\ g = \frac{w}{1 + e_2 w^2}, \tag{12}$$

where $-e_2 = \frac{1}{4} + c_1^2$ and $c_1 \in (0, \infty)$ is a parameter. Notice that when $c_1 \to 0$, the map $F_{c_1,0,0,e_2}$ goes to the map G_0, i.e. the one in type (I) when $e_2 = -\frac{1}{4}$.

(IIC) $F_{c_1,c_3,e_1,e_2} = (f, \phi_1, \phi_2, \phi_3, g)$ is of the form:

$$\begin{aligned} &f = \tfrac{z + (\frac{i}{2} + ie_1)zw}{1 + ie_1 w + e_2 w^2},\ \phi_1 = \tfrac{z^2}{1 + ie_1 w + e_2 w^2}, \\ &\phi_2 = \tfrac{c_1 zw}{1 + ie_1 w + e_2 w^2},\ \phi_3 = \tfrac{c_3 w^2}{1 + ie_1 w + e_2 w^2},\ g = \tfrac{w + ie_1 w^2}{1 + ie_1 w + e_2 w^2}, \end{aligned} \tag{13}$$

where $c_1, c_3 > 0, -e_1, -e_2 \geqslant 0,\ \ e_1e_2 = c_3^2,\ \ -e_1 - e_2 = \frac{1}{4} + c_1^2$.

For any map F_{c_1,c_3,e_1,e_2} in one of these four types, we denote F_{c_1,c_3,e_1,e_2}, or $(c_1, c_3, e_1,\ e_2)$, $\in \mathcal{K}_I$, $\mathcal{K}_{IIA}$, $\mathcal{K}_{IIB}$, and $\mathcal{K}_{IIC}$, respectively.

Recall from [1, (33)]

$$F \textit{ can be embedded into } \mathbb{H}^4 \ \Leftrightarrow\ c_3 = 0. \tag{14}$$

Concerning the proof of Theorem 1.1, our main idea to establish following formula (see (33)):

$$\mathcal{W}(F^{***}_{\Gamma(t+\Delta t)}) = \mathcal{W}(F^{***}_{\Gamma(t)}) + [4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2)](\Gamma(t))\Im(q_1(t))\Delta t + o(|\Delta t|). \tag{15}$$

One crucial point is that the term $[4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2)](\Gamma(t))$ is always non-negative so that it allows us to reduce the study of (9) into the study for the term $\Im(q_1(t))$.

We'll prove in Lemma 3.4 below that indeed

$$\textit{there is no map } F \textit{ satisfying both } (9) \textit{ and } (12), \tag{16}$$

and that a map

$$F \textit{ satisfies } (9) \textit{ and } (13) \ \ \Leftrightarrow\ \ F \textit{ satisfies } (13),\ (3) \textit{ and } (4), \tag{17}$$

which proves Theorem 1.1(i). To prove Theorem 1.1(ii), we first prove its local version (see Corollary 4.3). Then we shall find a way to reduce the global problem into the local one.

2 Notation and preliminaries

• Maps between balls Write $\mathbb{H}^n := \{(z,w) \in \mathbf{C}^{n-1} \times \mathbf{C} :\ \mathrm{Im}(w) > |z|^2\}$ for the Siegel upper-half space. Similarly, we can define the space $\mathrm{Rat}(\mathbb{H}^n, \mathbb{H}^N)$, $\mathrm{Prop}_k(\mathbb{H}^n, \mathbb{H}^N)$ and $\mathrm{Prop}(\mathbb{H}^n, \mathbb{H}^N)$ respectively. Since the Cayley transformation

$$\rho_n : \mathbb{H}^n \to \mathbb{B}^n,\ \ \rho_n(z,w) = \left(\frac{2z}{1 - iw},\ \frac{1 + iw}{1 - iw}\right)$$

is a biholomorphic mapping between $\mathbb{H}^n$ and $\mathbb{B}^n$, we can identify a map $F \in \mathrm{Prop}_k(\mathbb{B}^n, \mathbb{B}^N)$ or $\mathrm{Rat}(\mathbb{B}^n, \mathbb{B}^N)$ with $\rho_N^{-1} \circ F \circ \rho_n$ in the space $\mathrm{Prop}_k(\mathbb{H}^n, \mathbb{H}^N)$ or $\mathrm{Rat}(\mathbb{H}^n, \mathbb{H}^N)$, respectively.

Parametrize $\partial\mathbb{H}^n$ by $(z, \overline{z}, u)$ through the map $(z, \overline{z}, u) \to (z, u + i|z|^2)$. In what follows, we will assign the weight of z and u to be 1 and 2, respectively. For a non-negative integer m, a function $h(z, \overline{z}, u)$ defined over a small ball U of 0 in $\partial\mathbb{H}^n$ is said to be of quantity $o_{wt}(m)$ if $\frac{h(tz, t\overline{z}, t^2u)}{|t|^m} \to 0$ uniformly for (z,u) on any compact subset of U as $t (\in \mathbb{R}) \to 0$.

• Partial normalization of F Let $F = (f, \phi, g) = (\widetilde{f}, g) = (f_1, \cdots, f_{n-1}, \phi_1, \cdots, \phi_{N-n}, g)$ be a non-constant C^2-smooth CR map from $\partial\mathbb{H}^n$ into $\partial\mathbb{H}^N$ with $F(0) = 0$. For each $p \in \partial\mathbb{H}^2$, we write $\sigma_p^0 \in \mathrm{Aut}(\mathbb{H}^n)$ and $\tau_p^F \in \mathrm{Aut}(\mathbb{H}^N)$ for the maps

$$\begin{aligned} &\sigma_p^0(z,w) = (z + z_0, w + w_0 + 2i\langle z, \overline{z_0}\rangle), \\ &\tau_p^F(z^*, w^*) = (z^* - \widetilde{f}(z_0, w_0), w^* - \overline{g(z_0, w_0)} - 2i\langle z^*, \overline{\widetilde{f}(z_0, w_0)}\rangle). \end{aligned} \tag{18}$$

F is equivalent to $F_p = \tau_p^F \circ F \circ \sigma_p^0 = (f_p, \phi_p, g_p)$. Notice that $F_0 = F$ and $F_p(0) = 0$. The following is basic for the understanding of the geometric properties of F.

Lemma 2.1. *([6, §2, Lemma 5.3], [7, Lemma 2.0]): Let F be a C^2-smooth CR map from $\partial\mathbb{H}^n$ into $\partial\mathbb{H}^N$, $2 \leqslant n \leqslant N$ with $F(0) = 0$. For each $p \in \partial\mathbb{H}^n$, there is an automorphism $\tau_p^{**} \in Aut_0(\mathbb{H}^N)$ such that $F_p^{**} := \tau_p^{**} \circ F_p$ satisfies the following normalization:*

$$f_p^{**} = z + \frac{i}{2} a_p^{**(1)}(z) w + o_{wt}(3),\ \ \phi_p^{**} = \phi_p^{**(2)}(z) + o_{wt}(2),\ \ g_p^{**} = w + o_{wt}(4), \tag{19}$$

$$\langle \overline{z}, a_p^{**(1)}(z) \rangle |z|^2 = |\phi_p^{**(2)}(z)|^2.$$

Let $\mathcal{A}(p) = -2i(\frac{\partial^2 (f_p)_l^{**}}{\partial z_j \partial w}|_0)_{1 \leqslant j, l \leqslant (n-1)}$. We call the rank of $\mathcal{A}(p)$, which we denote by $Rk_F(p)$, the *geometric rank* of F at p. $Rk_F(p)$ depends only on p and F, and is a lower semi-continuous function on p. We define the *geometric rank* of F to be $Rk_F := max_{p \in \partial\mathbb{H}^n} Rk_F(p)$. Notice that we always have $0 \leqslant Rk_F \leqslant n-1$. We define the geometric rank of $F \in \mathrm{Prop}_2(\mathbb{B}^n, \mathbb{B}^N)$ to be the one for the map $\rho_N^{-1} \circ F \circ \rho_n \in \mathrm{Prop}_2(\mathbb{H}^n, \mathbb{H}^N)$. It is proved that F is linear fractional if and only if the geometric rank $Rk_F = 0$ (cf. [6, Theorem 4.3]). Hence, in all that follows, we assume that $Rk_F = \kappa_0 \geqslant 1$.

Denote by $\mathcal{S}_0 = \{(j,l) : 1 \leqslant j \leqslant \kappa_0, 1 \leqslant l \leqslant (n-1), j \leqslant l\}$ and write $\mathcal{S} := \{(j,l) :\ (j,l) \in \mathcal{S}_0,\ \text{or}\ \ j = \kappa_0 + 1, l \in \{\kappa_0 + 1, \cdots, \kappa_0 + N - n - \frac{(2n - \kappa_0 - 1)\kappa_0}{2}\}\}$. Then we further have the following normalization for F:

Lemma 2.2. *([7, Lemma 3.2]): Let F be a C^2-smooth CR map from an open piece $M \subset \partial\mathbb{H}^n$ into $\partial\mathbb{H}^N$ with $F(0) = 0$ and $Rk_F(0) = \kappa_0$. Let $P(n, \kappa_0) = \frac{\kappa_0(2n - \kappa_0 - 1)}{2}$. Then $N \geqslant n + P(n, \kappa_0)$ and there are $\sigma \in Aut_0(\partial\mathbb{H}^n)$ and $\tau \in Aut_0(\partial\mathbb{H}^N)$ such that $F_p^{***} = \tau \circ F \circ \sigma := (f, \phi, g)$ satisfies the following normalization conditions:*

$$\begin{cases} f_j = z_j + \dfrac{i\mu_j}{2} z_j w + o_{wt}(3), \ \ \dfrac{\partial^2 f_j}{\partial w^2}(0) = 0,\ j = 1 \cdots, \kappa_0,\ \mu_j > 0, \\ f_j = z_j + o_{wt}(3), \quad j = \kappa_0 + 1, \cdots, n-1, \\ g = w + o_{wt}(4), \\ \phi_{jl} = \mu_{jl} z_j z_l + o_{wt}(2),\ \textit{where}\ \ (j,l) \in \mathcal{S}\ \textit{with}\ \ \mu_{jl} > 0\ \textit{for}\ \ (j,l) \in \mathcal{S}_0 \\ \qquad \textit{and}\ \mu_{jl} = 0\ \textit{otherwise}. \end{cases} \tag{20}$$

Moreover $\mu_{jl} = \sqrt{\mu_j + \mu_l}$ for $j, l \leqslant \kappa_0$ $j \neq l$, $\mu_{jl} = \sqrt{\mu_j}$ if $j \leqslant \kappa_0$ and $l > \kappa_0$ or if $j = l \leqslant \kappa_0$.

Here we denote $Aut_0(\partial\mathbb{H}^n) = \{\psi \in Aut(\partial\mathbb{H}^n) \mid \psi(0) = 0\}$.

• **Degree of a rational map** For a rational holomorphic map $H = \frac{(P_1, ..., P_m)}{Q}$ over $\mathbb{C}^n$, where P_j, Q are holomorphic polynomials and $(P_1, ..., P_m, Q) = 1$, we define

$$deg(H) = max\{deg(P_j),\ 1 \leqslant j \leqslant m,\ deg(Q)\}.$$

For a rational map H and a complex affine subspace S of dimension k, we say that H is linear fractional along S, if S is not contained in the singular set of H and for any linear parametrization $z_j = z_j^0 + \sum_{l=1}^k a_{jl} t_l$ of S with $j = 1, \cdots, n$, $H^*(t_1, \cdots, t_k) := H(z_1^0 + \sum_{l=1}^k a_{1l} t_l, \cdots, z_n^0 + \sum_{l=1}^k a_{jn} t_j)$ has degree 1 in $(t_1, \cdots, t_k)$.

• **Actions of the isotropic groups of the Heisenberg hypersurfaces** Recall from [7, (2.4.1)] and [7, (2.4.2)], we define $\sigma \in Aut_0(\partial\mathbb{H}^2)$ and $\tau^* \in Aut_0(\partial\mathbb{H}^5)$ by

$$\sigma = \frac{(\lambda(z + aw) \cdot U,\ \lambda^2 w)}{q(z,w)},\ \ \tau^*(z^*, w^*) = \frac{(\lambda^*(z^* + a^* w^*) \cdot U^*, \lambda^{*2} w^*)}{q^*(z^*, w^*)}, \tag{21}$$

with $q(z,w) = 1 - 2i\langle \overline{a}, z \rangle + (r - i|a|^2)w$, $\lambda > 0$, $r \in \mathbb{R}$, $a, U \in \mathbb{C}$, $|U| = 1$, and $q^*(z^*, w^*) = 1 - 2i\langle \overline{a^*}, z^* \rangle + (r^* - i|a^*|^2)w^*$, $\lambda^* > 0$, $r^* \in \mathbb{R}$, $a^* = (a_1^*, a_2^*) \in \mathbb{C}^1 \times \mathbb{C}^3$ and U^* is an 4×4 unitary matrix, such that [7, ((2.5.1), (2.5.2)] holds:

$$\lambda^* = \lambda^{-1},\ a_1^* = -\lambda^{-1} a U,\ a_2^* = 0,\ r^* = -\lambda^{-2} r,\ U^* = \begin{pmatrix} U^{-1} & 0 \\ 0 & U_{22}^* \end{pmatrix}, \tag{22}$$

where $a^* = (a_1^*, a_2^*)$, U_{22}^* is an 3×3 unitary matrix. Define $F^* = \tau^*\circ F\circ\sigma$. By [7, Lemma 2.3(A)], we can write

$$\begin{aligned}
&f(z,w) = z + \tfrac{i}{2}zAw + o_{wt}(3),\ \ f^*(z,w) = z + \tfrac{i}{2}zA^*w + o_{wt}(3),\\
&\phi(z,w) = \tfrac{1}{2}z(B^1, B^2, B^3)z + z\mathcal{B}w + \tfrac{1}{2}\tfrac{\partial^2\phi}{\partial w^2}(0)w^2 + o(|(z,w)|^2),\\
&\phi^*(z,w) = \tfrac{1}{2}z(B^{*1}, B^{*2}, B^{*3})z + z\mathcal{B}^*w + \tfrac{1}{2}\tfrac{\partial^2\phi^*}{\partial w^2}(0)w^2 + o(|(z,w)|^2),
\end{aligned}\tag{23}$$

where $B^i = \frac{\partial^2\phi_i}{\partial z^2}(0)$, $B^{*i} = \frac{\partial^2\phi_i^*}{\partial z^2}(0)$ for $i = 1,2,3$ and $\mathcal{B} = (\frac{\partial^2\phi_1}{\partial z\partial w}, \frac{\partial^2\phi_2}{\partial z\partial w}, \frac{\partial^2\phi_3}{\partial z\partial w})$, $\mathcal{B}^* = (\frac{\partial^2\phi_1^*}{\partial z\partial w}, \frac{\partial^2\phi_2^*}{\partial z\partial w}, \frac{\partial^2\phi_3^*}{\partial z\partial w})$. Also, the same computation in [7, Lemma 2.3 (A)] gives the following:

$$\begin{aligned}
&\tfrac{\partial^2 g^*}{\partial z^2}(0) = 0,\ \tfrac{\partial^2 g^*}{\partial z\partial w}(0) = 0,\ \tfrac{\partial^2 g^*}{\partial w^2}(0) = 0,\ \tfrac{\partial^2 f^*}{\partial z^2}(0) = 0,\ \mathcal{A}^* = \lambda^2 U\mathcal{A}U^{-1},\\
&\tfrac{\partial^2 f^*}{\partial w^2}(0) = i\lambda^2 aU\mathcal{A}U^{-1} + \lambda^3\tfrac{\partial^2 f}{\partial w^2}(0)U^{-1},\\
&[B^{*1},\ B^{*2}, B^{*3}] = \lambda U[B^1,\ B^2,\ B^3]U^tU_{22}^*,\\
&\mathcal{B}^* = \lambda U[B^1,\ B^2,\ B^3]U^ta^tU_{22}^* + \lambda^2 U\mathcal{B}U_{22}^*,\\
&\tfrac{\partial^2\phi^*}{\partial w^2}(0) = \lambda aU[B^1,\ B^2,\ B^3]U^ta^tU_{22}^* + 2\lambda^2 aU\mathcal{B}U_{22}^* + \lambda^3\tfrac{\partial^2\phi}{\partial w^2}(0)U_{22}^*.
\end{aligned}\tag{24}$$

Lemma 2.3. *([8, theorem 4.1]) Let $F\in Rat(\partial\mathbb{H}^2, \partial\mathbb{H}^N)$ have degree 2 with $F(0) = 0$ and $Rk_F(0) = 1$ ($N\geqslant 4$). Then*

(1) F is equivalent to $(F^{***}, 0)$ where $F^{***} = (f, \phi_1, \phi_2, \phi_3, g)\in Rat(\partial\mathbb{H}^2, \partial\mathbb{H}^5)$ defined by

$$\begin{aligned}
f(z,w) &= \tfrac{z - 2ibz^2 + (\frac{i}{2} + ie_1)zw}{1 + ie_1w + e_2w^2 - 2ibz},\\
\phi_1(z,w) &= \tfrac{z^2 + bzw}{1 + ie_1w + e_2w^2 - 2ibz},\\
\phi_2(z,w) &= \tfrac{c_2w^2 + c_1zw}{1 + ie_1w + e_2w^2 - 2ibz},\\
\phi_3(z,w) &= \tfrac{c_3w^2}{1 + ie_1w + e_2w^2 - 2ibz},\\
g(z,w) &= \tfrac{w + ie_1w^2 - 2ibzw}{1 + ie_1w + e_2w^2 - 2ibz}.
\end{aligned}\tag{25}$$

Here $b, -e_1, -e_2, c_1, c_2, c_3$ are real non-negative numbers satisfying

$$e_1e_2 = c_2^2 + c_3^2,\ -e_1 - e_2 = \tfrac{1}{4} + b^2 + c_1^2,\ -be_2 = c_1c_2,\ c_3 = 0\ if\ c_1 = 0.\tag{26}$$

(2) $c_1, c_2, c_3, e_1, e_2, b$ are uniquely determined by F. Conversely, for any non-negative real numbers $c_1, c_2, c_3, e_1, e_2, b$ satisfying the relations in (26), the map F defined in (25) is an element in $Rat(\partial\mathbb{H}^2, \partial\mathbb{H}^5)$ of degree 2 with $F(0) = 0$ and $Rk_F(0) = 1$.

(3) If $e_2 = 0$, then F is equivalent to $(F_\theta, 0)$ with F_θ as in (1).

Remarks (i) The new normalized map in Lemma 2.3(1) can be obtained by $F^{***} = \tau^*\circ F^{**}\circ\sigma$ where F^{**} is as in Lemma 2.2 and σ and τ^* are as in (21).
(ii) For any map F in Lemma 2.3(1), $b = \sqrt{-e_1 - e_2 - \frac{1}{4} - c_1^2}$ and $c_2 = \sqrt{e_1e_2 - c_3^2}$ are determined by c_1, c_3, e_1 and e_2. Then c_1, c_3, e_1 and e_2 can be regarded as parameters, and we denote $F = F_{c_1,c_3,e_1,e_2}$.
(iii) We denote by $\mathcal{K}$ a subset of $\mathbb{R}^4$ such that (c_1, c_3, e_1, e_2), or $F_{c_1,c_3,e_1,e_2}\in\mathcal{K}$ if and only if F_{c_1,c_3,e_1,e_2} is a map as above.

Lemma 2.4. *([1, Lemma 2.5]) Let $F\in Rat(\partial\mathbb{H}^2, \partial\mathbb{H}^5)$ with $F(0) = 0$ and $deg(F) = 2$. Suppose that $p_m\in\partial\mathbb{H}^2$ is a sequence converging to $0\in\partial\mathbb{H}^2$ and F_{p_m} is of rank 1 at 0 for any m and $F_{p_m}^{***}$ converges such that $\frac{\partial^2\phi_{1,m}^{***}}{\partial z\partial w}|_0$, $\frac{\partial^2\phi_{2,m}^{***}}{\partial w^2}|_0$, $\frac{\partial^2\phi_{2,m}^{***}}{\partial z\partial w}|_0$ and $\frac{\partial^2\phi_{3,m}^{***}}{\partial w^2}|_0$ are bounded for all m. Then*

(i) F is of rank 1 at 0.

(ii) $F_{p_m}^{***}\to F^{***}$.

(iii) If we write $F_{p_m}^{***} = G_{2,m}\circ\tau_{p_m}\circ F\circ\sigma_{p_m}\circ G_{1,m}$ where σ_{p_m} and $\tau_{p_m} := \tau_{p_m}^F$ are as in (18), $G_{1,m}$ and $G_{2,m}$ are as in (21), then $G_{1,m}$ and $G_{2,m}$ are convergent to some $G_1\in Aut_0(\partial\mathbb{H}^2)$ and $G_2\in Aut_0(\partial\mathbb{H}^5)$ respectively.

Let F be as in Lemma 2.3 (1). By Lemma 2.3, F_p is equivalent to a map of the following form $F_p^{***} = (f_p^{***}, \phi_{1,p}^{***}, \phi_{2,p}^{***}, \phi_{3,p}^{***}, g_p^{***})$ for any $p\in\partial\mathbb{H}^2$ where $Rk_F(p) = 1$:

$$f_p^{***}(z,w) = \frac{z - 2ib(p)z^2 + (\frac{i}{2} + ie_1(p))zw}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z},$$

$$\phi_{1,p}^{***}(z,w) = \frac{z^2 + b(p)zw}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z},$$
$$\phi_{2,p}^{***}(z,w) = \frac{c_2(p)w^2 + c_1(p)zw}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z},$$

$$\phi_{3,p}^{***}(z,w) = \frac{c_3(p)w^2}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z},$$
$$g_p^{***}(z,w) = \frac{w + ie_1(p)w^2 - 2ib(p)zw}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z}.$$

Here $b(p), e_1(p), e_2(p), c_1(p), c_2(p), c_3(p)$ satisfy $e_2(p)e_1(p) = c_2^2(p) + c_3^2(p)$, $-e_2(p) = \frac{1}{4} + e_1(p) + b^2(p) + c_1^2(p)$, and $-b(p)e_2(p) = c_1(p)c_2(p)$, $c_3(p) = 0$ if $c_1(p) = 0$, with $c_1(p), c_2(p), b(p) \geqslant 0$, $e_2(p), e_1(p) \leqslant 0$.

Lemma 2.5. *Let F and F_p^{***} be as above. Let $p = (z_0, w_0) = (z_0, u_0 + i|z_0|^2) \in \partial\mathbb{H}^2$ near 0. Then the followings hold.*

(i) The real analytic functions have the formulas

$$b^2(p) = b^2 - 4b(2e_1 + c_1^2)\Im(z_0) + o(1),$$
$$c_1^2(p) = c_1^2 + 4c_1(bc_1 + 2c_2)\Im(z_0) + o(1),$$
$$e_2(p) + e_1(p) = e_2 + e_1 + 8b(e_1 + e_2)\Im(z_0) + o(1),$$
$$c_1^2(p) - e_1(p) - e_2(p) = c_1^2 - e_1 - e_2 + \Big(4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2)\Big)\Im(z_0) + o(1)$$

where we denote $o(k) = o(|(z_0, u_0)|^k)$.

(ii) If $c_1 > 0$, the real analytic function has the formula

$$c_3^2(p) = c_3^2 + 4(c_3)^2(5b - \frac{2c_2}{c_1})\Im(z_0) + o(1),$$

(iii) If $c_1 = 0$, then $c_3(p) \equiv 0$.

Proof: (1) All these formulas were proved in [1, lemma 3.1].

(ii) We use the formulas in [1, Step 3 and 4, § 5] and the notation to obtain

$$c_3^2 = \left|\frac{1}{2}\frac{\partial^2 \phi_{p3}^{***}}{\partial w^2}(0)\right|^2 = \left|\frac{1}{2}\frac{\partial^2 \phi_{pe3}^{**}}{\partial w^2}(0)\right|^2 = c_3^2 + 4(c_3)^2(5b - \frac{2c_2}{c_1})\Im(z_0) + o(1).$$

(iii) If $c_1 = 0$, by Lemma 2.3, $c_3 = 0$ and $F \in Rat(\mathbb{H}^2, \mathbb{H}^4)$. We modify slightly on the normalization F^{***} so that $\phi_{p3}^{***} \equiv 0$ and hence $c_3(p) \equiv 0$. $\square$

3 A Monotone Lemma

Recall that for any $(c_1, c_3, e_1, e_2) \in \mathcal{K}$, we denote

- $(c_1, c_3, e_1, e_2) \in \mathcal{K}_I$ (i.e. F_{c_1,c_3,e_1,e_2} is of the form of type (I)) if $c_1 = 0$ and $b = 0$;
- $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{II}$ (i.e. F_{c_1,c_3,e_1,e_2} is of the form of type (II)) if $c_1 > 0$ and $b = c_2 = 0$.

Also recall that for any map $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{II}$, we denote

- $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{IIA}$ (i.e. F_{c_1,c_3,e_1,e_2} is of the form of type (IIA)) if $c_1 > 0$, $b = c_2 = 0$ and $c_3 = e_2 = 0$;
- $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{IIB}$ (i.e. F_{c_1,c_3,e_1,e_2} is of the form of type (IIB)) if $c_1 > 0$, $b = c_2 = 0$ and $c_3 = e_1 = 0$;
- $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{IIC}$ (i.e. F_{c_1,c_3,e_1,e_2} is of the form of type (IIC)) if $c_1 > 0$, $b = c_2 = 0$ and $c_3 > 0$.

For any $(c_1, c_3, e_1, e_2) \in \mathcal{K}_I \cup \mathcal{K}_{II}$, we denote

- $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{I,II,1+4e_2+2c_1^2>0}$, if $1 + 4e_2 + 2c_1^2 > 0$;
- $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{I,II,1+4e_2+2c_1^2=0}$, if $1 + 4e_2 + 2c_1^2 = 0$;
- $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{I,II,1+4e_2+2c_1^2<0}$, if $1 + 4e_2 + 2c_1^2 < 0$.

For any $F_{c_1,c_3,e_1,e_2} \in \mathcal{K}$, we define $\mathcal{W}(F_{c_1,c_3,e_1,e_2}) := \mathcal{W}(c_1,c_3,e_1,e_2) := c_1^2 - e_1 - e_2$. We also consider curves

$$\Gamma(t) = (\alpha t, \beta_1 t + i|\alpha|^2 t^2) \in \partial\mathbb{H}^2, \quad \forall t \in [0,1], \quad |\alpha| \leqslant 1 \text{ and } |\beta_1| \leqslant 1 \tag{27}$$

where $\alpha = \alpha_1 + i\alpha_2, \alpha_j, \beta_1$ are real numbers.

Lemma 3.1. *Let Γ be any curve as in (27).*

(a) If $(c_1,c_3,e_1,e_2) \in \mathcal{K}_{I,II,1+4e_2+2c_1^2>0}$, then there exists $\delta = \delta(\Gamma) > 0$ such that

$$\mathcal{W}((F_{c_1,c_3,e_1,e_2})^{***}_{\Gamma(t_1)}) \leqslant \mathcal{W}((F_{c_1,c_3,e_1,e_2})^{***}_{\Gamma(t_2)}), \ \forall\ 0 \leqslant t_1 < t_2 \leqslant \delta. \tag{28}$$

(b) If $(c_1,c_3,e_1,e_2) \in \mathcal{K}_{I,II,1+4e_2+2c_1^2=0}$, then there exists $\delta = \delta(\Gamma) > 0$ such that

$$\mathcal{W}((F_{c_1,c_3,e_1,e_2})^{***}_{\Gamma(t)}) \equiv constant, \ \forall t. \tag{29}$$

(c) If $(c_1,c_3,e_1,e_2) \in \mathcal{K}_{I,II,1+4e_2+2c_1^2<0}$, then there exists $\delta = \delta(\Gamma) > 0$ such that

$$\mathcal{W}((F_{c_1,c_3,e_1,e_2})^{***}_{\Gamma(t_1)}) \geqslant \mathcal{W}((F_{c_1,c_3,e_1,e_2})^{***}_{\Gamma(t_2)}), \ \forall\ 0 \leqslant t_1 < t_2 \leqslant \delta. \tag{30}$$

Proof of Lemma 3.1: **Step a. The basic setup** The monotonicity (28) in (a) means

$$\frac{d\mathcal{W}(F^{***}_{\Gamma(t)}))}{dt} = \lim_{\Delta t \to 0} \frac{\mathcal{W}(F^{***}_{\Gamma(t+\Delta t)}) - \mathcal{W}(F^{***}_{\Gamma(t)})}{\Delta t} \geqslant 0, \ \forall t \in [0,\delta]. \tag{31}$$

For any $0 < t < \delta$ and sufficiently small $\Delta t > 0$, if we can write

$$F^{***}_{\Gamma(t+\Delta t)} = \left(F^{***}_{\Gamma(t)}\right)^{***}_{q(t,\Delta t)} \tag{32}$$

for some differentiable map $q(t,\Delta t) \in \partial\mathbb{H}^2$, then from Lemma 2.5 we should have

$$\mathcal{W}(F^{***}_{\Gamma(t+\Delta t)}) = \mathcal{W}(F^{***}_{\Gamma(t)}) + \left[4c_1(bc_1 + 2c_2) - 8b(e_1+e_2)\right](\Gamma(t))\Im(q_1(t))\Delta t + o(|\Delta t|), \tag{33}$$

where we write $q(t,\Delta t) := (q_1(t), q_2(t))\Delta t + o(|\Delta t|)$. Notice that $[4c_1(bc_1+2c_2) - 8b(e_1+e_2)](\Gamma(t)) \geqslant 0$ always holds because $c_1, c_2, -e_1 - e_2 \geqslant 0$. Then (31) follows if $\Im(q_1(t)) \geqslant 0$ holds. In particular, if $[4c_1(bc_1+2c_2) - 8b(e_1+e_2)](\Gamma(t)) \neq 0$ for any fixed $t \in [0,\delta)$, and if the following condition is satisfied:

$$\Im(q_1(t)) > 0, \qquad \forall t \in [0,\delta], \tag{34}$$

then the strict inequality (31) holds. To prove (31), it suffices to prove (34).

Step b. $\Gamma(t)$ determines $q(t,\Delta t)$ To prove (32), we define $q(t,\Delta t)$ by

$$\Gamma(t+\Delta t) = \sigma_{\Gamma(t)} \circ G_1(q(t,\Delta t)) \tag{35}$$

where $G_1 = G_1(t) \in Aut_0(\partial\mathbb{H}^2)$ and $G_2 \in Aut_0(\partial\mathbb{H}^5)$ are defined such that

$$(F_{\Gamma(t)})^{***} = G_2 \circ \tau^F_{\Gamma(t)} \circ F \circ \sigma_{\Gamma(t)} \circ G_1. \tag{36}$$

By (35), $q(t,\Delta t)$ is a function uniquely determined by $\Gamma(t)$ given by

$$q(t,\Delta t) = G_1^{-1} \circ \sigma^{-1}_{\Gamma(t)} \circ \Gamma(t+\Delta t). \tag{37}$$

The definition (37) will be justified in Step c. Here we derive a formula (39).

By the definition of σ (see (18)),

$$\sigma^{-1}_{\Gamma(t)}(z,w) = (z - z(t), w - w(t) - 2i\langle z, \overline{z(t)}\rangle + 2i|z(t)|^2),$$

and

$$\begin{aligned}\Gamma(t+\Delta t) &= \Big(\alpha(t+\Delta t), \beta_1(t+\Delta t) + i|\alpha|^2(t^2 + 2t\Delta t + \Delta t^2)\Big) \\ &= \Gamma(t) + (\alpha,\ \beta_1 + i|\alpha|^2(2t+\Delta t))\Delta t = \Gamma(t) + (\alpha\Delta t, (\beta_1 + 2i|\alpha|^2 t)\Delta t) + o(|\Delta t|).\end{aligned} \tag{38}$$

Then

$$\sigma^{-1}_{\Gamma(t)} \circ \Gamma(t+\Delta t) = (\alpha\Delta t,\ \beta_1\Delta t) + o(|\Delta t|).$$

We denote $G_1 \in Aut_0(\partial\mathbb{H}^2)$ as in (21), and we have

$$G_1(z,w) = \left(\frac{\lambda(z+\vec{a}w)U}{1-2i\langle\vec{a},z\rangle-(r+i|\vec{a}|^2)w},\ \frac{\lambda^2 w}{1-2i\langle\vec{a},z\rangle-(r+i|\vec{a}|^2)w}\right)$$

where $U = U(t) = e^{i\theta}$, $\theta = \theta(t) \in \mathbb{R}$, $\lambda = \lambda(t) > 0$ and $\vec{a} = \vec{a}(t) \in \mathbb{C}$, and $r = r(t) \in \mathbb{R}$, and

$$G_1^{-1}(z^*,w^*) = \left(\frac{\frac{1}{\lambda}(z-\frac{\vec{a}}{\lambda}Uw)U^{-1}}{1+2i\langle\frac{\vec{a}}{\lambda}U,z\rangle+(\frac{1}{\lambda^2}r-i|\frac{\vec{a}}{\lambda}|^2)w},\ \frac{\frac{1}{\lambda^2}w}{1+2i\langle\frac{\vec{a}}{\lambda}U,z\rangle+(\frac{1}{\lambda^2}r-i|\frac{\vec{a}}{\lambda}|^2)w}\right).$$

Therefore

$$\begin{aligned} q(t,\Delta t) &= G_1^{-1}\circ\sigma_{\Gamma(t)}^{-1}\circ\Gamma(t+\Delta t) = G_1^{-1}(\alpha\Delta t,\ \beta_1\Delta t)+o(|\Delta t|) \\ &= \left(\frac{1}{\lambda^2}(\lambda\alpha U^{-1}-\vec{a}\beta_1),\ \frac{1}{\lambda^2}\beta_1\right)\Delta t + o(|\Delta t|). \end{aligned}$$

By using the notation in (34), we have

$$\Im(q_1(t)) = \frac{1}{\lambda(t)^2}\Im\left(\lambda(t)\alpha U(t)^{-1}-\vec{a}(t)\beta_1\right). \tag{39}$$

Step c. The identity We want to prove that the identity (32) holds:

$$(F_{\Gamma(t+\Delta t)})^{***} = \left(\left((F_{\Gamma(t)})^{***}\right)_{q(t,\Delta t)}\right)^{***}, \tag{40}$$

for sufficiently small t and Δt, i.e., to prove the following identity

$$G_4\circ\tau^F_{\Gamma(t+\Delta t)}\circ F\circ\sigma_{\Gamma(t+\Delta t)}\circ G_3 = G_6\circ\tau^F_q\circ\left(G_2\circ\tau^F_{\Gamma(t)}\circ F\circ\sigma_{\Gamma(t)}\circ G_1\right)\circ\sigma_{q(t,\Delta t)}\circ G_5. \tag{41}$$

Here by abusing of notion, we still use τ^F_q to denote τ^H_q where $H = (F_{\Gamma(t)})^{***}$. Notice that $G_1, G_5, G_3 \in Aut_0(\partial\mathbb{H}_2)$, $\sigma_{\Gamma(t)}$, σ_q, $\sigma_{\Gamma(t+\Delta t)} \in Aut(\partial\mathbb{H}_2)$, and $G_2, G_6, G_4 \in Aut_0(\partial\mathbb{H}_5)$, $\tau^F_{\Gamma(t)}$, τ^F_q, $\tau^F_{\Gamma(t+\Delta t)} \in Aut(\partial\mathbb{H}_5)$ are uniquely determined by F, $\Gamma(t), q$ and $\Gamma(t+\Delta t)$ in the normalization process, respectively.

If we can write

$$\left(\left((F_{\Gamma(t)})^{***}\right)_{q(t,\Delta t)}\right)^{***} = B\circ(F_{\Gamma(t+\Delta t)})^{***}\circ A \tag{42}$$

for some $A \in Aut_0(\partial\mathbb{H}^2)$ and $B \in Aut_0(\partial\mathbb{H}^5)$, then (40) holds by Lemma 2.3(2).

In fact, we write

$$\begin{aligned} &\left(\left((F_{\Gamma(t)})^{***}\right)_{q(t,\Delta t)}\right)^{***} \\ &= G_6\circ\tau^F_q\circ\left(G_2\circ\tau^F_{\Gamma(t)}\circ F\circ\sigma_{\Gamma(t)}\circ G_1\right)\circ\sigma_{q(t,\Delta t)}\circ G_5 \\ &= \left(G_6\circ\tau^F_q\circ G_2\circ\tau^F_{\Gamma(t)}\circ(\tau^F_{\Gamma(t+\Delta t)})^{-1}\circ G_4^{-1}\right)\circ\left(G_4\circ\tau^F_{\Gamma(t+\Delta t)}\circ F\circ\sigma_{\Gamma(t+\Delta t)}\circ G_3\right)\circ \\ &\quad\circ\left(G_3^{-1}\circ\sigma^{-1}_{\Gamma(t+\Delta t)}\circ\sigma_{\Gamma(t)}\circ G_1\circ\sigma_{q(t,\Delta t)}\circ G_5\right) \\ &= B\circ(F_{\Gamma(t+\Delta t)})^{***}\circ A \end{aligned}$$

where $B = G_6\circ\tau^F_q\circ G_2\circ\tau^F_{\Gamma(t)}\circ(\tau^F_{\Gamma(t+\Delta t)})^{-1}\circ G_4^{-1}$ and $A = G_3^{-1}\circ\sigma^{-1}_{\Gamma(t+\Delta t)}\circ\sigma_{\Gamma(t)}\circ G_1\circ\sigma_{q(t,\Delta t)}\circ G_5$. Writing $A = G_3^{-1}\circ\left(\sigma^{-1}_{\Gamma(t+\Delta t)}\circ\sigma_{\Gamma(t)}\circ G_1\circ\sigma_{q(t,\Delta t)}\right)\circ G_5$. Notice G_3^{-1}, $G_5 \in Aut_0(\partial\mathbb{H}^2)$. By (35), we know $\sigma^{-1}_{\Gamma(t+\Delta t)}\circ\sigma_{\Gamma(t)}\circ G_1\circ\sigma_{q(t,\Delta t)} \in Aut_0(\partial\mathbb{H}^2)$. Then $A \in Aut_0(\partial\mathbb{H}^2)$. Similarly, we can show $B \in Aut_0(\partial\mathbb{H}^5)$.

Step d. Proof of (a) - the case $\alpha \neq 0$ Let α be as in (39). Suppose $\alpha \neq 0$. By our construction (see [1, Step 3 in § 5]), the vector $\vec{a}$ and the matrix U in (39) are given by

$$\vec{a} = \vec{a}(t) = i\frac{\partial^2 f^{**}_{pb}}{\partial w^2}(0) = i(e_1-2e_2)z_0 + 2ic_1c_2u_0 + (|p|) = i(e_1-2e_2)\alpha t + o(t), \tag{43}$$

$$U = U(t) = \begin{cases} e^{i\theta} = \frac{\partial^2 \phi^{**}_{pe1}}{\partial z \partial w}(0) / \left| \frac{\partial^2 \phi^{**}_{pe1}}{\partial z \partial w}(0) \right|, & if \ \frac{\partial^2 \phi^{**}_{pe1}}{\partial z \partial w}(0) \neq 0, \\ 1, & if \ \frac{\partial^2 \phi^{**}_{pe1}}{\partial z \partial w}(0) = 0, \end{cases} \tag{44}$$

and (see [1, Step 3 in § 5])

$$\begin{aligned} \frac{\partial^2 \phi^{**}_{pe1}}{\partial z \partial w}(0) = \frac{\partial^2 \phi^{**}_{pd1}}{\partial z \partial w}(0) = b - 2ib^3 u_0 - ibe_1 u_0 - 4ib^2 z_0 - \frac{i}{2} b u_0 \\ -iz_0 - 4ie_2 z_0 + 4ic_1 c_2 u_0 - 2ibc_1^2 u_0 - 2ic_1^2 z_0 = -i(1 + 4e_2 + 2c_1^2) z_0 + o(|p|), \end{aligned}$$

where $p = (z_0, w_0) = \Gamma(t) = (\alpha t, \beta_1 t + i|\alpha|^2 t^2) \in \partial \mathbb{H}^2$. Here we used the fact that $b = c_2 c_1 = 0$ because $(c_1, c_3, e_1, e_2) \in \mathcal{K}_I \cup \mathcal{K}_{II}$. Then we obtain

$$\frac{\partial^2 \phi^{**}_{pe1}}{\partial z \partial w}(0) = -i(1 + 4e_2 + 2c_1^2)\alpha t + o(t) \tag{45}$$

Now $1 + 4e_2 + 2c_1^2 > 0$. Since $\alpha \neq 0$, we have $\frac{\partial^2 \phi^{**}_{pe1}}{\partial z \partial w}(0) \neq 0$ by (45) so that $\vec{a}$, U^{-1} and q_1 are real analytic neat 0 from their construction (cf. [1]). Then

$$U(t)^{-1} = e^{-i\theta} = \frac{\overline{\frac{\partial^2 \phi^{**}_{pe1}}{\partial z \partial w}(0)}}{|\frac{\partial^2 \phi^{**}_{pe1}}{\partial z \partial w}(0)|} = \frac{i(1 + 4e_2 + 2c_1^2)\overline{\alpha} t + o(|t|)}{|\frac{\partial^2 \phi^{**}_{pe1}}{\partial z \partial w}(0)|} = \frac{i(1 + 4e_2 + 2c_1^2)\overline{\alpha}}{|(1 + 4e_2 + 2c_1^2)\overline{\alpha}|} + O(|t|).$$

and there exists a constant $\delta > 0$ such that

$$\begin{aligned} \Im(q_1(t)) = \tfrac{1}{\lambda(t)^2} \Im\Big(\lambda(t)\alpha U(t)^{-1} - \vec{a}(t)\beta_1\Big) = \tfrac{1}{\lambda(t)} \Im\Big(\alpha U(t)^{-1}\Big) + O(t) \\ = \tfrac{1}{\lambda} \Im\left(\tfrac{i(1+4e_2+2c_1^2)|\alpha|^2}{|(1+4e_2+2c_1^2)\alpha|}\right) + O(|t|) = |\alpha| + O(|t|), \quad \forall t \in [0, \delta] \end{aligned} \tag{46}$$

because $\lambda = \lambda(t) = 1 + O(|t|)$. This proves (34) as well as (28).

Step e. Proof of (a) - the case $\alpha = 0$ Next we will prove (a) for the case $\alpha = 0$. In this case $\Gamma(t) = (0, \beta_1 t)$, and $\Im(q_1(t)) = -\frac{\beta_1}{\lambda(t)^2}\Im(\vec{a}(t))$ and $\vec{a}(t) = i\frac{\partial^2 f^{**}_{pb}}{\partial w^2}(0)$. From [1, § 5, step 3 and step 2], we have $\frac{\partial^2 f^{**}_{pb}}{\partial w^2}(0) = \frac{\partial^2 f^{**}_{p}}{\partial w^2}(0) =$

$$= \frac{1}{\lambda(p)} T^2 \widetilde{f}(p) \cdot \overline{L\widetilde{f}(p)}^t - \frac{1}{\lambda(p)^2}(T\widetilde{f} \cdot \overline{L\widetilde{f}}^t)(T^2 g - 2iT^2\widetilde{f} \cdot \overline{\widetilde{f}}^t - 2i\|T\widetilde{f}\|^2)(p) \tag{47}$$

We want to prove $\vec{a}(t) \equiv 0$ which implies (28). This will be done by direct computation. Write F as in the following form:

$$f = zh + (\frac{i}{2} + ie_1)zwh, \phi_1 = z^2 h, \phi_2 = c_1 zwh, \phi_3 = c_3 w^2 h, g = wh + ie_1 w^2 h,$$

where $h = h(w) = \frac{1}{1 + ie_1 w + e_2 w^2}$. Then

$$h' = (-ie_1 - 2e_2 w)h^2, \ h'' = (-2e_2 - 2e_1^2 + 6ie_1 e_2 w + 6e_2^2 w^2)h^3.$$

From the definition of F_p where $p = (z, w)$, we have [1, § 5]

$$f(p) = zh + (\frac{i}{2} + ie_1)zwh,$$

$$Lf(p) = h + (\frac{i}{2} + ie_1)wh + 2i\overline{z}\Big(zh' + (\frac{i}{2} + ie_1)z(h + wh')\Big),$$

$$Tf(p) = zh' + (\frac{i}{2} + ie_1)z(h + wh'),$$

$$T^2 f(p) = zh'' + (\frac{i}{2} + ie_1)z(2h' + wh''),$$

$$\phi_1(p) = z^2 h, \quad L\phi_1(p) = 2zh + 2i\overline{z}z^2 h', \quad T\phi_1(p) = z^2 h',$$

$$\phi_2(p) = c_1 zwh, \quad L\phi_2(p) = c_1 wh + 2ic_1\overline{z}z(h + wh'), \quad T\phi_2(p) = c_1 z(h + wh'),$$
$$T^2\phi_1(p) = z^2 h'',$$
$$L^2\phi_2(p) = 2ic_1\overline{z}(h + wh') + 2i\overline{z}\Big[c_1(h + wh') + 2ic_1\overline{z}z(2h' + wh'')\Big]$$
$$= 4ic_1\overline{z}(h + wh') - 4c_1\overline{z}^2 z(2h' + wh''),$$

$$T^2\phi_2(p) = c_1 z(2h' + wh''),$$

$$\phi_3(p) = c_3 w^2 h, \quad L\phi_3(p) = 2ic_3\overline{z}(2wh + w^2h'), \quad T\phi_3(p) = c_3(2wh + w^2h'),$$

$$T^2\phi_3(p) = c_3(2h + 2wh' + 2wh' + w^2h'') = c_3(2h + 4wh' + w^2h''),$$

When $p = (0, t)$, we have
$$\lambda(p) = |Lf(p)|^2 + |L\phi_1(p)|^2 + |L\phi_2(p)|^2 + |L\phi_3(p)|^2 = |h(t)|^2 + |c_1 th(t)|^2 = 1 + o(t)$$
and $Tf(p) = T\phi_1(p) = T\phi_2(p) = L\phi_3(p) = T^2f(p) = T^2\phi_1(p) = T^2\phi_2(p) = 0$ so that $(T\widetilde{f}\cdot\overline{L\widetilde{f}}^t)(p) = 0$ and that $(T^2\widetilde{f}\cdot\overline{L\widetilde{f}}^t)(p) = 0$. Hence by (47) we obtain $\Im(q_1(t)) = -\frac{\beta_1}{\lambda(t)^2}\Im(\vec{a}(t)) \equiv 0$. The proof of (a) is complete.

Step f. Proof of (b) and (c) Similarly we can prove (c). To prove (b), we first consider the case when $\alpha \neq 0$. In this case, we can take a sequence of points $(c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)}) \in \mathcal{K}_{IIC,1+4e_2+2c_1^2>0}$ such that $(c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)}) \to (c_1, c_3, e_1, e_2)$. Then (46) holds for such maps $F_{c_1^{(k)},c_3^{(k)},e_1^{(k)},e_2^{(k)}}$:
$$\Im(q_1^{(k)}(t))) = |\alpha| + O(|t|), \qquad \forall t \in [0, \delta] \tag{48}$$
Also, we can take another sequence of points $(\widetilde{c}_1^{(k)}, \widetilde{c}_3^{(k)}, \widetilde{e}_1^{(k)}, \widetilde{e}_2^{(k)}) \in \mathcal{K}_{IIC,1+4e_2+2c_1^2<0}$ such that $(\widetilde{c}_1^{(k)}, \widetilde{c}_3^{(k)}, \widetilde{e}_1^{(k)}, \widetilde{e}_2^{(k)}) \to (c_1, c_3, e_1, e_2)$. Then by letting $k \to \infty$ and the same argument in the proof for (c), we get
$$\Im(\widetilde{q}_1^{(k)}(t))) = -|\alpha| + O(|t|), \qquad \forall t \in [0, \delta] \tag{49}$$
for maps $F_{\widetilde{c}_1^{(k)},\widetilde{c}_3^{(k)},\widetilde{e}_1^{(k)},\widetilde{e}_2^{(k)}}$. Such estimate is uniform for all k. Notice that the function $[4c_1(bc_1+2c_2)-8b(e_1+e_2)](\Gamma(t))\Im(q_1(t))$ in (33) is real analytic but $4c_1(bc_1+2c_2)-8b(e_1+e_2)$ and $\Im(q_1)$ may be not (see Remark (a) following the proof of Lemma 3.1 below). Then by (48) and (49) and by letting $k \to \infty$, we must have
$$[4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2)](\Gamma(t))\Im(q_1(t)) \equiv 0, \quad \forall t \in [0, \delta]$$
for the map F_{c_1,c_3,e_1,e_2} so that $\Im(q_1(t)) \equiv 0$ is proved.

Next we consider the case when $\alpha = 0$, by Step e, we have $\Im(q_1(t)) \equiv 0$ so that (c) is proved $\square$.

Remark (a) We notice that if $1 + 4e_2 + 2c_1^2 = 0$, $\frac{\partial^2\phi^{**}_{pe1}}{\partial z\partial w}(0)$ may be zero so that U and hence U^{-1} may not be differentiable. By the way, $\mathcal{W}(F_p^{***}) = c_1^2(p) - e_1(p) - e_2(p) = \frac{1}{4} + 2c_1^2(p) + b^2(p)$ is real analytic but $c_1(p)$ and $b(p)$ may not be differentiable; this is because of some definitions such as (44) (cf. [1, p.1521-1522]). Then the function $[4c_1(bc_1+2c_2)-8b(e_1+e_2)](\Gamma(t))\Im(q_1(t))$ in (33) is real analytic but $4c_1(bc_1+2c_2)-8b(e_1+e_2)$ and $\Im(q_1)$ may be not.

(b) If we replace the curve $\Gamma(t) = (\alpha t, \beta_1 t + i|\alpha|^2t^2)$ by another curve
$$\Gamma(t) = (\alpha t, \beta_0 + \beta_1 t + i|\alpha|^2 t^2), \tag{50}$$
then (38) and hence (46) holds.

Recall $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{II} \iff$ (5) holds with $c_1 > 0$ and $b = c_2 = 0 \iff c_1 > 0$ and either
$$e_1 = \frac{-(\frac{1}{4} + c_1^2) - \sqrt{(\frac{1}{4} + c_1^2)^2 - 4c_3^2}}{2}, \; e_2 = \frac{-(\frac{1}{4} + c_1^2) + \sqrt{(\frac{1}{4} + c_1^2)^2 - 4c_3^2}}{2}, \tag{51}$$

where $4c_3^2 \leqslant (\frac{1}{4}+c_1^2)^2$, or

$$e_1 = \frac{-(\frac{1}{4}+c_1^2)+\sqrt{(\frac{1}{4}+c_1^2)^2-4c_3^2}}{2},\ e_2 = \frac{-(\frac{1}{4}+c_1^2)-\sqrt{(\frac{1}{4}+c_1^2)^2-4c_3^2}}{2}, \tag{52}$$

where $4c_3^2 \leqslant (\frac{1}{4}+c_1^2)^2$. Here c_1 and c_3 are parameters.

We can write a disjoint union $\mathcal{K}_{II} = \mathcal{K}_{II,e_1<e_2} \cup \mathcal{K}_{II,e_1=e_2} \cup \mathcal{K}_{II,e_1>e_2}$, where

$$\mathcal{K}_{II,e_1<e_2} = \{(c_1,c_3,e_1,e_2) \in \mathcal{K}_{II} \mid e_1 < e_2\}$$

$$\mathcal{K}_{II,e_1=e_2} = \{(c_1,c_3,e_1,e_2) \in \mathcal{K}_{II} \mid e_1 = e_2\},$$

and

$$\mathcal{K}_{II,e_1>e_2} = \{(c_1,c_3,e_1,e_2) \in \mathcal{K}_{II} \mid e_1 > e_2\}.$$

Then $\mathcal{K}_{II,e_1<e_2} = \{(c_1,c_3,e_1,e_2) \in \mathcal{K}_{II} \mid (51)\ \textit{and}\ 4c_3^2 < (\frac{1}{4}+c_1^2)^2\ \textit{hold}\}$, $\mathcal{K}_{II,e_1=e_2} = \{(c_1,c_3,e_1,e_2) \in \mathcal{K}_{II} \mid (51) \textit{or}\ (52)\ \textit{and}\ 4c_3^2 = (\frac{1}{4}+c_1^2)^2\ \textit{hold}\}$, and $\mathcal{K}_{II,e_1<e_2} = \{(c_1,c_3,e_1,e_2) \in \mathcal{K}_{II} \mid (52)\ \textit{and}\ 4c_3^2 < (\frac{1}{4}+c_1^2)^2\ \textit{hold}\}$.

Lemma 3.2. *(i)* $\mathcal{K}_{II,e_1<e_2} \subseteq \mathcal{K}_{I,II,1+4e_2+2c_1^2>0}$, *and* $\mathcal{K}_{II,e_1=e_2} \subseteq \mathcal{K}_{I,II,1+4e_2+2c_1^2>0}$.

(ii) Let $(c_1,c_3,e_1,e_2) \in \mathcal{K}_{II,e_1>e_2}$. *Then*
(a) $(c_1,c_3,e_1,e_2) \in \mathcal{K}_{I,II,1+4e_2+2c_1^2>0}$ *if and only if* $\frac{1}{2}c_1^2+c_1^4 < 4c_3^2 < (\frac{1}{4}+c_1^2)^2$ *holds.*
(b) $(c_1,c_3,e_1,e_2) \in \mathcal{K}_{I,II,1+4e_2+2c_1^2=0}$ *if and only if* $\frac{1}{2}c_1^2+c_1^4 = 4c_3^2$ *holds.*
(c) $(c_1,c_3,e_1,e_2) \in \mathcal{K}_{I,II,1+4e_2+2c_1^2<0}$ *if and only if* $0 \leqslant 4c_3^2 < \frac{1}{2}c_1^2+c_1^4$ *holds.*

Proof of Lemma 3.2: (i) For any $(c_1,c_3,e_1,e_2) \in \mathcal{K}_{II,e_1<e_2} \cup \mathcal{K}_{II,e_1=e_2}$, by $-e_1-e_2 = \frac{1}{4}+c_1^2$ and (51), we have

$$1+4e_2+2c_1^2 = \frac{1}{2}+2e_2-2e_1 = \frac{1}{2}+2\sqrt{\left(\frac{1}{4}+c_1^2\right)^2-4c_3^2} \geqslant \frac{1}{2} > 0.$$

(ii) For any $(c_1,c_3,e_1,e_2) \in \mathcal{K}_{II,e_1>e_2}$, we know that $1+4e_2+2c_1^2 > 0$ is equivalent to $\frac{1}{2}+2e_2-2e_1 = \frac{1}{2}-2\sqrt{(\frac{1}{4}+c_1^2)^2-4c_3^2} > 0$, i.e., $\frac{1}{2}c_1^2+c_1^4 < 4c_3^2$, so that (a) is proved. (b) and (c) are proved similarly. □.

Lemma 3.3. *Let* $\mathcal{E} := \{(c_1,c_3,e_1,e_2) \in \mathcal{K}_I \cup \mathcal{K}_{II} \mid (F_{c_1,c_3,e_1,e_2})_p^{***} \equiv F_{c_1,c_3,e_1,e_2},\ \ \forall p \in \partial\mathbb{H}^2\ \textit{near}\ 0\}$. *Then* $F_{c_1,c_3,e_1,e_2} \in \mathcal{E}$ *if and only if for any curve* Γ *as in (27),*

$$(4c_1(bc_1+2c_2)-8b(e_1+e_2))(\Gamma(t)) \equiv 0,\ \forall t \in [0,1]. \tag{53}$$

Proof: It is clear

$$F_{c_1,c_3,e_1,e_2} \in \mathcal{E} \iff c_1(p), c_3(p)\ \textit{are constant},\ \forall p \in \partial\mathbb{H}^2\ \textit{near}\ 0. \tag{54}$$

If $F_{c_1,c_3,e_1,e_2} \in \mathcal{E}$, then either $c_1(p) = b(p) = 0$ or $c_1(p) > 0, b(p) = c_2(p) = 0$, $\forall p \in \partial\mathbb{H}^2$ near 0 (i.e., the case (I) or (IIA), (IIB) and (IIC)). Then the equality in (53) holds.

Conversely, suppose that $(4c_1(bc_1+2c_2)-8b(e_1+e_2))(\Gamma(t)) \equiv 0$ for any choice of curve $\Gamma(t)$ and for any (c_1,c_3) in some open subset of $\mathbb{R}^2$. Then $b_1(p) = 0$ and $c_1(p)c_2(p) = 0$, $\forall p \in \partial\mathbb{H}^2$ near 0. If $c_1 \equiv 0$, then by Lemma 2.5(iii), $c_3(p) = 0, \forall p$ so that $F_{c_1,c_3,e_1,e_2} \in \mathcal{E}$. If $c_1(p) > 0$ for any p in some open subset of $\partial\mathbb{H}^2$, then $c_2(p) = 0$, $\forall p$. Then we apply Lemma 2.5(ii) to know

$$c_3^2(p) = c_3^2+4(c_3)^2(5b-\frac{2c_2}{c_1})\Im(z_0)+o(|p|) = c_3^2+o(|p|), \quad \textit{where}\ p = (z_0,w_0) \in \partial\mathbb{H}^2 \tag{55}$$

which implies as in (33) that $c_3(p) = \textit{constant}$, $\forall p$. Also, by (33), from $(4c_1(bc_1+2c_2)-8b(e_1+e_2))(\Gamma(t)) \equiv 0$ it implies $\mathcal{W}((F_{c_1,c_3,e_1,e_2})^{***}_{\Gamma(t)}) = \textit{constant}$, $\forall\Gamma$ and $\forall t$. Then

$$\mathcal{W}((F_{c_1,c_3,e_1,e_2})^{***}_{\Gamma(t)}) = (c_1^2-e_1-e_2)(\Gamma(t)) = (\frac{1}{4}+2c_1^2)(\Gamma(t)) = \textit{constant},$$

which implies that $c_1(\Gamma(t)) = \textit{constant}$ for any $t \in [0,t_0]$, i.e., $c_1 \equiv \textit{constant}$. By (54), we obtain $F_{c_1,c_3,e_1,e_2} \in \mathcal{E}$. Claim (53) is proved. □

Theorem 1.1(i) will follow by Lemma 3.2 and the following lemma.

Lemma 3.4. *Let $(c_1, c_3, e_1, e_2) \in \mathcal{K}_I \cup \mathcal{K}_{II}$. Then F_{c_1,c_3,e_1,e_2} satisfies (9) if and only if $F_{c_1,c_3,e_1,e_2} \in \mathcal{K}^* := \mathcal{K}_I \cup \mathcal{K}_{II} - \mathcal{K}_{I,II,1+4e_2+2c_1^2<0}$.*

Proof: $(\Longleftarrow)$ It follows from Lemma 3.1.

$(\Longrightarrow)$ Take any map $F_{c_1,c_3,e_1,e_2} \in \mathcal{K}_{I,II,1+4e_2+2c_1^2<0}$ satisfying the minimum property (9). We first show that $F_{c_1,c_3,e_1,e_2} \in \mathcal{E}$ where $\mathcal{E}$ was defined in above lemma.

By Step d in the proof of Lemma 3.1, we know that for any curve Γ as in Lemma 3.1, there is $\delta > 0$ such that

$$\Im(q_1(t)) = -|\alpha| + O(|t|), \ \forall t \in [0, \delta].$$

Suppose that F_{c_1,c_3,e_1,e_2} satisfies (9). By (33), it implies $(4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2))(\Gamma(t)) \equiv 0$ for any such curves $\Gamma(t)$ and for any (c_1, c_3) with $0 \leqslant 4c_3^2 \leqslant (\frac{1}{4} + c_1^2)^2$. Then by above lemma, $F_{c_1,c_3,e_1,e_2} \in \mathcal{E}$.

$\mathcal{E} \cap \mathcal{K}_{I,II,1+4e_2+2c_1^2<0}$ is a real analytic set in $\mathcal{K}_{I,II,1+4e_2+2c_1^2<0}$. We claim:

$$\mathcal{E} \cap \mathcal{K}_{I,II,1+4e_2+2c_1^2<0} = \emptyset. \tag{56}$$

Suppose (56) is not true. Then we can take

$$(c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) \in \mathcal{K}_{I,II,1+4e_2+2c_1^2<0} \cap \mathcal{E}. \tag{57}$$

We can take a sequence of points $(c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)}) \in \mathcal{K}_{I,II,1+4e_2+2c_1^2<0} - \mathcal{E}$ such that

$$(c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)}) \to (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}).$$

By our choice of $(c_1^{(k)}, c_3^{(k)}, e_1^{(k)}, e_2^{(k)})$, the corresponding maps $F_{c_1^{(k)},c_3^{(k)},e_1^{(k)},e_2^{(k)}}$ has the property that the associated function $\mathcal{W}\big((F_{c_1^{(k)},c_3^{(k)},e_1^{(k)},e_2^{(k)}})^{***}_{\Gamma(t)}\big)$ is strictly decreasing as t goes from 0 to 1. Then $F_{c_1^{(k)},c_3^{(k)},e_1^{(k)},e_2^{(k)}}$ is equivalent to some map $F_{\tilde{c}_1^{(k)},\tilde{c}_3^{(k)},\tilde{e}_1^{(k)},\tilde{e}_2^{(k)}} \in \mathcal{K}^* = \mathcal{K}_I \cup \mathcal{K}_{II} - \mathcal{K}_{I,II,1+4e_2+2c_1^2<0}$ with the minimum $\mathcal{W}$ value. Since the function value $\mathcal{W}((F_{c_1^{(k)},c_3^{(k)},e_1^{(k)},e_2^{(k)}})^{***}_{\Gamma})$ is decreasing, the sequence of points $(\tilde{c}_1^{(k)}, \tilde{c}_3^{(k)}, \tilde{e}_1^{(k)}, \tilde{e}_2^{(k)})$ is also bounded in $\mathcal{K}$. By taking subsequence, we may assume that $(\tilde{c}_1^{(k)}, \tilde{c}_3^{(k)}, \tilde{e}_1^{(k)}, \tilde{e}_2^{(k)}) \to (\tilde{c}_1^{(0)}, \tilde{c}_3^{(0)}, \tilde{e}_1^{(0)}, \tilde{e}_2^{(0)}) \in \mathcal{K}^*$. Then $F_{c_1^{(0)},c_3^{(0)},e_1^{(0)},e_2^{(0)}}$ is equivalent to $F_{\tilde{c}_1^{(0)},\tilde{c}_3^{(0)},\tilde{e}_1^{(0)},\tilde{e}_2^{(0)}} \in \mathcal{K}^*$, i.e.,

$$F_{\tilde{c}_1^{(0)},\tilde{c}_3^{(0)},\tilde{e}_1^{(0)},\tilde{e}_2^{(0)}} = \left(F_{c_1^{(0)},c_3^{(0)},e_1^{(0)},e_2^{(0)}}\right)^{***}_q \tag{58}$$

for some non zero $q \in \partial\mathbb{H}^2$, by the same argument as in (7) and (8) (or [1, Step 1, § 4]). On the other hand, since $F_{c_1^{(0)},c_3^{(0)},e_1^{(0)},e_2^{(0)}} \in \mathcal{E}$, by the definition of $\mathcal{E}$, (58) cannot occur. This contradiction shows that (57) cannot occur. Thus Claim (56) is proved. $\square$

4 Local version of Theorem 1.1(ii)

For each point $p = (a, b + i|a|^2) \in \partial\mathbb{H}^2$ where $b \in \mathbb{R}$ and $a \in \mathbb{C}$, we denote $\pi(p) = \pi(a, b + i|a|^2) := (|a|, |b|) \in \mathbb{R}^2$. We denote by $\square_c := [0, c] \times [0, c]$ a square and $\triangle_c := \{(x, y) \mid 0 \leqslant x \leqslant c, \ 0 \leqslant y \leqslant x\}$ a triangle inside $\square_c$. Let $\Gamma(t) = (\alpha t, \beta_1 t + i|\alpha|^2 t^2)$ with $t \in [0, 1]$ be line segments, The set $\{\pi(\Gamma(t)) = \pi(\alpha t, \beta_1 t + i|\alpha|^2 t^2) \mid |\alpha| = 1, \ |\beta_1| \leqslant 1, \ \ 0 \leqslant t \leqslant t_0\}$ is equal to $\triangle_{t_0}$. Notice that $\pi(a, b + i|a|^2) \in \triangle_{t_0}$ if and only if there exists such a line segment $\Gamma(t)$ so that $(a, b + i|a|^2) = \Gamma(t^*)$ for some $t^* \in [0, t_0]$.

Lemma 4.1. *For any $P^{(0)} = (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) \in \mathcal{K}^*$, there is a neighborhood U of $P^{(0)}$ in $\mathcal{K}^*$ and a constant $c > 0$ such that for any point $(c_1', c_3', e_1', e_2'), (c_1'', c_3'', e_1'', e_2'') \in U$ with $F_{c_1'',c_3'',e_1'',e_2''} = (F_{c_1',c_3',e_1',e_2'})^{***}_p$ where $p = (a, b + i|a|^2) \in \partial\mathbb{H}^2$, $a \in \mathbb{C}$, $b \in \mathbb{R}$, $|p| := \max\{|a|, |b|\} \leqslant c$, we have*

$$(c_1'', c_3'', e_1'', e_2'') = (c_1', c_3', e_1', e_2'). \tag{59}$$

Proof of Lemma 4.1: **Step 1. Choose U and c** For the point $P^{(0)} \in \mathcal{K}^*$, by Lemma 3.1 and the uniform estimate (46), there exists a neighborhood U of this point and a constant $0 < t_0 < 1$ such that for any point $(c_1', c_3', e_1', e_2') \in U$ and for any curve $\Gamma(t) = \{(\alpha t, \beta_1 t + i|\alpha|^2 t^2)\}$ with $\alpha \in \mathbb{C}$, $\beta_1 \in \mathbb{R}$ with $|\beta_1| \leqslant 1$, $|\alpha| = 1$, $0 \leqslant t \leqslant t_0$, we have the property

$$\mathcal{W}((F_{c_1',c_3',e_1',e_2'})^{***}_{\Gamma(t)}) \ \textit{ is nondecreasing}, \quad \forall t \in [0, t_0]. \tag{60}$$

Since $F_{c_1'',c_3'',e_1'',e_2''}=(F_{c_1',c_3',e_1',e_2'})^{***}_p=H\circ\tau\circ F_{c_1',c_3',e_1',e_2'}\circ\sigma_p\circ G$ where $G\in Aut_0(\partial\mathbb{H}^2)$, $H\in Aut_0(\partial\mathbb{H}^5)$, τ and and σ_p are as in (18), we can write

$$F_{c_1',c_3',e_1',e_2'}=(F_{c_1'',c_3'',e_1'',e_2''})^{***}_q,$$

where $q=G^{-1}(-z_0,-\overline{w_0})$. Since $G(0)=0$ and $G^{-1}(0)=0$, by continuity, $q\to 0$ as $p\to 0$. Then we can choose a number $0<c<t_0$ such that $\forall p=(a,b+i|a|^2)\in\partial\mathbb{H}^2$ with $|p|\leqslant c$, the point $q=(A,B+i|A|^2)$ satisfies $|q|\leqslant t_0$. Let us verify that c is the desired number.

Step 2. There exists a curve from 0 to p with monotone property We have to put the condition $|\alpha|=1$ in (60); otherwise we may not be able to find the t_0 for all curves. We want to remove this condition by adding one more piece of the line segment, namely, we claim that for any p and (c_1',c_3',e_1',e_2') as above, there is a curve $\Gamma(t)$, $t\in[0,t^*]$, consisting of one or two pieces of line segments, such that (60) is still true: $\mathcal{W}((F_{c_1',c_3',e_1',e_2'})^{***}_{\Gamma(t)})$ is nondecreasing along Γ.

Write $p=(a,b+i|a|^2)\in\partial\mathbb{H}^2$. We distinguish two cases: (i) $\pi(a,b+i|a|^2)\in\triangle_c$; and (ii) $\pi(a,b+i|a|^2)\in\square_c-\triangle_c$.

In the first case (i): for any $p=(a,b+i|a|^2)$ with $|a|\leqslant c$ and $|b|\leqslant|a|c$, assuming $p\neq 0$, we have $p=\Gamma(t^*)$ for some curve $\Gamma(t)=(\alpha t,\beta_1 t+i|\alpha|^2t^2)$ with $0\leqslant\beta_1\leqslant 1$ and $|\alpha|=1$ as above with some $t^*\in[0,c]$. In fact, we have $\alpha=\frac{a}{|a|}$, $\beta_1=\frac{b}{|a|}$ and $t^*=|a|$. By (60) the function $\mathcal{W}((F_{c_1',c_3',e_1',e_2'})^{***}_{\Gamma(t)})$ is increasing as t varies from 0 to t^*.

In the second case (ii): $p=(a,b+i|a|^2)$ with $|a|\leqslant c$ and $|a|<|b|\leqslant c$. Let us assume $b>0$; otherwise it can be proved by the same argument. In this case, we cannot find Γ such that it connects 0 and p as in the case (i). However, we can define two pieces of curves:

$$\Gamma(t)=\begin{cases}\Gamma_1(t), & 0\leqslant t\leqslant b-|a|,\\ \Gamma_2(t), & b-|a|\leqslant t\leqslant b.\end{cases}$$
$$:=\begin{cases}(0,t), & 0\leqslant t\leqslant b-|a|,\\ \left(\frac{a}{|a|}(t-b+|a|),t+i\left|t-b+|a|\right|^2\right), & b-|a|\leqslant t\leqslant t^*:=b.\end{cases}$$

Here $\pi(\Gamma_1)=\{0\}\times[0,b-|a|]$ is a vertical line segment; and $\pi(\Gamma_2)$ is another line segment connecting $\Gamma_1(b-|a|)$ and the point p.

By Step e in § 3, the function $\mathcal{W}((F_{c_1',c_3',e_1',e_2'})^{***}_{\Gamma_1(t)})$ is constant for $0\leqslant t\leqslant b-|a|$. Next we consider $\mathcal{W}((F_{c_1',c_3',e_1',e_2'})^{***}_{\Gamma_2(t)})$. If we use a new variable $u=t-b+|a|$, then $\Gamma_2(t)$ can be written as

$$\Gamma_2(u)=\left(\frac{a}{|a|}u,\quad (b-|a|)+u+iu^2\right),\quad 0\leqslant u\leqslant|a|.$$

By the remark (b) in (50), (46) is still valid for $\Gamma_2(u)$ so that $\mathcal{W}((F_{c_1',c_3',e_1',e_2'})^{***}_{\Gamma_2(t)})$ is nondecreasing for any $b-|a|\leqslant t\leqslant t^*$. Our claim is proved.

Step 3. The $\mathcal{W}$ function is constant We claim:

$$\mathcal{W}((F_{c_1',c_3',e_1',e_2'})^{***}_{\Gamma})=constant. \tag{61}$$

In fact, since $F_{c_1'',c_3'',e_1'',e_2''}=(F_{c_1',c_3',e_1',e_2'})^{***}_p$ and $F_{c_1',c_3',e_1',e_2'}=(F_{c_1'',c_3'',e_1'',e_2''})^{***}_q$. We have $F_{c_1',c_3',e_1',e_2'}=((F_{c_1',c_3',e_1',e_2'})^{***}_p)^{***}_q$.

Since $\pi(p)\in\square_c$, by our choice of c, $q=(A,B+i|A|^2)$ satisfies $\pi(q)\in\square_{t_0}$, i.e., $|A|\leqslant t_0$ and $|B|\leqslant t_0$. Then by Step 2, there exists a curve $\widetilde{\Gamma}(\widetilde{t})$, $0\leqslant\widetilde{t}\leqslant\widetilde{t}^*$, connecting 0 and q such that the function $\mathcal{W}((F_{c_1',c_3',e_1',e_2'})^{***}_{\widetilde{\Gamma}(\widetilde{t})})$ is nondecreasing along $\widetilde{\Gamma}$. Then we obtain

$$\mathcal{W}(F_{c_1',c_3',e_1',e_2'})=\mathcal{W}((F_{c_1',c_3',e_1',e_2'})^{***}_{\Gamma(0)})\leqslant\mathcal{W}((F_{c_1',c_3',e_1',e_2'})^{***}_{\Gamma(t^*)})=\mathcal{W}(F_{c_1'',c_3'',e_1'',e_2''}), \tag{62}$$

and

$$\mathcal{W}(F_{c_1'',c_3'',e_1'',e_2''})=\mathcal{W}((F_{c_1'',c_3'',e_1'',e_2''})^{***}_{\widetilde{\Gamma}(0)})\leqslant\mathcal{W}((F_{c_1'',c_3'',e_1'',e_2''})^{***}_{\widetilde{\Gamma}(\widetilde{t}^*)})=\mathcal{W}((F_{c_1',c_3',e_1',e_2'}). \tag{63}$$

By (62) and (63), Claim (61) is proved.

Step 4. Proof of the uniqueness We next claim that $(F_{c_1',c_3',e_1',e_2'})^{***}_{\Gamma(t)}$ is constant:

$$(F_{c_1',c_3',e_1',e_2'})^{***}_{\Gamma(t)}\equiv F_{c_1',c_3',e_1',e_2'},\quad\forall t\in[0,t_0]. \tag{64}$$

Let us consider the case (i) in Step 2. From (31) and Lemma 2.5, it implies that $\big(4c_1'(b'c_1' + 2c_2') - 8b'(e_1' + e_2')\big)\Gamma(t) = 0$ for any $t \in [0, t^*]$. Thus by the argument in (55), we proved $c_1'(\Gamma(t)) = c_3'(\Gamma(t)) = 0$ for any $t \in [0, t^*]$. This implies that $(F_{c_1',c_3',e_1',e_2'})^{***}_{\Gamma(t)}$ is the same map for any $t \in [0, t_0]$. Claim (64) is proved. The case (ii) will be proved by similar argument as the case (i) and by the remark (b) in (50). $\square$

Lemma 4.2. *For any point $P^{(0)} = (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) \in \mathcal{K}^* - \mathcal{E}$ where $\mathcal{E}$ is defined in Lemma 3.3, there is a neighborhood V of $P^{(0)}$ in $\mathcal{K}$, a neighborhood U of $P^{(0)}$ in $\mathcal{K}^* - \mathcal{E}$ and a neighborhood E of 0 in $\partial\mathbb{H}^2$ such that the map $\Psi : U \times E \to V$, $(F, p) \mapsto F_p^{***}$ is surjective.*

Proof: We first claim that for any $F_{c_1,c_3,e_1,e_2} \in \mathcal{K}^* - \mathcal{E}$, the set $N := \{(F_{c_1,c_3,e_1,e_2})^{***}_p \mid p \in \partial\mathbb{H}^2\}$ is of real dimension $\geqslant 2$. In fact, consider a function $\mathcal{W}((F_{c_1,c_3,e_1,e_2})^{***}_\Gamma)$ on N where $\Gamma(t) = (\alpha t, \beta_1 t + |\alpha|^2 t^2)$ is a curve in $\partial\mathbb{H}^2$ as (27). By (46), we have $\Im(q_1(t)) = |\alpha| + O(|t|)$ for $t > 0$ sufficiently small. Since $F_{c_1,c_3,e_1,e_2} \in \mathcal{K}^* - \mathcal{E}$, by Lemma 3.3, we have $(4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2))(\Gamma(t)) \not\equiv 0$ holds for some curve Γ. Then from (33),

$$\mathcal{W}(F^{***}_{\Gamma(t+\Delta t)}) = \mathcal{W}(F^{***}_{\Gamma(t)}) + \Big[4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2)\Big](\Gamma(t))|\alpha|\Delta t + o(|\Delta t|), \tag{65}$$

Since $\alpha \in \mathbb{C} \cong \mathbb{R}^2$, our claim is proved.

It remains to prove $\dim_{\mathbb{R}} \Psi(U \times E) = 4$. Notice that $\dim_{\mathbb{R}} \mathcal{K} = 4$, $\dim_{\mathbb{R}}(\mathcal{K}^*) \geqslant 2$, and that the map defined by $(\mathcal{K}^* - \mathcal{E}) \times \partial\mathbb{H}^2 \to \mathcal{K}$, $(F, p) \mapsto F_p^{***}$ is (Nash) algebraic. Then it suffices to show that this map is injective, i.e., for any two distinct points (c_1, c_3, e_1, e_2), $(\widetilde{c}_1, \widetilde{c}_3, \widetilde{e}_1, \widetilde{e}_2) \in \mathcal{K}^*$, which are sufficiently close to $(c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)})$, and for any two points $p, \widetilde{p} \in \partial\mathbb{H}^2$, which are sufficiently close to $0 \in \partial\mathbb{H}^2$,

$$(F_{c_1,c_3,e_1,e_2})^{***}_p \neq (F_{\widetilde{c}_1,\widetilde{c}_3,\widetilde{e}_1,\widetilde{e}_2})^{***}_{\widetilde{p}}. \tag{66}$$

If this can be proved, it follows $\dim_{\mathbb{R}} \Psi(U \times E) = 4$.

Recall that for a fixed F, we write

$$F_p^{***} = H_p \circ \tau_p \circ F \circ \sigma_p \circ G_p, \tag{67}$$

where $\sigma_p \in Aut(\mathbb{H}^2)$ and $\tau_p \in Aut(\mathbb{H}^5)$ are defined in (18), $G_p \in Aut_0(\mathbb{H}^2)$ and $H_p \in Aut_0(\partial\mathbb{H}^5)$.

In case (66) does not hold, i.e., we have $(F_{c_1,c_3,e_1,e_2})^{***}_p = (F_{\widetilde{c}_1,\widetilde{c}_3,\widetilde{e}_1,\widetilde{e}_2})^{***}_{\widetilde{p}}$. By (67), we write

$$H_p \circ \tau_p \circ F_{c_1,c_3,e_1,e_2} \circ \sigma_p \circ G_p = \widetilde{H}_p \circ \widetilde{\tau}_p \circ F_{\widetilde{c}_1,\widetilde{c}_3,\widetilde{e}_1,\widetilde{e}_2} \circ \widetilde{\sigma}_p \circ \widetilde{G}_p,$$

i.e.,

$$F_{c_1,c_3,e_1,e_2} = \tau_p^{-1} \circ H_p^{-1} \circ \widetilde{H}_p \circ \widetilde{\tau}_p \circ F_{\widetilde{c}_1,\widetilde{c}_3,\widetilde{e}_1,\widetilde{e}_2} \circ \widetilde{\sigma}_p \circ \widetilde{G}_p \circ G_p^{-1} \circ \sigma_p^{-1} = (F_{\widetilde{c}_1,\widetilde{c}_3,\widetilde{e}_1,\widetilde{e}_2})^{***}_{p_0}, \tag{68}$$

where $p_0 = \widetilde{\sigma}_p \circ \widetilde{G}_p \circ G_p^{-1} \circ \sigma_p^{-1}(0)$.

Notice from (67) that there is $\delta > 0$ such that as $p \to 0$, $\sigma_p, G_p, \tau_p, H_p$ all converge to the identity maps in $Aut(\mathbb{H}^2)$ and $Aut(\mathbb{H}^5)$ respectively. We apply this fact to (68) to conclude that for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $(c_1, c_3, e_1, e_2), (\widetilde{c}_1, \widetilde{c}_3, \widetilde{e}_1, \widetilde{e}_2) \in \mathcal{K}^*$ with

$$dist\big((c_1, c_3, e_1, e_2), (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)})\big) < \delta, \quad dist\big((\widetilde{c}_1, \widetilde{c}_3, \widetilde{e}_1, \widetilde{e}_2), (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)})\big) < \delta,$$

we must have $|p_0| < \epsilon$. We can choose ϵ to be the c as in Lemma 4.1. By applying Lemma 4.1 to (68) to conclude $F_{c_1,c_3,e_1,e_2} = F_{\widetilde{c}_1,\widetilde{c}_3,\widetilde{e}_1,\widetilde{e}_2}$. This contracts with the fact that (c_1, c_3, e_1, e_2) and $(\widetilde{c}_1, \widetilde{c}_3, \widetilde{e}_1, \widetilde{e}_2)$ are distinct. Hence (66) is proved. $\square$

Corollary 4.3. *(Local version of Theorem 1.1(ii)) For any $P^{(0)} = (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) \in \mathcal{K}^* - \mathcal{E}$ where $\mathcal{E}$ is defined in Lemma 3.3, there is a neighborhood U of $P^{(0)}$ in $\mathcal{K}^* - \mathcal{E}$ such that $\forall (c_1', c_3', e_1', e_2'), (c_1'', c_3'', e_1'', e_2'') \in U$ such that $F_{c_1'',c_3'',e_1'',e_2''}$ and $F_{c_1',c_3',e_1',e_2'}$ are equivalent, we have $(c_1'', c_3'', e_1'', e_2'') = (c_1', c_3', e_1', e_2')$.*

Proof: Let U_1 be a neighborhood of $P^{(0)}$ in $\mathcal{K}^* - \mathcal{E}$, E a neighborhood of 0 in $\partial\mathbb{H}^2$ and V a neighborhood of $P^{(0)}$ in $\mathcal{K}$ as in Lemma 4.2. Let U be a neighborhood of $P^{(0)}$ in $\mathcal{K}^* - \mathcal{E}$ and $c > 0$ be a constant as in Lemma 4.1. We choose $U_1, E = \{(z, u + i|z|^2) \in \partial\mathbb{H}^2 \mid |z| < c, |u| < c\}$, V such that $U_1 \subset U$ and $V \cap (\mathcal{K}^* - \mathcal{E}) \subset U$. Then by Lemma 4.2, we have $F_{c_1'',c_3'',e_1'',e_2''} = (F_{c_1',c_3',e_1',e_2'})^{***}_p$ with $|p| < c$, and by Lemma 4.1, $(c_1'', c_3'', e_1'', e_2'') = (c_1', c_3', e_1', e_2')$. $\square$

5 The proof of Theorem 1.1

Before proving Theorem 1.1, we mention a fact. Let σ_a and $\sigma_b \in Aut(\partial\mathbb{H}^2)$ defined as in (18) and $F \in Rat(\mathbb{H}^2, \mathbb{H}^5)$, then we can define a family of automorphism $\Theta_s = \sigma_{sb+(1-s)a}$, $0 \leqslant s \leqslant 1$, and $\Psi_s = \tau^F_{sb+(1-s)a} \in Aut(\partial\mathbb{H}^5)$ defined as in (18) so that $\Psi_s \circ F \circ \Theta_s \in Rat(\mathbb{H}^2, \mathbb{H}^5)$ satisfies $\Theta_0 = \sigma_a$, $\Theta_1 = \sigma_b$ and

$$\Psi_s \circ F \circ \Theta_s(0) = 0, \quad \forall s \in [0,1]. \tag{69}$$

Proof of Theorem 1.1: For any $F \in Rat(\mathbb{H}^2, \mathbb{H}^5)$ with degree 2, by [1] and Lemma 3.3, F is equivalent to another map $F_{\widetilde{c}_1,\widetilde{c}_3,\widetilde{e}_1,\widetilde{e}_2} \in \mathcal{K}^*$ with the minimum property (9). By Lemma 3.2 and 3.4, Theorem 1.1(i) is proved.

It remains to prove Theorem 1.1(ii). We need to show: if $F_{c_1^{(0)},c_3^{(0)},e_1^{(0)},e_2^{(0)}}$ and $F_{\widetilde{c}_1^{(0)},\widetilde{c}_3^{(0)},\widetilde{e}_1^{(0)},\widetilde{e}_2^{(0)}}$ in $\mathcal{K}^*$ are equivalent, then

$$(\widetilde{c}_1^{(0)}, \widetilde{c}_3^{(0)}, \widetilde{e}_1^{(0)}, \widetilde{e}_2^{(0)}) = (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}). \tag{70}$$

We assume that $(c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) \notin \mathcal{E}$ where $\mathcal{E}$ is defined in Lemma 3.3; otherwise $F_{c_1^{(0)},c_3^{(0)},e_1^{(0)},e_2^{(0)}}$ and $F_{\widetilde{c}_1^{(0)},\widetilde{c}_3^{(0)},\widetilde{e}_1^{(0)},\widetilde{e}_2^{(0)}}$ cannot be equivalent.

Step 1. Construct a curve $\hat{L}_0$ Since $F_{c_1^{(0)},c_3^{(0)},e_1^{(0)},e_2^{(0)}}$ and $F_{\widetilde{c}_1^{(0)},\widetilde{c}_3^{(0)},\widetilde{e}_1^{(0)},\widetilde{e}_2^{(0)}}$ are equivalent,

$$F_{\widetilde{c}_1^{(0)},\widetilde{c}_3^{(0)},\widetilde{e}_1^{(0)},\widetilde{e}_2^{(0)}} = \Psi \circ F_{c_1^{(0)},c_3^{(0)},e_1^{(0)},e_2^{(0)}} \circ \Theta \tag{71}$$

where $\Theta \in Aut(\mathbb{H}^2)$ and $\Psi \in Aut(\mathbb{H}^5)$. Notice $\Psi \circ F_{c_1^{(0)},c_3^{(0)},e_1^{(0)},e_2^{(0)}} \circ \Theta(0) = 0$ holds.

We take a real analytic curve $L = L(s) \in \mathcal{K}^* - \mathcal{E}$, $0 \leqslant s < 1$, such that $L(0) = (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)})$. In fact, since $(c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) \notin \mathcal{E}$ and $\mathcal{E}$ is closed, L could be taken in a neighborhood of $(c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)})$.

By using automorphisms of balls, Cayley transformations and (69), we can take a real analytic family of automorphisms $\Theta_s \in Aut(\partial\mathbb{H}^2)$, $\Psi_s \in Aut(\partial\mathbb{H}^5)$, $s \in [0,1]$, such that when $s = 0$, $\Theta_0 = \Theta$, $\Psi_0 = \Psi$; when $s \in (0,1)$, $\Theta_s(0) \neq \infty$, $\Psi_s \circ F_{L(s)} \circ \Theta_s(0) = 0$; when $s = 1$, $\Theta_1 = Id$, $\Psi_1 = Id$. Then we define

$$\hat{L}_0(s) := \Psi_s \circ F_{L(s)} \circ \Theta_s \in Rat(\mathbb{H}^2, \mathbb{H}^5), \quad 0 \leqslant s \leqslant 1,$$

such that $\hat{L}_0(s)(0) = 0$ for all s, $F_{\hat{L}_0(0)} = \Psi \circ F_{L(0)} \circ \Theta$ and $\hat{L}_0(1) = L(1)$. Our goal is to show: $\hat{L}_0(s) = L(s)$, $\forall s \in [0,1]$, so that $\hat{L}_0(0) = L(0)$, i.e., (70) holds.

Step 2. Define a curve $\hat{L}(s)$ Notice that $\hat{L}_0$ must be in $\mathcal{K}$, namely, $F_{\hat{L}_0(s)}$ may geometric rank one at the origin for all $s \in [0,1]$, so that $(F_{\hat{L}_0(s)})^{***}$ is well defined for all $s \in [0,1]$.

Recall $\Theta_s(0) \neq \infty$ for any $s \in (0,1]$ and $\Theta_1 = Id$. Then for any $s \in (0,1]$, we denote $\psi(s) := \Theta_s(0) \in \partial\mathbb{H}^2$ with $\psi(1) = 0$, so that $\Theta_s = \sigma_{\psi(s)} \circ G_s$ where $\sigma_{\psi(s)}$ is defined as in (18) and $G_s \in Aut_0(\partial\mathbb{H}^2)$, i.e., we have a continuous map $\psi(s) \in \partial\mathbb{H}^2$ such that $\psi(1) = 0$ and

$$(F_{\hat{L}_0(s)})^{***} = \Big(F_{L(s)}\Big)^{***}_{\psi(s)}, \quad \forall s \in (0,1], \quad \textit{and} \quad (F_{\hat{L}_0(1)})^{***} = F_{L(1)}. \tag{72}$$

Even though $(F_{\hat{L}_0(s)})^{***}$ is in $\mathcal{K}$ for any $s \in (0,1]$, it may not be in $\mathcal{K}^*$ because the minimum property (9) may not be satisfied. We claim that $(F_{\hat{L}_0(s)})^{***}$ is equivalent to another map $F_{\hat{L}(s)} \in \mathcal{K}^*$. More precisely, we want to find $q(s) \in \partial\mathbb{H}^2$ so that

$$F_{\hat{L}(s)} := (F_{\hat{L}_0(s)})^{***}_{q(s)} \in \mathcal{K}^*, \quad \forall s \in (0,1]. \tag{73}$$

To define such $q(s)$, we consider several cases below.

If $s = 1$, since $F_{L(1)} \in \mathcal{K}^*$ and $\psi(1) = 0$, we define $q(1) = 0$.

If $s \in (0,1]$ at which the minimum property (9) holds, we define $q(s) = 0$.

If $s \in (0,1]$ at which (9) does not hold, we consider a continuous curve $\Gamma^{(s)}(t) \in \partial\mathbb{H}^2 - \Xi_F$, $0 \leqslant t \leqslant 1$, with $\Gamma^{(s)}(0) = 0$ such that the function value $\mathcal{W}((F_{\hat{L}_0(s)})^{***}_{\Gamma^{(s)}(t)})$ is decreasing along $\Gamma^{(s)}$. We denote by ℓ_s the infimum of $\mathcal{W}((F_{\hat{L}_0(s)})^{***}_{\Gamma^{(s)}})$ over all such curves. Then there exists a sequence of curves $\Gamma^{(s)}_m$ in $\partial\mathbb{H}^2$ such that

$$\ell_s = \lim_{m\to\infty} \mathcal{W}\Big((F_{L(s)})^{***}_{\Gamma^{(s)}_m(1)}\Big). \tag{74}$$

Since $\mathcal{W}((F_{\hat{L}_0(s)})^{***}_p) = c_1(p)^2 - e_1(p) - e_2(p)$, the decreasing property implies $c_1(p), -e_1(p)$ and $-e_2(p)$ are bounded (cf. [1, Step 1, §4]), so that $(F_{\hat{L}_0(s)})^{***}_{\Gamma^{(s)}_m(t)}$, regarded as a point, is inside $\mathcal{K}$ and is contained a compact subset of $\mathcal{K}$ that is independent of $\Gamma^{(s)}_m$. Therefore, by taking subsequences, we may assume that the limit $\lim_{m\to\infty}(F_{\hat{L}_0(s)})^{***}_{\Gamma^{(s)}_m(1)}$ exists as a point in $\mathcal{K}^*$ and that $\lim_{m\to\infty}\Gamma^{(s)}_m(1) \in \overline{\partial\mathbb{H}^2}$ exists. Let us define

$$F_{\hat{L}(s)} := \lim_{m\to\infty}(F_{\hat{L}_0(s)})^{***}_{\Gamma^{(s)}_m(1)} \in \mathcal{K}^*. \tag{75}$$

It remains to show that $q(s) \in \partial\mathbb{H}^2$ can be defined such that $F_{\hat{L}(s)} = (F_{\hat{L}_0(s)})^{***}_{q(s)}$.

By the choice of $L(1)$ and Corollary 4.3, there exists a neighborhood U of $L(1)$ in $\mathcal{K}^*$, such that if a point $(c_1, c_3, e_1, e_2) \in U$ such that F_{c_1,c_3,e_1,e_2} and $F_{L(1)}$ are equivalent, then $(c_1, c_3, e_1, e_2) = L(1)$.

Let us consider $\mathcal{K}\cap\mathbb{B}^4(\hat{L}_0(s), r)$, the intersection of $\mathcal{K}$ with the sphere in $\mathbb{C}^4$ which is centered at $\hat{L}_0(s)$ with radius r. We also consider $\mathcal{K}^*\cap\mathbb{B}^2(\hat{L}_0(s), r)$, the intersection of $\mathcal{K}^*$ with the sphere in $\mathbb{C}^2$ which is centered at $\hat{L}_0(s)$ with radius r. We take r so small that $\mathcal{K}^*\cap\mathbb{B}^2(\hat{L}_0(s), r) \subset U$.

Step 3. Claim on $F_{\hat{L}(s)} \to F_{\hat{L}_0(s)}$ Regarding $F_{\hat{L}(s)}$ as points in $\mathcal{K}$, we claim:

$$dist\Big(F_{\hat{L}(s)},\quad F_{\hat{L}_0(s)}\Big) \to 0, \quad as\ s \to 1. \tag{76}$$

Suppose (76) is not true. Then there exists a sequence $s_k \to 1$ such that

$$dist\Big(F_{\hat{L}(s_k)},\quad F_{\hat{L}_0(s_k)}\Big) \geqslant \delta_0, \quad as\ k \to \infty. \tag{77}$$

for a certain $\delta_0 > 0$. By (75), we can take integer m_{s_k} for each s_k such that

$$0 \leqslant \mathcal{W}((F_{\hat{L}_0(s_k)})^{***}_{\Gamma^{(s_k)}_{m_{s_k}}(1)}) - \ell_{s_k} < \frac{1}{k},\ and\ dist\Big((F_{\hat{L}_0(s_k)})^{***}_{\Gamma^{(s_k)}_{m_{s_k}}(1)},\quad F_{\hat{L}(s_k)}\Big) < \frac{1}{k}. \tag{78}$$

By (77) we have

$$dist\Big((F_{\hat{L}_0(s_k)})^{***}_{\Gamma^{(s_k)}_{m_{s_k}}(1)},\quad F_{\hat{L}_0(s_k)}\Big) \geqslant \frac{\delta_0}{2}. \tag{79}$$

Then we can choose $r < \frac{\delta_0}{2}$. Then $\{(F_{\hat{L}_0(s_k)})^{***}_{\Gamma^{(s_k)}_{m_{s_k}}}\}_{t\in[0,1]}$, regarded as a curve in $\mathcal{K}$ initiated from the point $F_{\hat{L}_0(s_k)}$, must be across through the sphere $(\mathcal{K}\cap\partial B^4(\hat{L}_0(s_k), r))$, i.e.,

$$\{(F_{\hat{L}_0(s_k)})^{***}_{\Gamma^{(s_k)}_{m_{s_k}}}\}_{t\in[0,1]} \cap (\mathcal{K}\cap\partial B^4(\hat{L}_0(s_k), r)) \neq \emptyset. \tag{80}$$

Let $Q^{(s_k)}$ be a point in the intersection (80) and then $Q^{(s_k)} = (F_{\hat{L}_0(s_k)})^{***}_{\Gamma^{s_k}_{m_{s_k}}(t_k)}$ for some $t_k \in [0, 1]$. By taking subsequences, we assume $Q := \lim_{k\to\infty} Q^{(s_k)}$ exists. By the construction, we see that the F_Q is equivalent to $F_{L(1)}$ and

$$Q \in \mathcal{K}^*, \quad and \quad dist(Q, L(1)) = r.$$

Since $Q \in \mathcal{K}^*\cap\partial B^2(\hat{L}_0(1), r) \subset U$, by Corollary 4.3, $Q = L(1)$, i.e., $dist(Q, L(1)) = 0$, but this is a contradiction. Claim (76) is proved.

Step 4. Proof of $\hat{L}(s) \equiv L(s)$ From (76), we have continuous.

$$dist\Big(F_{\hat{L}(s)},\quad F_{L(s)}\Big) \to 0, \quad as\ s \to 1.$$

Since both $F_{\hat{L}(s)} \in \mathcal{K}^*$ and $F_{L(s)} \in \mathcal{K}^* - \mathcal{E}$ where $s \in (s_0, 1]$ for some $s_0 > 0$ such that $0 \leqslant 1 - s_0$ is sufficiently small, by Corollary 4.3 and the choice of $L(1)$, we conclude

$$F_{\hat{L}(s)} = F_{L(s)}, \quad \forall s \in (s_0, 1].$$

Repeating this process. Finally by continuity $F_{\hat{L}(s)} = F_{L(s)}$, $\forall s \in [0, 1]$. When restricted at 0, $F_{\hat{L}_0(0)} = F_{\hat{L}(0)} = F_{L(0)}$, so that (70) is proved. $\square$

Acknowledgments This work was started when the first author, with Professor Xiaojun Huang, was visiting the School of Mathematical Sciences, Wuhan University, China, in Summer 2004. The authors would like to thank Professor H. Chen for arrangement and help when they were working at Wu Han and to thank Professor X. Huang for lots of helpful discussion.

Added note In a recent paper [5, theorem 3.1], it is proved that any map $F \in Rat(\mathbb{B}^2, \mathbb{B}^N)$ with degree 2 must be equivalent to a polynomial map.

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