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UPPER BOUNDARY POINTS OF THE GAP INTERVALS FOR RATIONAL MAPS BETWEEN BALLS*

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Dedicated to Professor Ngaiming Mok on the occasion of his 60th birthday

Abstract. The paper focuses on the study of rational proper holomorphic maps from $\mathbb{B}^n$ to $\mathbb{B}^N$. We classify these maps when N is the upper boundary point of the gap interval I_k, $k \le n-2$ and the geometric rank of the map is k.

Key words. Proper holomorphic maps, holomorphic classification, geometric rank, Chern-Moser equation.

Mathematics Subject Classification. 32H35.

1. Introduction. Let us denote by $\mathbb{B}^n$ the unit ball in $\mathbb{C}^n$ and $Rat(\mathbb{B}^n, \mathbb{B}^N)$ the set of all proper holomorphic rational maps F from $\mathbb{B}^n$ to $\mathbb{B}^N$. We say that $f, g \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ are *equivalent* if there are $\sigma \in Aut(\mathbb{B}^n)$ and $\tau \in Aut(\mathbb{B}^N)$ such that $f = \tau \circ g \circ \sigma$. Let $K(n) := \max\{t \in \mathbb{Z}^+ \mid \frac{t(t+1)}{2} < n\}$. For any integer k with $1 \le k \le K(n)$, we define the *gap interval*

$$\mathcal{I}_k := \left(kn,\ (k+1)n - \frac{k(k+1)}{2}\right). \tag{1.1}$$

Let us recall the *gap conjecture* [HJY09]: Any proper holomorphic rational map $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ $(n \ge 3)$ is equivalent to a map of the form $(G, 0')$ where $G \in Rat(\mathbb{B}^n, \mathbb{B}^{N'})$ where $N' < N$ if and only if $N \in \mathcal{I}_k$ for some $1 \le k \le K(n)$. For the sake of simplicity, we call F is equivalent to G. Recently, P. Ebenfelt proposed a SOS conjecture (i.e., the Sums of Squares of Polynomial conjecture) [E16] and proved that if the SOS conjecture is true, then it implies the gap conjecture.

The "only if" part of the gap conjecture was proved in [HJY09]. For the "if" part, the cases for $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$ have been proved by Huang [Hu99], Huang-Ji[HJ01], Hamada [H05], Huang-Ji-Xu[HJX06] and Huang-Ji-Yin [HJY14].

Moreover, if N is the boundary point of the interval $\mathcal{I}_k$ for $k = 1$ and 2, maps in $Rat(\mathbb{B}^n, \mathbb{B}^N)$ have been determined, up to equivalence, as follows.

- When $N = n$ which is the lower boundary point of $\mathcal{I}_1 = (n, 2n-1)$, it was proved by Alexanda [A77]: any map $F \in Rat(\mathbb{B}^n, \mathbb{B}^n)$ is an automorphism.
- When $N = 2n-1$ which is the upper boundary point of $\mathcal{I}_1$, it was proved by [HJ01]: any map $F \in Rat(\mathbb{B}^n, \mathbb{B}^{2n-1})$ is either the linear map, or the Whitney map $W_{n,1}$.
- When $N = 2n$ which is the lower boundary point of $\mathcal{I}_2 = (2n, 3n-3)$, it was proved by Hamada [Ha05] that any map $F \in Rat(\mathbb{B}^n, \mathbb{B}^{2n})$ is the linear map, or Whitney map, or in the D'Angelo map family.

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- Recently, when $N = 3n-3$ which is the upper boundary point of $\mathcal{I}_2$, it was proved by Andrews-Huang-Ji-Yin [AHJY16] that any map $F \in Rat(\mathbb{B}^n, \mathbb{B}^{3n-3})$ is linear, or Whitney map $W_{n,1}$, or in the D'Angelo family, or a generalized Whitney map $W_{n,2}$.

For any integer N in the closed interval $\overline{\mathcal{I}_k}$ and for any map $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$, its geometric rank $\kappa_0 \le k$. In fact, for any $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ with the geometric rank κ_0, it is known [Hu03]

$$N \ge n + \frac{(2n-\kappa_0-1)\kappa_0}{2} = (\kappa_0+1)n - \frac{\kappa_0(\kappa_0+1)}{2}, \tag{1.2}$$

which implies $\kappa_0 \le k$.

In this paper, we show that when N is the upper boundary point of $\mathcal{I}_k$, we can determine maps in $Rat(\mathbb{B}^n, \mathbb{B}^N)$ with geometric rank $\kappa_0 = k$. This gives a new proof for the above mentioned result in [HJ01] when $k=1$ and the above mentioned result in [AHJY16] when $k=2$.

THEOREM 1.1. *Let $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ with $n \ge 3$. Let N be the upper boundary point of the gap interval $\mathcal{I}_k$ and $k \le n-2$. Suppose that the geometric rank of F is $\kappa_0 = k$. Then F is equivalent to the generalized Whitney map $W_{n,\kappa}$.*

Notice that in above theorem, $N = (k+1)n - \frac{k(k+1)}{2}$ and that $\kappa_0 = k \le K(n)$ by the definition of $\mathcal{I}_k$, i.e., $\frac{\kappa_0(\kappa_0+1)}{2} < n$ holds. Here we recall [HJY09] that a map $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ is called the *generalized Whitney map* $W_{n,k}$ if $N = (k+1)n - \frac{(k+1)k}{2}$ and

$$W_{n,k}(z) = W_{n,k}(z_1, ..., z_n) = (z_1\psi_1, ..., z_k\psi_k, \psi_{k+1}) \tag{1.3}$$

where

$$\begin{cases} \psi_1 = (z_1, \sqrt{2}z_2, ..., \sqrt{2}z_k, z_{k+1}, ..., z_n), \\ \psi_2 = (z_2, z\sqrt{2}z_3, ..., \sqrt{2}z_k, z_{k+1}, ..., z_n), \\ ......, \\ \psi_{k-1} = (z_{k-1}, \sqrt{2}z_k, z_{k+1}, ..., z_n), \\ \psi_k = (z_k, z_{k+1}, ..., z_n), \\ \psi_{k+1} = (z_{k+1}, ..., z_n). \end{cases} \tag{1.4}$$

We may have another interpretation for the above theorem. If $N = n + \frac{(2n-\kappa_0-1)\kappa_0}{2}$, from (1.2), we say that N is the *minimum*. Based on the semi-linearity property, the following two problems are formulated in [JX04].

Problem (A): Study and classify maps in $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ with N minimum and $1 \le \kappa_0 \le n-2$.

Problem (B): Study and classify maps in $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ with N minimum and $\kappa_0 = n-1$.

For Problem (B), when $n=2$, it is $Rat(\mathbb{B}^2, \mathbb{B}^3)$ which was solved by Faran's theorem. The next case $n=3$, $Rat(\mathbb{B}^3, \mathbb{B}^6)$ is unsolved. In general, Problem (B) should be much more difficult than Problem (A) because it lacks of the semi-linearity property (see [Hu03]). For Problem (A), when $\kappa_0 = 1$, it is $Rat(\mathbb{B}^n, \mathbb{B}^{2n-1})$ which was solved by [HJ01] that F is the Whitney map $W_{n,1}$; when $\kappa_0 = 2$, it is $Rat(\mathbb{B}^n, \mathbb{B}^{3n-3})$, it is recently solved by [AHJY16] that the map F must be the generalized Whiney map $W_{n,2}$. Then Theorem 1.1 covers the remaining part of Problem (A) under the condition $\frac{\kappa_0(\kappa_0+1)}{2} < n$.

THEOREM 1.2. *Let $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ be with geometric rank $\kappa_0 \leq n-2$, where $N = n + \frac{2n-\kappa_0-1}{2}\kappa_0$ is minimum and $\frac{\kappa_0(\kappa_0+1)}{2} < n$. Then F is equivalent to the generalized Whitney map W_{n,κ_0}.*

Note that Theorem 1.1 and Theorem 1.2 are equivalent, and that the condition $\frac{\kappa_0(\kappa_0+1)}{2} < n$ in both of the above theorems will be used in Lemma 2.3 below.

2. Preliminaries.

2a. The associated maps F_p^{**}. Let $F = (f, \phi, g) = (\widetilde{f}, g) = (f_1, \cdots, f_{n-1}, \phi_1, \cdots, \phi_{N-n}, g)$ be a non-constant rational CR map from an open subset M of $\partial\mathbb{H}_n$ into $\partial\mathbb{H}_N$ with $F(0) = 0$. For each $p \in M$ close to 0, we write $\sigma_p^0 \in \mathrm{Aut}(\mathbb{H}_n)$ for the map sending (z, w) to $(z + z_0, w + w_0 + 2i\langle z, \overline{z_0}\rangle)$ and $\tau_p^F \in \mathrm{Aut}(\mathbb{H}_N)$ by defining

$$\tau_p^F(z^*, w^*) = (z^* - \widetilde{f}(z_0, w_0), w^* - \overline{g(z_0, w_0)} - 2i\langle z^*, \overline{\widetilde{f}(z_0, w_0)}\rangle).$$

Then F is equivalent to $F_p = \tau_p^F \circ F \circ \sigma_p^0 = (f_p, \phi_p, g_p)$. Notice that $F_0 = F$ and $F_p(0) = 0$. The following is fundamentally important for the understanding of the geometric properties of F. Let us denote $Prop(\mathbb{H}_n, \mathbb{H}_N) := \{$holomorphic proper maps from $\mathbb{H}_n$ into $\mathbb{H}_N\}$ and $Prop_k(\mathbb{H}_n, \mathbb{H}_N) := Prop(\mathbb{H}_n, \mathbb{H}_N) \cap C^k(\overline{\mathbb{H}_n})$.

LEMMA 2.1. *([Hu99]) Let $F \in Prop_2(\mathbb{H}_n, \mathbb{H}_N)$ with $2 \leq n \leq N$. For each $p \in \partial\mathbb{H}_n$, there is an automorphism $\tau_p^{**} \in Aut_0(\mathbb{H}_N)$ such that $F_p^{**} := \tau_p^{**} \circ F_p$ satisfies the following normalization:*

$$f_p^{**} = z + \frac{i}{2}a_p^{**(1)}(z)w + o_{wt}(3),\ \phi_p^{**} = \phi_p^{**(2)}(z) + o_{wt}(2),\ g_p^{**} = w + o_{wt}(4),$$

with $\langle \overline{z}, a_p^{**(1)}(z)\rangle|z|^2 = |\phi_p^{**(2)}(z)|^2$.

2b. Geometric rank. Write $\mathcal{A}(p) := -2i(\frac{\partial^2 (f_p)_l^{**}}{\partial z_j \partial w}|_0)_{1\leq j,l\leq(n-1)}$. We call the rank of the $(n-1)\times(n-1)$ matrix $\mathcal{A}(p)$, which we denote by $Rk_F(p)$, the *geometric rank* of F at p. $Rk_F(p)$ depends only on p and F, and is a lower semi-continuous function on p, and is independent of the choice of $\tau_p^{**}(p)$ [Hu03]. Define the *geometric rank* of F to be $\kappa_0(F) = max_{p\in\partial\mathbb{H}_n} Rk_F(p)$. Notice that it always holds that $0 \leq \kappa_0 \leq n-1$. Define the geometric rank of $F \in \mathrm{Prop}_2(\mathbb{B}^n, \mathbb{B}^N)$ to be the one for the map $\rho_N^{-1} \circ F \circ \rho_n \in \mathrm{Prop}_2(\mathbb{H}_n, \mathbb{H}_N)$. By [Hu03], $\kappa_0(F)$ depends only on the equivalence class of F and when $N < \frac{n(n+1)}{2}$, the geometric rank $\kappa_0(F)$ of F is precisely the κ_0 mentioned in the introduction.

Under the condition $1 \leq \kappa_0 \leq n-2$, the following theorem was proved in [Hu03] and [HJX06].

THEOREM 2.2 ([HJX06]). *Suppose that $F \in Prop_3(\mathbb{H}_n, \mathbb{H}_N)$ has geometric rank $1 \leq \kappa_0 \leq n-2$ with $F(0) = 0$. Then there are $\sigma \in Aut(\mathbb{H}_n)$ and $\tau \in Aut(\mathbb{H}_N)$ such that $\tau \circ F \circ \sigma$ takes the following form, which is still denoted by $F = (f, \phi, g)$ for convenience of notation:*

$$\begin{cases} f_l = \sum_{j=1}^{\kappa_0} z_j f_{lj}^*(z,w); \ l \le \kappa_0 \\ f_j = z_j, \ for \ \kappa_0+1 \le j \le n-1; \\ \phi_{lk} = \mu_{lk} z_l z_k + \sum_{j=1}^{\kappa_0} z_j \phi_{lkj}^* \ for \quad (l,k) \in \mathcal{S}_0, \\ \phi_{lk} = O_{wt}(3), \quad (l,k) \in \mathcal{S}_1, \\ g = w; \\ f_{lj}^*(z,w) = \delta_l^j + \frac{i\delta_l^j \mu_l}{2} w + b_{lj}^{(1)}(z)w + O_{wt}(4), \\ \phi_{lkj}^*(z,w) = O_{wt}(2), \quad (l,k) \in \mathcal{S}_0, \\ \phi_{lk} = \sum_{j=1}^{\kappa_0} z_j \phi_{lkj}^* = O_{wt}(3) \quad for \ (l,k) \in \mathcal{S}_1 \end{cases} \tag{2.1}$$

Here, for $1 \le \kappa_0 \le n-2$, we write $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$, the index set for all components of ϕ, where $\mathcal{S}_0 = \{(j,l) : 1 \le j \le \kappa_0, 1 \le l \le n-1, j \le l\}$, $\mathcal{S}_1 = \Big\{(j,l) : j = \kappa_0+1, \kappa_0+1 \le l \le \kappa_0 + N - n - \frac{(2n-\kappa_0-1)\kappa_0}{2}\Big\}$, and

$$\mu_{jl} = \begin{cases} \sqrt{\mu_j + \mu_l} & for \ j < l \le \kappa_0; \\ \sqrt{\mu_j} & if \ j \le \kappa_0 < l \ or \ if \ j = l \le \kappa_0. \end{cases} \tag{2.2}$$

2c. A family of affine hyperspaces L_ϵ. Let us review some background materials on the semi-linearity properties on $Rat(\mathbb{B}^n, \mathbb{B}^N)$ (cf. [Hu03] and [HJX06]). Let $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ with $1 \le \kappa_0 \le n-2$. Let E_0 be the proper complex variety consisting of poles and the non-immerse points of F. We define

$\mathcal{V}_F := \{(Z, S_Z) \in (\mathbb{C}^n - E_0) \times Gr_{n,k^0}(\mathbb{C})\}$, F is linear fractional when restricted to $S_Z + Z\}$.

Here $Gr_{n,k^0}(\mathbb{C})$ is the Grassmannian manifold consisting of all $k^0 := n - \kappa_0$-dimensional complex subspaces in $\mathbb{C}^n$. Then $\mathcal{V}_F$ is a complex analytic variety with the projection

$$\pi : \mathcal{V}_F \to \mathbb{C}^n - E_0, \quad (Z, S_Z) \mapsto Z \tag{2.3}$$

is proper holomorphic. There is another proper complex variety $E_1 \subset \mathbb{C}^n - E_0$ such that for any $Z \in \mathbb{C}^n - E_0 \cup E_1$, π has a unique preimage in $\mathcal{V}_F$, i.e., for any $Z \in \mathbb{C}^n - E_0 \cup E_1$, there is a unique complex subspace S_Z of dimension k^0 such that F is linear fractional when restricted to $S_Z + Z$. In particular, if F satisfies the normalization condition as in Theorem 2.2, the restriction of F on S_Z is affine linear. Write $\mathcal{V}_F = \cup_j \mathcal{V}_F^{(j)}$ for the irreducible decomposition of $\mathcal{V}_F$. Then there is only one irreducible component, say $\mathcal{V}_F^{(1)}$, whose projection to $\mathbb{C}^n - E_0$ contains a sufficiently small domain inside $\mathbb{H}_n$ and has a small piece of $\partial\mathbb{H}_n$ containing 0 as part of its boundary. If necessary, we can assume that $0 \notin E_1$ and thus π is biholomorphic near $(0, S_0) \in \mathcal{V}_F$.

By [HJX06, p. 520], we can assume that for any $\epsilon = (\epsilon_1, \epsilon_2, ..., \epsilon_{\kappa_0}) (\in \mathbb{C}^{\kappa_0}) \approx 0$, there is a unique affine subspace L_ϵ of codimension κ_0 defined by equations of the form:

$$z_j = \sum_{i=\kappa_0+1}^{n-1} a_{ji}(\epsilon) z_i + a_{jn}(\epsilon) w + \epsilon_j, \quad 1 \le j \le \kappa_0. \tag{2.4}$$

such that F is a linear map on L_ϵ, where $a_{ji}(\epsilon)$ are holomorphic functions in ϵ near 0 with $a_{ji}(0, ..., 0) = 0$ for all j.

2d. Basic notation. Let $F=(f,\phi,g)\in Rat(\mathbb{B}^n,\mathbb{B}^N)$ be as in Theorem 2.2 with geometric rank κ_0. We have $N=\sharp(f)+\sharp(\phi)+\sharp(g)$ and $\sharp(\phi)=\sharp(\mathcal{S}_0)+\sharp(\mathcal{S}_1)$ where we denote by $\sharp(A)$ the number of elements of a set A, and $\sharp(f)=n-1$, $\sharp(g)=1$, $\sharp(\mathcal{S}_0)=\frac{(2n-1-\kappa_0)\kappa_0}{2}=n\kappa_0-\frac{(\kappa_0+1)\kappa_0}{2}$, $\sharp(\mathcal{S}_1)=N-n-\sharp(\mathcal{S}_0)$.

We denote by $P^{(j,k)}(z,w)$ the polynomial of (z,w) with degree $deg(z)=j$ and degree $deg(w)=k$ and denote $P^{(j,k)}(z)$ the coefficient of w. For example, $P^{(1,1)}(z,w)=\sum_{j=1}^{\kappa_0}a_jz_jw=P^{(1,1)}(z)w,\quad P^{(1,1)}(z)=\sum_{j=1}^{\kappa_0}a_jz_j$.

For any rational holomorphic map $H=\frac{(P_1,...,P_m)}{Q}$ on $\mathbb{C}^n$, where P_j,Q are holomorphic polynomials with $(P_1,...,P_m,Q)=1$, the *degree* of H is defined to be $deg(H):=\max\{\deg(P_j),deg(Q),1\le j\le m\}$.

- **The part** ϕ. Write $\phi=(\Phi_0,\Phi_1)$, $\Phi_0=(\phi_{lk})_{(l,k)\in\mathcal{S}_0}$ and $\Phi_1=(\phi_{lk})_{(l,k)\in\mathcal{S}_1}$. Here $\sharp(\Phi_0)=\sharp(\mathcal{S}_0)$ and $\sharp(\Phi_1)=\sharp(\mathcal{S}_1)$.
  When $N=n+\frac{2n-\kappa_0-1}{2}\kappa_0$ and the geometric rank being κ_0, then
$$\begin{aligned}\sharp(\Phi_0)&=(n-1)+\cdots+(n-\kappa_0)=\frac{2n-\kappa_0-1}{2}\kappa_0,\\ \sharp(\Phi_1)&=N-n-\sharp(\Phi_0)=0\end{aligned}\tag{2.5}$$
  Namely, there is no Φ_1 term.
- **The part** $f_j^{(1,1)}(z)$. Write $f^{(1,1)}(z)=(f_1^{(1,1)}(z),...,f_{\kappa_0}^{(1,1)}(z),0,\cdots,0)$. By Theorem 2.2, we have $f_j^{(1,1)}(z)=\frac{i\mu_j}{2}z_j$, $\mu_j>0$ for $1\le j\le\kappa_0$.
- **The part** $\Phi_0^{(2,0)}(z)$. One important portion of Φ_0 is the z-quadric part (see Theorem 3.2): $\Phi_0^{(2,0)}(z)=\{\phi_{jl}^{(2)}(z)=\mu_{jl}z_jz_l\}_{(j,l)\in\mathcal{S}_0}$.
- **The part** $\Phi_0^{(1,1)}(z)$. Another portion of Φ_0 is $\Phi_0^{(1,1)}(z)w$ which are not mentioned in Theorem 2.2:
$$\Phi_0^{(1,1)}(z)=\sum_{j=1}^{\kappa_0}e_jz_j,\quad e_j\in\mathbb{C}^{\sharp(\mathcal{S}_0)}.$$
- **The part** $f^{(2,1)}(z)$. Write $f^{(2,1)}(z)=(f_1^{(2,1)}(z),...,f_{\kappa_0}^{(2,1)}(z),0,\cdots,0)$. We see from ([HJY, (3.5)]) that
$$f_j^{(2,1)}(z)=-\xi_j.\tag{2.6}$$
  Here
$$\xi_j=\Phi^{(2,0)}\cdot\overline{e_j}=\sum_{1\le k,l\le\kappa_0}\phi_{kl}^{(2,0)}\overline{e_{j,kl}}+\sum_{1\le k\le\kappa_0<\alpha\le n-1}\phi_{k\alpha}^{(2,0)}\overline{e_{j,k\alpha}}.\tag{2.7}$$

2e. The components of $\Phi_1^{(3,0)}$.

LEMMA 2.3. *Let $\kappa_0\ge 2$ and $(\kappa_0+1)n-\frac{\kappa_0(\kappa_0+1)}{2}\le N\le(\kappa_0+2)n-\kappa_0(\kappa_0+1)+\kappa_0-2$. Then*
$$\Phi_1^{(3,0)}(z)=\left(\frac{2}{\sqrt{\mu_j+\mu_l}}\Big(\sqrt{\frac{\mu_j}{\mu_l}}z_j\xi_l-\sqrt{\frac{\mu_l}{\mu_j}}z_l\xi_j\Big),\quad 0'\right)_{1\le j<l\le\kappa_0}\tag{2.8}$$
$|\phi^{(3,0)}(z)|^2=4\big(\sum_{j\le\kappa_0}\frac{1}{\mu_j}|\xi_j(z)|^2\big)|z|^2$.

The above result was proved in the third gap paper [HJY14], Corollary 3.4, under the condition "$(\kappa_0+1)n-\kappa_0 \le N \le (\kappa_0+2)n-\kappa_0(\kappa_0+1)+\kappa_0-2$." By checking the proof in [HJY14], we find that this is still valid when the condition is replaced by

$$(\kappa_0+1)n-\frac{\kappa_0(\kappa_0+1)}{2} \le N \le (\kappa_0+2)n-\kappa_0(\kappa_0+1)+\kappa_0-2. \tag{2.9}$$

Geometrically we notice that the lower bound in (2.9) is the right end point of the gap interval $\mathcal{I}_{\kappa_0}$ and the upper bound in (2.9) is less than the left end point of the gap interval $\mathcal{I}_{\kappa_0+1}$.

When the condition in Theorem 1.1 or Theorem 1.2 is satisfied, the inequality $N=(\kappa_0+1)n-\frac{\kappa_0(\kappa_0+1)}{2} \le (\kappa_0+2)n-\kappa_0(\kappa_0+1)+\kappa_0-2$ holds because of the condition $\frac{\kappa_0(\kappa_0+1)}{2}<n$. Lemma 2.3 can be applied to the maps in Theorem 1.1.

3. Properties of the semi-linear subspace. In this section, we will use the automorphisms of the balls to normalize the semi-linear subspace achieved in [Hu03]. More precisely, by (2.4), we will prove the following:

Proposition 3.1. *Let $F \in Prop_3(\mathbb{H}_n,\mathbb{H}_N)$ be as in Theorem 2.2, and the semi-linear subspace L_ϵ be given by*

$$z_j=\sum_{\alpha=\kappa_0+1}^{n-1} a_{j\alpha}(\epsilon)z_\alpha+a_{jn}(\epsilon)w+\epsilon_j,\ 1\le j\le \kappa_0. \tag{3.1}$$

Then there are automorphisms $\sigma \in Aut(\mathbb{H}_n)$ and $\tau \in Aut(\mathbb{H}_N)$ such that $\hat{F}=\tau\circ F\circ\sigma$ still takes the form (2.1), and the semi-linear subspace still has the form (3.1). Moreover, we have

$$a_{1\alpha}^{(I_1)}=0 \text{ for } \kappa_0+1\le\alpha\le n-1 \text{ and } Re(a_{1n}^{(I_1)})=0 \tag{3.2}$$

where we denote $a_{jk}^{(1)}(\epsilon)=a_{jk}^{(I_1)}\epsilon_1+\ldots+a_{jk}^{(I_{\kappa_0})}\epsilon_{\kappa_0}$.

Proof. Consider the image $\hat{L}_\epsilon := \hat{\sigma}_{\vec{c}}(L_\epsilon)$ given by

$$Z_j=\sum_{\alpha=\kappa_0+1}^{n-1} A_{j\alpha}(\epsilon)Z_\alpha+A_{jn}(\epsilon)W+\rho_j(\epsilon), \tag{3.3}$$

where the inverse of the the automorphism is given by

$$\begin{aligned}\hat{\sigma}_{\vec{c}}^{-1}(Z,W):=&\frac{(Z_1,\cdots,Z_{\kappa_0},Z_{\kappa_0+1}+c_{\kappa_0+1}W,\ldots,Z_{n-1}+c_{n-1}W,W)}{q_c}\\ =&(z_1,z_2,\ldots,z_{n-1},z_n)\end{aligned} \tag{3.4}$$

and $q_{\vec{c}} := 1-2i\overline{\vec{c}}\cdot Z+(r-i|\vec{c}|^2)W$, where $\vec{c}=(0,\ldots,0,c_{\kappa_0+1},\ldots,c_{n-1})$. Substituting (3.4) into (3.1), we obtain

$$Z_j=\sum_{\alpha=\kappa_0+1}^{n-1} a_{j\alpha}(\epsilon)(Z_\alpha+c_\alpha W)+a_{jn}(\epsilon)W+\epsilon_j q_{\vec{c}},$$

Combining this with (3.3), we get

$$\begin{aligned}&\sum_{\alpha=\kappa_0+1}^{n-1} A_{j\alpha}(\epsilon)Z_\alpha + A_{jn}(\epsilon)W + \rho_j(\epsilon)\\ =&\sum_{\alpha=\kappa_0+1}^{n-1} a_{j\alpha}(\epsilon)(Z_\alpha + c_\alpha W) + a_{jn}(\epsilon)W + \epsilon_j(1 - 2i\overline{\vec{c}}\cdot Z + (r - i|\vec{c}|^2)W).\end{aligned} \tag{3.5}$$

By considering the coefficients of Z_j, $\kappa_0+1 \le j \le n-1$ and W terms, we obtain

$$A_{j\alpha}(\epsilon) = a_{j\alpha}(\epsilon) - \epsilon_j(2i\overline{c_\alpha}),\ A_{jn}(\epsilon) = a_{j\alpha}(\epsilon)c_\alpha + a_{jn}(\epsilon) + \epsilon_j(r - i|c|^2).$$

Thus we can choose $\vec{c}$ and r such that (3.2) holds true. From [Hu03] Lemma 2.2, there is a corresponding $\tau \in \mathrm{Aut}(\mathbb{H}_N)$ such that $\hat{F} = \tau \circ F \circ \sigma$ still takes the form (2.1). □

Next, we give some applications of the semi-linear subspace defined as above, which will be used later:

Let H be an affine linear function along L_ϵ. Write

$$\begin{aligned}&H|_{L_\epsilon}\\ =&H\Big(\sum_{\alpha=\kappa_0+1}^{n-1} a_{1\alpha}z_\alpha + a_{1n}w + \epsilon_1, \cdots, \sum_{\alpha=\kappa_0+1}^{n-1} a_{\kappa_0\alpha}z_\alpha + a_{\kappa_0 n}w + \epsilon_{\kappa_0},\ z_{\kappa_0+1}, ..., z_{n-1},\ w\Big).\end{aligned}$$

Then we must have $\frac{\partial^2 H|_{L_\epsilon}}{\partial z_\alpha \partial w} \equiv 0$ and $\frac{\partial^2 H|_{L_\epsilon}}{\partial w^2} \equiv 0$ for $\kappa_0+1 \le \alpha \le n-1$, from which we infer

$$\begin{aligned}\frac{\partial^2 H|_{L_\epsilon}}{\partial z_\alpha \partial w} =&\frac{\partial^2 H}{\partial z_\alpha \partial w} + \sum_{j=1}^{\kappa_0} \frac{\partial^2 H}{\partial z_j \partial w} a_{j\alpha} + \sum_{j=1}^{\kappa_0} \frac{\partial^2 H}{\partial z_j \partial z_\alpha} a_{jn} + \sum_{j=1}^{\kappa_0} \frac{\partial^2 H}{\partial z_j^2} a_{j\alpha}a_{jn}\\ &+ \sum_{i<j,i,j=1}^{\kappa_0} \frac{\partial^2 H}{\partial z_i \partial z_j}(a_{i\alpha}a_{jn} + a_{j\alpha}a_{in}) = 0, \quad at\ (\epsilon, 0)\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\frac{\partial^2 H|_{L_\epsilon}}{\partial w^2} =&\frac{\partial^2 H}{\partial w^2} + \sum_{j=1}^{\kappa_0} \frac{\partial^2 H}{\partial z_j \partial w} 2a_{jn} + \sum_{j=1}^{\kappa_0} \frac{\partial^2 H}{\partial z_j^2} a_{jn}^2\\ &+ \sum_{i<j,i,j=1}^{\kappa_0} \frac{\partial^2 H}{\partial z_i \partial z_j} 2a_{in}a_{jn} = 0, \quad at\ (\epsilon, 0).\end{aligned} \tag{3.7}$$

Choosing $H = f_h$ for $1 \le h \le \kappa_0$ in (3.6), and collecting ϵ_j, $1 \le j \le \kappa_0$ terms in the above equation, we get

$$\frac{i}{2}\mu_h a_{h\alpha}^{(1)} + \sum_{j=1}^{\kappa_0} f_h^{(I_j+I_\alpha+I_n)}\epsilon_j = 0, \qquad at\ (\epsilon, 0). \tag{3.8}$$

This, together with $h = 1$ in (3.2), gives $f_1^{(I_1+I_\alpha+I_n)}(\epsilon, 0) = 0$. On the other hand, by (2.6), we have $f_1^{(2,1)}(z) = -\xi_1$. Recall that $\xi_1^{(I_1+I_\alpha)} = \sqrt{\mu_1}\overline{e_{1,1\alpha}}$ in (2.7). Thus we obtain $f_1^{(I_1+I_\alpha+I_n)} = -\sqrt{\mu_1}\overline{e_{1,1\alpha}}$. Hence

$$e_{1,1\alpha} = 0 \text{ for } \kappa_0 + 1 \le \alpha \le n-1. \tag{3.9}$$

Setting $H = f_j$ for $1 \le j \le \kappa_0$ or ϕ in (3.7), respectively, we obtain

$$\frac{i}{2}\mu_j a_{jn}^{(1)}(\epsilon) + f_j^{(1,2)}(\epsilon, 0, \cdots, 0) = 0, \ \phi^{(1,2)}(\epsilon, 0, \cdots, 0) + \sum_{j=1}^{\kappa_0} e_j a_{jn}^{(1)}(\epsilon) = 0. \tag{3.10}$$

4. Some applications of the Chern-Moser equation. Let $F = (f, \phi, g) : \mathbb{H}_n \to \mathbb{H}_N$ with $N = n + \frac{2n-\kappa_0-1}{2}\kappa_0$ and geometric rank κ_0. Moreover, F satisfies the normalization as in Theorem 2.2. We will derive some basic relations from the Chern-Moser equation, which is based on the calculations in Section 4 of [HJY14].

When $N = n + \frac{2n-\kappa_0-1}{2}\kappa_0$ and the geometric rank being κ_0, as shown in (2.5), we have $\Phi_1 = 0$. In particular, $\Phi_1^{(3,0)} = 0$, thus (2.8) gives

$$\mu_j z_j \xi_l = \mu_l z_l \xi_j \text{ for } 1 \le j, l \le \kappa_0. \tag{4.1}$$

Denote $e_{i,jk} := e_{i,kj}$ and $\phi_{jk} := \phi_{kj}$ when $j > k$. Then for any j with $1 \le j \le \kappa_0$,

$$\begin{aligned}
\xi_j = \Phi^{(2,0)} \cdot \overline{e_j} &= \sum_{(i,k)\in\mathcal{S}_0, i=j, \ or, k=j} \phi_{ik}^{(2,0)} \cdot \overline{e_{e_{j,ik}}} + \sum_{(i,k)\in\mathcal{S}_0, i,k\neq j} \phi_{ik}^{(2,0)} \cdot \overline{e_{e_{j,ik}}} \\
&= \sum_{j<k\le n-1} \phi_{jk}^{(2,0)} \cdot \overline{e_{j,jk}} + \sum_{1\le i\le j} \phi_{ij}^{(2,0)} \cdot \overline{e_{j,ij}} + \sum_{(i,k)\in\mathcal{S}_0, i,k\neq j} \phi_{ik}^{(2,0)} \overline{e_{j,ik}} \\
&= \sum_{j<i\le n-1} \phi_{ji}^{(2,0)} \cdot \overline{e_{j,ji}} + \sum_{1\le i\le j} \phi_{ij}^{(2,0)} \cdot \overline{e_{j,ij}} + \sum_{(i,k)\in\mathcal{S}_0, i,k\neq j} \phi_{ik}^{(2,0)} \overline{e_{j,ik}} \\
&= \sum_{1\le i\le n-1} \phi_{ij}^{(2,0)} \cdot \overline{e_{j,ij}} + \sum_{(i,k)\in\mathcal{S}_0, i,k\neq j} \phi_{ik}^{(2,0)} \overline{e_{j,ik}}.
\end{aligned}$$

Observe that when $j \neq l$, the terms $z_l \sum_{(i,k)\in\mathcal{S}_0, i,k\neq j} \phi_{ik}^{(2,0)} \overline{e_{j,ik}}$ are not divided by z_j. Thus (4.1) implies $e_{j,ik} = 0$ for $(i,k) \in \mathcal{S}_0, i, k \neq j$. Now (4.1) is of the form

$$\mu_j z_j \sum_{1\le i\le n-1} \phi_{il}^{(2,0)} \overline{e_{l,il}} = \mu_l z_l \sum_{1\le i\le n-1} \phi_{ij}^{(2,0)} \overline{e_{j,ij}}. \tag{4.2}$$

We can write the above identity as

$$\mu_j z_j \sum_{1\le i\le n-1} \phi_{li}^{(2,0)} \overline{e_{l,li}} = \mu_l z_l \sum_{1\le i\le n-1} \phi_{ji}^{(2,0)} \overline{e_{j,ji}}.$$

Setting $l = 1$ and making use of (3.9), we obtain

$$\mu_j z_j \sum_{1\le i\le \kappa_0} \phi_{1i}^{(2,0)} \overline{e_{1,1i}} = \mu_1 z_1 \sum_{1\le i\le n-1} \phi_{ji}^{(2,0)} \overline{e_{j,ji}}$$

which implies $e_{j,j\alpha} = 0$ for any $\kappa_0 + 1 \le \alpha \le n-1$. Combining this with (2.6)-(2.7), we know $f_h^{(I_j+I_\alpha+I_n)} = 0$. Together with (3.8), we obtain

$$a_{h\alpha}^{(1)} = 0, \quad \forall 1 \le h \le \kappa_0, \ \kappa_0 + 1 \le \alpha \le n-1. \tag{4.3}$$

The rest relations in (4.2) are

$$\mu_j z_j \phi_{il}^{(2,0)} \overline{e_{l,il}} = \mu_l z_l \phi_{ij}^{(2,0)} \overline{e_{j,ij}} \text{ for } 1 \le i, j, l \le \kappa_0.$$

Namely, we obtain

$$\mu_j \mu_{il} e_{l,il} = \mu_l \mu_{ij} e_{j,ij} \text{ for } 1 \le i, j, l \le \kappa_0. \tag{4.4}$$

5. Proof of Theorem 1.1. This section is devoted to the proof of Theorem 1.1. The key point is to prove that the degree of the map in Theorem 1.1 is less than or equals to 2. Then we can apply Lebl's Theorem [L11] to complete the proof of our main theorem.

LEMMA 5.1. *Keep the notations and assumptions in Theorem 1.1, then deg*$(F) \leq 2$.

Proof. From the basic Chern-Moser equation, we have

$$\frac{g(z,w)-\overline{g(z,w)}}{2i} = f(z,w)\cdot\overline{f(z,w)} + \phi(z,w)\cdot\overline{\phi(z,w)}, \quad \forall \mathrm{Im}(w) = |z|^2.$$

By complexification, we write

$$\frac{g(z,w)-\overline{g(\overline{\chi},\overline{\eta})}}{2i} = \sum_{l=1}^{n-1} f_l(z,w)\overline{f_l(\overline{\chi},\overline{\eta})} + \sum \phi_t(z,w)\overline{\phi_t(\overline{\chi},\overline{\eta})}, \ \ \forall \frac{w-\eta}{2i} = z\cdot\chi.$$

Applying $\mathcal{L}_j := \frac{\partial}{\partial z_j} + 2i\chi_j\frac{\partial}{\partial w}$ for $z=0$ and $w=\eta=0$ to the both sides of the above identity, we obtain

$$\frac{\mathcal{L}_j g(0,0)}{2i} = \sum_{l=1}^{n-1} \mathcal{L}_j f_l(0,0)\overline{f_l(\overline{\chi},0)} + \sum \mathcal{L}_j\phi_t(0,0)\overline{\phi_t(\overline{\chi},0)}$$

and

$$\frac{\mathcal{L}_j\mathcal{L}_k g(0,0)}{2i} = \sum_{l=1}^{n-1} \mathcal{L}_j\mathcal{L}_k f_l(0,0)\overline{f_l(\overline{\chi},0)} + \sum \mathcal{L}_j\mathcal{L}_k\phi_t(0,0)\overline{\phi_t(\overline{\chi},0)}.$$

In terms of matrix, they take the form

$$\begin{pmatrix} \chi_1 \\ \cdots \\ \chi_{\kappa_0} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = B\begin{pmatrix} \overline{f_1(\overline{\chi},0)} \\ \cdots \\ \overline{f_{\kappa_0}(\overline{\chi},0)} \\ \overline{\phi(\overline{\chi},0)} \end{pmatrix} \tag{5.1}$$

where $B(F)$ is a $\frac{\kappa_0}{2}(2n-\kappa_0+1)\times\frac{\kappa_0}{2}(2n-\kappa_0+1)$ matrix:

$$B(F) := \begin{pmatrix} \mathcal{L}_j f_h & \mathcal{L}_j\phi_{hl} & \mathcal{L}_j\phi_{h\alpha} \\ \mathcal{L}_j\mathcal{L}_k f_h & \mathcal{L}_j\mathcal{L}_k\phi_{hl} & \mathcal{L}_j\mathcal{L}_k\phi_{h\alpha} \\ \mathcal{L}_j\mathcal{L}_\beta f_h & \mathcal{L}_j\mathcal{L}_\beta\phi_{hl} & \mathcal{L}_j\mathcal{L}_\beta\phi_{h\alpha} \end{pmatrix}_{1\leq j,h,k,l\leq\kappa_0,\kappa_0+1\leq\alpha,\beta\leq n-1}\Bigg|_{(0,0,\chi,0)} \tag{5.2}$$

Write $A_j = \frac{\mu_{1j}e_{1,1j}}{\mu_1}$, and let $\widetilde{F}:\mathbb{C}^{n-1}\backslash\{1-2i\sum_{j=1}^{\kappa_0}\overline{A_j}z_j=0\}\to\mathbb{C}^{N-1}$ be defined as follows:

$$\begin{aligned} &\widetilde{f_\mu}(z) = z_\mu \text{ for } 1\leq\mu\leq n-1, \\ &\widetilde{\phi_{jj}}(z) = \frac{\mu_{jj}z_j^2}{1-2i\sum_{j=1}^{\kappa_0}\overline{A_j}z_j} \text{ for } 1\leq j\leq\kappa_0, \\ &\widetilde{\phi_{jk}}(z) = \frac{\mu_{jk}z_jz_k}{1-2i\sum_{j=1}^{\kappa_0}\overline{A_j}z_j} \text{ for } 1\leq j<k\leq\kappa_0, \\ &\widetilde{\phi_{j\alpha}}(z) = \frac{\mu_{j\alpha}z_jz_\alpha}{1-2i\sum_{j=1}^{\kappa_0}\overline{A_j}z_j} \text{ for } 1\leq j\leq\kappa_0<\alpha\leq n-1. \end{aligned} \tag{5.3}$$

We claim that $F(z,0) = \widetilde{F}$, which follows from the following identity:

$$\begin{pmatrix} \chi_1 \\ \vdots \\ \chi_{\kappa_0} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = B \begin{pmatrix} \overline{\widetilde{f}_1(\overline{\chi})} \\ \vdots \\ \overline{\widetilde{f}_{\kappa_0}(\overline{\chi})} \\ \overline{\widetilde{\phi}(\overline{\chi})} \end{pmatrix}, \tag{5.4}$$

In fact, once (5.4) is achieved, we infer from (5.1) that

$$B \begin{pmatrix} \overline{f_1(\overline{\chi},0)} - \overline{\widetilde{f}_1(\overline{\chi})} \\ \vdots \\ \overline{f_{\kappa_0}(\overline{\chi},0)} - \overline{\widetilde{f}_{\kappa_0}(\overline{\chi})} \\ \overline{\phi(\overline{\chi},0)} - \overline{\widetilde{\phi}(\overline{\chi})} \end{pmatrix} = 0.$$

Notice that

$$B = \mathrm{diag}\big(1, \cdots, 1, A_1, \cdots, A_{\kappa_0}, B_1, \cdots, B_{\kappa_0}\big) + O(|\chi|).$$

Here

$$A_j = (2\sqrt{\mu_j}, \sqrt{\mu_j + \mu_{j+1}}, \cdots, \sqrt{\mu_j + \mu_{\kappa_0}}) \in \mathbb{C}^{\kappa_0 - j+1},$$
$$B_j = (\sqrt{\mu_j}, \cdots, \sqrt{\mu_j}) \in \mathbb{C}^{n-\kappa_0}.$$

Thus B is nonsingular and we derive the claim $F(z,0) = \widetilde{F}(z)$. Hence $deg(F(z,0)) \leq 2$. Replacing F by F_p^{***} for any $p \in \partial\mathbb{H}_n$ near the origin, we can show $deg(F_p^{**}(z,0)) \leq 2$ in a similar manner. By [HJ01, Section 5], we have that $deg(F) \leq 2$.

The identity (5.4) follows from the following direct computations:

• **Calculate** $(\mathcal{L}_h H)(0,0)$ **with** $1 \leq h \leq \kappa_0$ **for** $H = f_j, \phi_{jk}$**.** At the point $(0,0)$, we have

$$(\mathcal{L}_h f_j)(0,0) = \delta_h^j,$$
$$(\mathcal{L}_h \phi_{jk})(0,0) = 0 \text{ for } (j,k) \in \mathcal{S}_0.$$

Then

$$\sum_{j=1}^{\kappa_0} (\mathcal{L}_h f_j)(0,0) \cdot \overline{\widetilde{f}_j(\overline{\chi})} + \sum_{(\mu,\nu)\in\mathcal{S}_0} (\mathcal{L}_h \phi_{\mu\nu})(0) \cdot \overline{\widetilde{\phi}_{\mu\nu}(\overline{\chi})} = \sum_{j=1}^{\kappa_0} \delta_h^j \cdot \chi_j = \chi_h. \tag{5.5}$$

• **Calculate** $(\mathcal{L}_h^2 H)(0,0)$ $(1 \leq h \leq \kappa_0)$ **for** $H = f_j, \phi_{jk}$**.** A direct computation shows that $\mathcal{L}_h^2 = \frac{\partial^2}{\partial z_h^2} + 4i\chi_h \frac{\partial^2}{\partial z_h \partial w} + (2i\chi_h)^2 \frac{\partial^2}{\partial w^2}$. At the point $(0,0)$, we have

$$(\mathcal{L}_h^2 f_j)(0,0) = 4i\chi_h \cdot \frac{i}{2}\mu_j \cdot \delta_h^j = -2\delta_h^j \mu_j \chi_j,$$
$$(\mathcal{L}_h^2 \phi_{hh})(0,0) = 2\sqrt{\mu_h} + 4i\chi_h e_{h,hh},$$
$$(\mathcal{L}_h^2 \phi_{hj})(0,0) = 4i\chi_h e_{h,hj} \text{ for } 1 \leq j \leq n-1,\ j \neq h,$$
$$(\mathcal{L}_h^2 \phi_{jk})(0,0) = 0 \text{ for } j,k \neq h.$$

Setting $l = 1$ in (4.4), we obtain

$$\mu_{hj} e_{h,hj} = \frac{\mu_h}{\mu_1} \mu_{1j} e_{1,1j} = \mu_h A_j. \tag{5.6}$$

Thus we get

$$\begin{aligned}
&\sum_{j=1}^{\kappa_0} (\mathcal{L}_h^2 f_j)(0,0) \cdot \overline{\widetilde{f}_j(\overline{\chi})} + \sum_{(\mu,\nu)\in\mathcal{S}_0} (\mathcal{L}_h^2 \phi_{\mu\nu})(0,0) \cdot \overline{\widetilde{\phi}_{\mu\nu}(\overline{\chi})} \\
=&(-2\mu_h \chi_h) \cdot \chi_h + \big(2\sqrt{\mu_h} + 4i\chi_h e_{h,hh}\big) \cdot \frac{\mu_{hh}\chi_h^2}{1 + 2i\sum_{j=1}^{\kappa_0} A_j \chi_j} \\
&+ \sum_{j\neq h, 1\leq j\leq n-1} 4i\chi_h e_{h,hj} \cdot \frac{\mu_{hj}\chi_h\chi_j}{1 + 2i\sum_{j=1}^{\kappa_0} A_j\chi_j} \\
=&\frac{1}{1 + 2i\sum_{j=1}^{\kappa_0} A_j\chi_j} \Big(-2\mu_h\chi_h^2 \cdot 2i \sum_{j=1}^{\kappa_0} A_j\chi_j + 4i\chi_h^2 \sum_{j=1}^{\kappa_0} \mu_{hj} e_{h,hj}\chi_j \Big) = 0.
\end{aligned} \tag{5.7}$$

• **Calculate** $(\mathcal{L}_h\mathcal{L}_l H)(0,0)$ $(1 \leq h < l \leq \kappa_0)$ **for** $H = f_j, \phi_{jk}$**.** A direct computation shows that $\mathcal{L}_h\mathcal{L}_l = \frac{\partial^2}{\partial z_h \partial z_l} + 2i\chi_l \frac{\partial^2}{\partial z_h \partial w} + 2i\chi_h \frac{\partial^2}{\partial z_l \partial w} - 4\chi_h\chi_l \frac{\partial^2}{\partial w^2}$. At the point $(0,0)$, we have

$$\begin{aligned}
&(\mathcal{L}_h\mathcal{L}_l f_j)(0,0) = 2i\chi_l \cdot \frac{i}{2}\mu_j\delta_h^j + 2i\chi_h \cdot \frac{i}{2}\mu_j\delta_l^j = -\mu_j\chi_l\delta_h^j - \mu_l\chi_h\delta_l^j, \\
&(\mathcal{L}_h\mathcal{L}_l \phi_{hl})(0,0) = \mu_{hl} + 2i\chi_l \cdot e_{h,hl} + 2i\chi_h \cdot e_{l,lh}, \\
&(\mathcal{L}_h\mathcal{L}_l \phi_{jk})(0,0) = 2i\chi_l \cdot e_{h,jk} + 2i\chi_h \cdot e_{l,jk} \ for \ (j,k) \neq (h,l) \text{ or } (l,h).
\end{aligned}$$

Combining this with (5.6), we get

$$\begin{aligned}
&\sum_{j=1}^{\kappa_0} (\mathcal{L}_h\mathcal{L}_l f_j)(0,0) \cdot \overline{\widetilde{f}_j(\overline{\chi})} + \sum_{(\mu,\nu)\in\mathcal{S}_0} (\mathcal{L}_h\mathcal{L}_l \phi_{\mu\nu})(0,0) \cdot \overline{\widetilde{\phi}_{\mu\nu}(\overline{\chi})} \\
=& -\mu_h\chi_l \cdot \chi_h - \mu_l\chi_h \cdot \chi_l + \big(\mu_{hl} + 2i\chi_l \cdot e_{h,hl} + 2i\chi_h \cdot e_{l,lh}\big) \cdot \frac{\mu_{hl}\chi_h\chi_l}{1 + 2i\sum_{j=1}^{\kappa_0} A_j\chi_j} \\
&+ \sum_{1\leq j<k\leq\kappa_0, (j,k)\neq(h,l)} \big(2i\chi_l \cdot e_{h,jk} + 2i\chi_h \cdot e_{l,jk}\big) \cdot \frac{\mu_{jk}\chi_j\chi_k}{1 + 2i\sum_{j=1}^{\kappa_0} A_j\chi_j} \\
=& \frac{1}{1 + 2i\sum_{j=1}^{\kappa_0} A_j\chi_j} \Big(-(\mu_h + \mu_l)\chi_h\chi_l \cdot 2i \sum_{j=1}^{\kappa_0} A_j\chi_j + 2i\chi_l \sum_{1\leq k\leq\kappa_0} e_{h,hk}\mu_{hk}\chi_h\chi_k \\
&+ 2i\chi_h \sum_{1\leq j\leq\kappa_0} e_{l,lj}\mu_{lj}\chi_j\chi_l \Big) = 0.
\end{aligned} \tag{5.8}$$

• **Calculate** $(\mathcal{L}_h\mathcal{L}_\alpha H)(0,0)$ $(1 \leq h \leq \kappa_0 < \alpha \leq n-1)$ **for** $H = f_i, \phi_{jl}$**.** A direct computation shows that $\mathcal{L}_h\mathcal{L}_\alpha = \frac{\partial^2}{\partial z_h \partial z_\alpha} + 2i\chi_\alpha \frac{\partial^2}{\partial z_h \partial w} + 2i\chi_h \frac{\partial^2}{\partial z_\alpha \partial w} + 2i\chi_h \cdot 2i\chi_\alpha \frac{\partial^2}{\partial w^2}$. At the point $(0,0)$, we have

$$\begin{aligned}
&(\mathcal{L}_h\mathcal{L}_\alpha f_j)(0,0) = 2i\chi_\alpha \cdot \frac{i}{2}\mu_j\delta_j^h = -\mu_h\chi_\alpha\delta_j^h, \\
&(\mathcal{L}_h\mathcal{L}_\alpha \phi_{jk})(0,0) = 2i\chi_\alpha e_{h,jk}, \\
&(\mathcal{L}_h\mathcal{L}_\alpha \phi_{j\beta})(0,0) = \mu_{h\alpha}\delta_h^j\delta_\alpha^\beta.
\end{aligned}$$

We get

$$\begin{aligned}
&\sum_{j=1}^{\kappa_0}(\mathcal{L}_h\mathcal{L}_\alpha f_j)(0,0)\cdot\overline{\widetilde{f}_j(\overline{\chi})}+\sum_{(\mu,\nu)\in\mathcal{S}_0}(\mathcal{L}_h\mathcal{L}_\alpha\phi_{\mu\nu})(0,0)\cdot\overline{\widetilde{\phi}_{\mu\nu}(\overline{\chi})}\\
=&-\mu_h\chi_\alpha\cdot\chi_h+\sum_{1\le j\le k\le\kappa_0}2i\chi_\alpha e_{h,jk}\cdot\frac{\mu_{jk}\chi_j\chi_k}{1+2i\sum_{j=1}^{\kappa_0}A_jz_j}+\mu_{h\alpha}\frac{\mu_{h\alpha}\chi_h\chi_\alpha}{1+2i\sum_{j=1}^{\kappa_0}A_jz_j}\\
=&\frac{1}{1+2i\sum_{j=1}^{\kappa_0}A_jz_j}\Big(-\mu_h\chi_h\chi_\alpha\cdot 2i\sum_{j=1}^{\kappa_0}A_jz_j+\sum_{j=1}^{\kappa_0}2i\chi_\alpha e_{h,jh}\cdot\mu_{jh}\chi_j\chi_h\Big)=0.
\end{aligned}\tag{5.9}$$

By all of the above, (5.4) is proved. This also finishes the proof of Lemma 5.1. □

Now we are in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemma 5.1, the degree of the map in Theorem 1.1 is at most 2. By Lebl's Theorem [L11, Theorem 1.5], it must have the following form:

$$(\sqrt{t_1}z_1,\sqrt{t_2}z_2,...,\sqrt{t_n}z_n,\sqrt{1-t_1}z_1^2,\sqrt{1-t_2}z_2^2,...,\sqrt{1-t_n}z_n^2,\sqrt{2-t_i-t_j}z_iz_j)_{i\neq j}\tag{5.10}$$

where $0\le t_1\le ...\le t_n\le 1$, $(t_1,t_2,...,t_n)\neq(1,1,...,1)$.

Suppose that $t_j=0$ for $1\le j\le h$, $t_j=1$ for $k+1\le j\le n$ and $t_j\in(0,1)$ for $h\le j\le k$. Here $0\le h\le k\le n$ and $h=0$ means that there is no $t_j=0$. Then it has the following form:

$$\big(\sqrt{t_{h+1}}z_{h+1},\cdots,\sqrt{t_n}z_n,\sqrt{1-t_1}z_1^2,\cdots,\sqrt{1-t_k}z_k^2,(\sqrt{2-t_i-t_j}z_iz_j)_{1\le i\le k,1\le j\le n,i<j}\big)\tag{5.11}$$

Notice that this map is linear on $z_j=c_j$ for $1\le j\le k$ and can not be linear on any lower dimensional linear subspace. By [Hu03], the geometric rank of this map is k. Thus we have $k=\kappa_0$. By counting the dimension of the map, we have

$$(n-h)+k+\frac{n(n-1)}{2}-\frac{(n-k)(n-k-1)}{2}=n+\frac{2n-\kappa_0-1}{2}\kappa_0.$$

Then we get $h=k=\kappa_0$. In this case, (5.11) is exactly the generalized Whitney map defined by (1.3)-(1.4). This completes the proof of Theorem 1.1. □

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