J Geom Anal (2014) 24:1912–1928
DOI 10.1007/s12220-013-9402-0

CR Embeddings and Kähler Manifolds with Pseudo-Conformally Flat Curvature Tensors

Xiaojun Huang · Shanyu Ji · Brandon Lee

Received: 9 October 2012 / Published online: 15 March 2013
© Mathematica Josephina, Inc. 2013

Abstract Under certain conditions on co-dimension and curvature tensors, the image of some CR or holomorphic maps are proved to be totally geodesic.

Keywords CR embedding · Kähler manifold · Pseudo-conformally flat curvature tensor

Mathematics Subject Classification Primary 30V30 · Secondary 30V20

1 Introduction

In Several Complex Variables, understanding when a CR manifold can be embedded into a sphere is a subtle problem. Forstneric [9] and Faran [7] proved the existence of real analytic strictly pseudoconvex hypersurfaces in $\mathbb{C}^{n+1}$ which do not admit any germ of non-constant holomorphic map taking M into the sphere $\partial\mathbb{B}^{N+1}$ for any positive integer N. Zaitsev constructed explicit examples for the Forstneric–Faran phenomenon [26]. Meanwhile, there has been much work done to prove the uniqueness of such embeddings up to the action of automorphisms. For instance, a well-known rigidity theorem says that if M^{2n+1} is a CR spherical immersion inside $\partial\mathbb{B}^{N+1}$ with

Communicated by Peter Ebenfelt.

X. Huang
Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway Township, NJ 08854, USA
e-mail: huangx@math.rutgers.edu

S. Ji (✉) · B. Lee
Department of Mathematics, University of Houston, Houston, TX 77204, USA
e-mail: shanyuji@math.uh.edu

B. Lee
e-mail: bmlee68@math.uh.edu

$N \le 2n-1$, then M must be totally geodesic (i.e., M is the image of $\partial\mathbb{B}^{n+1}$ by a linear fractional holomorphic map). Ebenfelt, Huang, and Zaitsev ([5], Theorem 1.2) proved that if $d < \frac{n}{2}$, any smooth CR-immersion $f : M \to \partial\mathbb{B}^{n+d+1}$, where M is a smooth CR hypersurface of dimension $2n+1$, is rigid. Oh in [21] obtained a very interesting result on the non-embeddability for real hyperboloids into spheres of low codimension. Kim and Oh [17] found a necessary and sufficient condition for the local holomorphic embeddability into a sphere of a generic strictly pseudoconvex pseudo-Hermitian CR manifold in terms of its Chern–Moser curvatures. Along these lines, we mention recent studies in the papers of Huang and Zhang [15], Ebenfelt and Sun [6] and Huang and Zaitsev [14]. We also refer the reader to a recent survey paper [12] by the first two authors and many references therein. Our first goal in this paper is to study the non-embeddability property for a class of hypersurfaces, called real hypersurfaces of involution type, in the low codimensional case, by the properties of a naturally related Gauss curvature. (For some other studies on degenerate hypersurfaces of involution type in $\mathbb{C}^2$, we also mention a recent paper by Kolar and Lambel [18].)

We first recall that a connected real hypersurface M in $\mathbb{C}^n$ is called a *real hypersurface of revolution type* in $\mathbb{C}^n$ if it can be defined by an equation of the following form:

$$\begin{aligned} &M = \left\{(z,w) \in \mathbb{C}^n \times \mathbb{C} \mid r = 0\right\} \\ &r = p(z,\overline{z}) + q(w,\overline{w}), \quad q(w,\overline{w}) = \overline{q}(w,\overline{w}),\ \mathrm{d}(q)|_{\{q=0\}} \neq 0, \\ &p(z,\overline{z}) = \sum_{1\le\alpha,\beta\le n} h_{\alpha\overline{\beta}} z^{\alpha}\overline{z}^{\beta}. \end{aligned} \tag{1}$$

Here $(h_{\alpha\overline{\beta}})$ is a positive definite (constant) Hermitian matrix, $q(w,\overline{w})$ is a smooth function in $\mathbb{C}$ and takes negative values somewhere in $\mathbb{C}$. Such a real hypersurface admits a $U(n)$-action and was first studied by Webster in [24]. Associated with such a real hypersurface is a domain D_0 in $\mathbb{C}$ defined by $D_0 := \{w \in \mathbb{C} :\ q(w,\overline{w}) < 0\}$. Assume that M is strongly pseudoconvex in a certain small neighborhood M_0 of $(z_0, w_0) \in M$ with $w_0 \in D_0$. Write π for the projection from $M \setminus \{q(w,\overline{w}) = 0\}$ into D_0. And assume that U_0 is a neighborhood of w_0 in D_0 with $\pi(M_0) = U_0$. Then, by the symmetry, $\pi^{-1}(U_0)$ is strongly pseudoconvex. Hence, without loss of generality, we assume that $\pi^{-1}(U_0) = M_0$. Webster observed that then $h := -(\log q)_{w\overline{w}} > 0$ in U_0 and thus we have a well-defined Hermitian metric $ds^2 = h dw d\overline{w}$. Write the Gauss curvature of such a metric as K, which is given by $K = -\frac{1}{h}\frac{\partial^2}{\partial w \partial \overline{w}} \log h$. We first prove the following result, which reveals the connection between the Hermitian geometry over U_0 and the local smooth CR embeddability of M into a sphere with lower codimension:

Theorem 1.1 *Let D_0, U_0, K, and M_0 be just defined as above. Let $M_0 = \pi^{-1}(U_0)$ be a (connected) strongly pseudoconvex open piece of the real hypersurface of revolution in $\mathbb{C}^{n+1}$ defined as in (1) with $2 \le n \le N \le 2n-2$. Suppose that the Gauss curvature $K \ge -2$ over U_0 and for any $p \in M_0$, there is a non-constant smooth CR map from a neighborhood of p in M_0 into $\partial\mathbb{B}^{N+1}$. Then $K \equiv -2$ over U_0 and the embedding* *image in* $\partial\mathbb{B}^{N+1}$ *is totally geodesic, namely, a CR transversal intersection of an affine complex subspace of dimension* $(n+1)$ *with an open piece of* $\partial\mathbb{B}^{N+1}$.

Example 1.2 Let $q = |w|^2 + \epsilon|w|^4 - 1$ and $(h_{\alpha\overline{\beta}}) = I_{n\times n}$ in (1). Then, for $\epsilon > 0$, M admits a non-totally geodesic holomorphic embedding into the unit sphere in $\mathbb{C}^{n+2}$ through the map $(z, w) \mapsto (z, w, \sqrt{\epsilon}w^2)$. However, for $\epsilon < 0$, the Gauss curvature K of $ds^2 = -(\log q)_{w\overline{w}} dw \otimes d\overline{w}$ is given by $K = -2 - 4\epsilon + o(1) > -2$ near a neighborhood of $w = 0$. (See Example 7.1.) Thus, by Theorem 1.1 and the algebraicity theorem of the first author in [10], M in this setting cannot be local holomorphically embedded into $\partial\mathbb{B}^{N+1}$ with $N \le 2n - 2$. Hence the curvature assumption is needed in Theorem 1.1. Similarly, let $q = |w|^2 + \epsilon|w|^4 + |w|^6 - 1$ with $\epsilon < 0$, $|\epsilon| \ll 1$. Then M defined by $r = |z|^2 + |w|^2 + \epsilon|w|^4 + |w|^6 - 1 = 0$ is now compact and strongly pseudoconvex. Since the Gauss curvature K defined above now is larger than -2 in a neighborhood of 0 in D_0, combining Theorem 1.1 with the algebraicity theorem of the first author in [10], we also see that any open piece of M cannot be smoothly CR embedded into $\partial\mathbb{B}^{N+1}$ with $N \le 2n - 2$. However, it appears to be a very interesting problem to find out if the assumption $N \le 2n - 2$ can be dropped in this specific algebraic and compact strongly pseudoconvex example.

Remark 1.3 There is a very nice connection of the study of real hypersurfaces of involution type and Hermitian vector bundles over a Riemann surface. Indeed, M_0 in (1) can be regarded as the Grauert tube (or sphere bundle) of the trivial holomorphic vector bundle of rank n over D_0 with the Hermitian metric $d^2h = -q(w, \overline{w})d^2Eucl$. For this reason, it may be interesting to consider further the case when D_0 is a domain in $\mathbb{C}^m$ with $m > 1$.

Our proof of Theorem 1.1 is based on the framework established in [5], computations of pseudo-Hermitian curvature tensors in [24], and the following rigidity lemma of the first author:

Rigidity Lemma [11] *Let* $g_1, \ldots, g_k, f_1, \ldots, f_k$ *be holomorphic functions in* $z \in \mathbb{C}^n$ *near* 0. *Assume* $g_j(0) = f_j(0) = 0$ *for all* j. *Let* $A(z, \overline{z})$ *be real-analytic near the origin such that*

$$\sum_{j=1}^{k} g_j(z)\overline{f_j(z)} = |z|^2 A(z, \overline{z}). \tag{2}$$

If $k \le n - 1$, *then* $A(z, \overline{z}) \equiv 0$ *and* $\sum_{j=1}^{k} g_j(z)\overline{f_j(z)} \equiv 0$.

This rigidity lemma played an important role in understanding many other problems in CR geometry. For instance, the proof of the third gap theorem [13] is obtained by repeatedly applying this lemma in subtle ways. In [5], a different formulation of the above lemma was formulated. A new formulation of this rigidity lemma is presented in Lemma 2.1 of Sect. 2, and will be used in this paper.

Along the same lines of applying the above rigidity lemma, we also study rigidity problems for conformal maps between a class of Kähler manifolds with pseudo-conformally flat metrics. More precisely, we prove the following:

Theorem 1.4 *Let $f : (X, \omega) \to (Y, \sigma)$ be a holomorphic conformal embedding, where (X, ω) and (Y, σ) are Kähler manifolds with $\dim_{\mathbb{C}} X = n$ and $\dim_{\mathbb{C}} Y = N$. Suppose $2 \le n \le N \le 2n - 1$ and that the curvature tensors of (X, ω) and (Y, σ) are pseudo-conformally flat. Then $f(X)$ is a totally geodesic submanifold of Y.*

Here we mention that a holomorphic map $f : (M, \omega) \to (N, \sigma)$ between Hermitian manifolds M and N is called *conformal* if $f^*\sigma = k\omega$ holds for some positive function k on M. When $\dim(M) > 1$ and both M and N are Kähler, the conformal factor k is always a positive constant. Hence, in our consideration, we always assume that k is a positive constant. A tensor $T_{\alpha\overline{\beta}\mu\overline{\nu}}$ over a complex manifold is called *pseudo-conformally flat* if in any holomorphic chart, we have

$$T_{\alpha\overline{\beta}\mu\overline{\nu}} = H_{\alpha\overline{\beta}} g_{\mu\overline{\nu}} + \hat{H}_{\mu\overline{\beta}} g_{\alpha\overline{\nu}} + H^*_{\alpha\overline{\nu}} g_{\mu\overline{\beta}} + \widetilde{H}_{\mu\overline{\nu}} g_{\alpha\overline{\beta}}, \tag{3}$$

where $(H_{\alpha\overline{\beta}})$, $(\hat{H}_{\alpha\overline{\beta}})$, $(H^*_{\alpha\overline{\beta}})$, and $(\widetilde{H}_{\alpha\overline{\beta}})$ are smoothly varied Hermitian matrices, and $(g_{\alpha\overline{\beta}})$ is the smoothly varied Hermitian metric over the chart.

Basic examples for Hermitian manifolds with pseudo-conformally flat curvature tensors are the complex space forms: $\mathbb{C}^n$ with the Euclidean metric, $\mathbb{CP}^n$ with the Fubini–Study metric and $\mathbb{B}^n$ with the Poincaré metric. Other more complicated examples contain the Bochner–Kähler manifolds [1].

Concerning the dimension condition $N \le 2n - 1$ in Theorem 1.2, we recall some related results on global holomorphic immersions. For $\mathbb{CP}^n$, Feder proved in 1965 [8] that any holomorphic immersion $f : \mathbb{CP}^n \to \mathbb{CP}^N$ with $N \le 2n - 1$ has totally geodesic image (realizing $\mathbb{CP}^n$ as a linear subvariety). For $X = \mathbb{B}^n/\Gamma$, Cao and Mok proved in 1990 [3] that if $f : X \to Y$ is a holomorphic immersion where X and Y are complex hyperbolic space forms of complex dimension n and N respectively, such that X is compact and $N \le 2n - 1$, then f has totally geodesic image. In CR geometry, we have the rigidity theorem [11]: If $F : \partial\mathbb{B}^{n+1} \to \partial\mathbb{B}^{N+1}$ is a CR map which is C^2-smooth with $1 \le n \le N \le 2n - 1$, then F must be linear fractional. Also, Mok had constructed an example [20] of a non-totally geodesic holomorphic isometric embedding from the disc Δ into Δ^p with $p > 1$. For other related rigidity results, we refer the reader to the papers by Calabi [2], Mok and Ng [19], Mok [20], Yuan and Zhang [25] and many references therein.

2 A Tensor Version of the Rigidity Lemma

We first reformulate the rigidity lemma mentioned in (2) into the following version (see also related formulations in [5]):

Lemma 2.1 *Let $A^a_{\alpha\beta}$ and $B^a_{\alpha\beta}$ be complex numbers where $1 \le \alpha, \beta \le n$, $n + 1 \le a \le N$. Let $(g_{\alpha\overline{\beta}})$ and $(G_{a\overline{b}})$ be Hermitian matrices with $(g_{\alpha\overline{\beta}})$ positive definite. Let $(H^{(l)}_{\alpha\overline{\beta}})$, $(\hat{H}^{(l)}_{\alpha\overline{\beta}})$, $(H^{*(l)}_{\alpha\overline{\beta}})$, and $(\widetilde{H}^{(l)}_{\alpha\overline{\beta}})$ be Hermitian matrices where $1 \le l \le k$. Suppose* *that $N-n \le n-1$ and that*

$$\sum_{a,b=n+1}^{N} G_{a\overline{b}} A^{a}_{\alpha\beta} X^{\alpha} X^{\beta} \overline{B^{b}_{\mu\nu}}\, \overline{X^{\mu} X^{\nu}}$$
$$= \sum_{l=1}^{k} \big(H^{(l)}_{\alpha\overline{\beta}} g_{\mu\overline{\nu}} + \hat{H}^{(l)}_{\mu\overline{\beta}} g_{\alpha\overline{\nu}} + H^{*(l)}_{\alpha\overline{\nu}} g_{\mu\overline{\beta}} + \widetilde{H}^{(l)}_{\mu\overline{\nu}} g_{\alpha\overline{\beta}} \big) X^{\alpha} \overline{X^{\beta}} X^{\mu} \overline{X^{\nu}} \tag{4}$$

holds for any $X = (X^{\alpha}) = (X^{\beta}) = (X^{\mu}) = (X^{\nu}) \in \mathbb{C}^n$. Then

$$\sum_{a,b=n+1}^{N} G_{a\overline{b}} A^{a}_{\alpha\overline{\beta}} X^{\alpha} X^{\beta} \overline{B^{b}_{\mu\nu}}\, \overline{X^{\mu} X^{\nu}} \equiv 0, \quad \forall X \in \mathbb{C}^n. \tag{5}$$

Proof The right-hand side of (4) is equal to

$$\sum_{l=1}^{k} \big(H^{(l)}_{\alpha\overline{\beta}} g_{\mu\overline{\nu}} + \hat{H}^{(l)}_{\mu\overline{\beta}} g_{\alpha\overline{\nu}} + H^{*(l)}_{\alpha\overline{\nu}} g_{\mu\overline{\beta}} + \widetilde{H}^{(l)}_{\mu\overline{\nu}} g_{\alpha\overline{\beta}} \big) X^{\alpha} X^{\mu} \overline{X^{\beta}}\, \overline{X^{\nu}}$$
$$= \sum_{l=1}^{k} \big(H^{(l)}_{\alpha\overline{\beta}} X^{\alpha} \overline{X^{\beta}} |X|^2 + \hat{H}^{(l)}_{\mu\overline{\beta}} X^{\mu} \overline{X^{\beta}} |X|^2 + H^{*(l)}_{\alpha\overline{\nu}} X^{\alpha} \overline{X^{\nu}} |X|^2 + \widetilde{H}^{(l)}_{\mu\overline{\nu}} X^{\mu} \overline{X^{\nu}} |X|^2 \big)$$
$$= |X|^2 \sum_{l=1}^{k} \big(H^{(l)}_{\alpha\overline{\beta}} X^{\alpha} \overline{X^{\beta}} + \hat{H}^{(l)}_{\mu\overline{\beta}} X^{\mu} \overline{X^{\beta}} + H^{*(l)}_{\alpha\overline{\nu}} X^{\alpha} \overline{X^{\nu}} + \widetilde{H}_{\mu\overline{\nu}} X^{\mu} \overline{X^{\nu}} \big)$$
$$= |X|^2 A(X, \overline{X}), \tag{6}$$

where $A(X, \overline{X})$ is some real analytic function of X and $|X|^2 = g_{\alpha\overline{\beta}} X^{\alpha} \overline{X^{\beta}}$. Notice that the left-hand side of (4) is equal to

$$\sum_{a,b=n+1}^{N} G_{ab} A^{a}_{\alpha\beta} X^{\alpha} X^{\beta} \overline{B^{b}_{\mu\nu}}\, \overline{X^{\mu}}\, \overline{X^{\nu}} = \sum_{a=n+1}^{N} g_a(X) \overline{h_a(X)} \tag{7}$$

where $g_a(X) = \sum_{\alpha,\beta} A^{a}_{\alpha\beta} X^{\alpha} X^{\beta}$ and $h_a(X) = \sum_{b=n+1}^{N} \sum_{\alpha,\beta} \overline{G_{ab}} B^{b}_{\alpha\beta} X^{\alpha} X^{\beta}$ are holomorphic functions. Namely, we have

$$\sum_{a=n+1}^{N} g_a(X) \overline{h_a(X)} = |X|^2 A(X, \overline{X}), \quad \forall X \in \mathbb{C}^n.$$

By the hypothesis, $N - n < n$, and it follows from the rigidity lemma (2) that $A(X, \overline{X}) \equiv 0$. Thus (5) holds. □

3 Pseudo-Hermitian Geometry

CR Submanifold of Hypersurface Type Let M be a smooth strictly pseudoconvex $(2n+1)$-dimensional CR submanifold in $\mathbb{C}^{n+1}$. A non-zero real smooth 1-form θ along M is said to be a *contact* form of M if $\theta|_p$ annihilates $T_p^{(1,0)}M \oplus T_p^{(0,1)}M$ for any $p \in M$. Let r be a local defining function of M. Then $\theta = i\partial_z r$ is a contact form of M and any other contact form is a multiple of θ: $k\theta$ with $k \neq 0$ a smooth function along M. Fix a contact form θ. Then there is a unique smooth vector field T, called the *Reeb vector field* such that (i) $\theta(T) \equiv 1$, (ii) $d\theta(T, X) \equiv 0$ for any smooth tangent vector field X over M. We have the complexified tangent bundle CTM which admits the decomposition $CTM = T^{(1,0)}M \oplus T^{(0,1)}M \oplus \mathbb{R}T$.

$$L_\theta(u, v) := -id\theta(u \wedge \overline{v}) = i\theta\big([u, \overline{v}]\big), \quad \forall u, v \in T_p^{1,0}(M),\ \forall p \in M. \tag{8}$$

Recall that we say (M, θ) is *strictly pseudoconvex* if the Levi-form L_θ is positive definite for all $z \in M$.

Let $T'M$ be the annihilator bundle of $\mathcal{V} := T^{(0,1)}M$ which is a rank $n+1$ subbundle of $\mathbb{C}T^*M$.

Admissible Coframe If we choose a local basis L_α, $\alpha = 1, \ldots, n$, of $(1,0)$ vector fields (i.e., sections of $\overline{\mathcal{V}} = T_M^{1,0}$), so that $(T, L_\alpha, L_{\overline{\alpha}})$ is a frame for $\mathbb{C}TM := \mathbb{C} \otimes TM$ where $L_{\overline{\alpha}} = \overline{L_\alpha}$. Then the equation in (ii) above is equivalent to

$$d\theta = ig_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}}. \tag{9}$$

Here $\theta^{\overline{\beta}} = \overline{\theta^\beta}$, $(g_{\alpha\overline{\beta}})$ is the (Hermitian) *Levi-form matrix* and $(\theta, \theta^\alpha, \theta^{\overline{\alpha}})$ is the *coframe* dual to $(T, L_\alpha, L_{\overline{\alpha}})$. (For brevity, we shall say that (θ, θ^α) is the coframe dual to (T, L_α).) Note that θ and T are real, whereas θ^α and L_α always have non-trivial real and imaginary parts.

Without mentioning T, we can complete θ to a coframe (θ, θ^α) by adding $(1,0)$-cotangent vectors (the cotangent vectors that annihilate $\mathcal{V}$) θ^α. The coframe is called *admissible* if $\langle \theta^\alpha, T\rangle = 0$, for $\alpha = 1, \ldots, n$. As other equivalent definitions, (θ, θ^α) is admissible if (9) holds.

Pseudo-Hermitian Geometry on M Observe that (by the uniqueness of the Reeb vector field) for a given contact form θ on M, the admissible coframes are determined up to transformations

$$\widetilde{\theta}^\alpha = u_\beta^\alpha \theta^\beta, \quad \big(u_\beta^{\ \alpha}\big) \in GL\big(\mathbb{C}^n\big).$$

Every choice of a contact form θ on M is called a *pseudo-Hermitian structure* and defines a Hermitian metric on $\mathcal{V}$ (and on $\overline{\mathcal{V}}$) via the (positive-definite) Levi-form (see (8)). For every such θ, Tanaka [22] and Webster [23] defined a *pseudo-Hermitian connection* $\triangledown$ on $\overline{\mathcal{V}}$ (and also on $\mathbb{C}TM$) which is expressed relative to an admissible coframe (θ, θ^α) by

$$\triangledown L_\alpha = \omega_\alpha^{\ \beta} \otimes L_\beta$$

where the 1-forms ω_β^α on M are uniquely determined by the structure equations:

$$d\theta^\beta = \theta^\alpha \wedge \omega_\alpha{}^\beta \bmod \left(\theta \wedge \theta^\alpha\right), \qquad dg_{\alpha\overline{\beta}} = \omega_{\alpha\overline{\beta}} + \omega_{\overline{\beta}\alpha}. \tag{10}$$

We may rewrite the first condition in (10) as

$$d\theta^\beta = \theta^\alpha \wedge \omega_\alpha^\beta + \theta \wedge \tau^\beta, \quad \tau^\beta = A_{\overline{\nu}}^\beta \theta^{\overline{\nu}}, \; A^{\alpha\beta} = A^{\beta\alpha} \tag{11}$$

for a suitably determined torsion matrix $(A_{\overline{\nu}}^\beta)$, where the last symmetry relation holds automatically (see [23]).

The *pseudo-Hermitian curvature* $R_{\alpha\mu\overline{\nu}}^\beta$ and $W_{\alpha\mu}^\beta$ of the pseudo-Hermitian connection is given, in view of [23, (1.27), (1.41)], by

$$d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta = R_{\alpha\mu\overline{\nu}}^\beta \theta^\mu \wedge \theta^{\overline{\nu}} + W_{\alpha\mu}^\beta \theta^\mu \wedge \theta - W_{\alpha\overline{\nu}}^\beta \theta^{\overline{\nu}} \wedge \theta + i\theta_\alpha \wedge \tau^\beta - i\tau_\alpha \wedge \theta^\beta. \tag{12}$$

4 Local CR Embeddings

Coframes on $f : M \to \hat{M}$ Let $f : M \to \hat{M}$ be a local CR embedding where M is a strictly pseudoconvex hypersurface in $\mathbb{C}^{n+1}$ and $\hat{M}$ is a strictly pseudoconvex hypersurface in $\mathbb{C}^{\hat{n}+1}$. We use a $\hat{}$ to denote objects associated with $\hat{M}$. We shall also omit the $\hat{}$ over frames and coframes if there is no ambiguity. It will be clear from the context if a form is pulled back to M or not. Under the above assumptions, we identify M with the submanifold $f(M)$ and write $M \subset \hat{M}$. Capital Latin indices A, B, etc., will run over the set $\{1, \ldots, \hat{n}\}$; Greek indices α, β, etc., will run over $\{1, \ldots, n\}$; small Latin indices a, b, etc., will run over the complementary set $\{n+1, \ldots, \hat{n}\}$.

Let (θ, θ^α) and $(\hat{\theta}, \hat{\theta}^A)$ be coframes on M and $\hat{M}$, respectively, and recall that f is a *CR mapping* if

$$f^*(\hat{\theta}) = a\theta, \qquad f^*\left(\hat{\theta}^A\right) = E_\alpha^A \theta^\alpha + E^A \theta,$$

where a is a real-valued function and E_α^A, E^A are complex-valued functions.

We identify M with the submanifold $f(M)$ of $\hat{M}$ and write $M \subset \hat{M}$. Then the CR bundle $\mathcal{V} = T^{0,1}M$ is a rank n subbundle of $\hat{\mathcal{V}} = T^{0,1}\hat{M}$ along M. Then there is a rank $(\hat{n} - n)$ subbundle $N'M$ consisting of 1-forms on $\hat{M}$ whose pullbacks to M by f vanish. The subbundle $N'M$ is called the *holomorphic conormal bundle* of M in $\hat{M}$.

We write i^* for the standard pullback map and i_* for the pushforward map. Notice that our consideration is purely local. We let $p \in M$ and fix a local admissible coframe $\{\theta, \theta^\alpha\}$ for M. Let T be the Reeb vector field associated with θ. Assume that $\widehat{M}$ is a small neighborhood of 0 in $\mathbb{R}^{\widehat{m}}$, $p = 0$, and M is defined near 0 by $x_j = 0$ with $j = m+1, \ldots, \widehat{m}$. First, we can extend θ to a contact form of $\widehat{M}$ in a neighborhood of 0. Write $x' = (x_1, \ldots, x_m)$. Define $\widehat{\theta} = u\theta$, with $u(x', 0) \equiv 1$. Then $d\widehat{\theta} = du \wedge \theta + ud\theta$. We want $d\widehat{\theta} \lrcorner T = 0$ along M. For this, we write $ud\theta \lrcorner T = \sum_{j=1}^{\widehat{m}} d_j(x', 0)dx_j$. Then, we need to have, along M: $du = \sum_{j=1}^{\widehat{m}} d_j(x', 0)dx_j$. Since T is the Reeb vector field for θ along M, we have $d_j(x', 0) = 0$ for $j \leq m$. Thus, choose $u =$

$1+\sum_{j=m+1}^{\widehat{m}} d_j(x',0)x_j$. Then we have $d\widehat{\theta}\lrcorner T=0$ along M. Now, by the uniqueness of the Reeb vector field, we see that the Reeb vector field $\widehat{T}$ of $\widehat{\theta}$, when restricted to M, coincides with T. Extend θ^α to a neighborhood of 0 in $\widehat{M}$ to get $\widehat{\theta^\alpha}$, and add $\widehat{\theta^a}$ so that $\{\widehat{\theta},\widehat{\theta^\alpha},\widehat{\theta^a}\}$ forms a basis for $T'\widehat{M}$ near 0. After a linear change for the forms $\{\widehat{\theta^\alpha},\widehat{\theta^a}\}$, we can assume that the pullback of $\widehat{\theta^a}$ to M is zero for each $a=n+1,\ldots,\widehat{n}$, the pullback of $\widehat{\theta^\alpha}$ to M is θ^α for $\alpha=1,\ldots,n$, $\widehat{\theta}$ remains the same, and $\{\widehat{\theta},\widehat{\theta^\alpha},\widehat{\theta^a}\}$ is an admissible coframe along $\widetilde{M}$ near 0.

Next, suppose that $d\theta=\sqrt{-1}g_{\alpha\overline{\beta}}\theta^\alpha\wedge\theta^{\overline{\beta}}$ with $g_{\alpha\overline{\beta}}=\delta_{\alpha\beta}$ along M. We can even make the Levi-form of $\widehat{M}$ with respect to the coframe $\{\widehat{\theta},\widehat{\theta^\alpha},\widehat{\theta^a}\}$ also the identical matrix along M. Indeed, let $\{T,L_\alpha\}$ be the dual frame of $\{\theta,\theta^\alpha\}$ along M. Extend L_α to a vector field $\widetilde{L_\alpha}$ of type $(1,0)$ in a neighborhood of 0 in $\widehat{M}$. Find $\{\widehat{L_a}\}$ so that $\{\widetilde{L_\alpha},\widehat{L_a}\}$ forms a base of vector fields of type $(1,0)$ over $\widehat{M}$ with its Levi form along $\widehat{M}$ near 0 the identical matrix. Let $\widehat{\theta}$ be as constructed above such that its Reeb vector field $\widehat{T}$ is T, when restricted to M. Then we can find $\{\widehat{\theta},\widehat{\theta^\alpha},\widehat{\theta^a}\}$, which are the dual coframe of $\{\widehat{T},\widehat{L_A}\}$. Then along M, $\langle i^*(\widehat{\theta^a}),L_\alpha\rangle=\langle\widehat{\theta^a},\widehat{L}_\alpha\rangle|_M=0$; $\langle i^*(\widehat{\theta^a}),T\rangle=\langle\widehat{\theta^a},\widehat{T}|_M\rangle=0$. Hence the pullback of $\widehat{\theta^a}$ to M is zero. Clearly, the pullback of $\widehat{\theta^\alpha}$ to M is θ^α and $i^*(\widehat{\theta})=\theta$. Assume that

$$d\widehat{\theta}=\sqrt{-1}g_{A\overline{B}}\widehat{\theta}^A\wedge\widehat{\theta}^{\overline{B}}+\sum_{A=1}^{\widehat{n}}\left(e_A(x)\widehat{\theta}^A+\overline{e_A(x)}\widehat{\theta}^{\overline{A}}\right)\wedge\widehat{\theta}.$$

Contracting along $\widehat{T}$, we see that $e_A\equiv 0$. Hence, we see that $\{\widehat{\theta},\widehat{\theta^\alpha},\widehat{\theta^a}\}$ is an admissible coframe. Now, the Levi-form of $\widehat{M}$ along M is the identity with respect to such a frame.

We say that the pseudo-Hermitian structure $(\hat{M},\hat{\theta})$ is *admissible* for the pair $(M,\hat{M})$ if the Reeb vector field $\hat{T}$ for $\hat{\theta}$ is tangent to M. With the just-obtained coframe $(\hat{\theta},\hat{\theta}^A)$ on $\hat{M}$ where $A=1,2,\ldots,\hat{n}$, the holomorphic conormal bundle $N'M$ is spanned by the linear combinations of the $\hat{\theta}^a$. Summarizing the above, we see the following basic fact from [5]:

Proposition 4.1 [5], Corollary 4.2 *Let M and $\hat{M}$ be strictly pseudoconvex CR-manifolds of dimensions $2n+1$ and $2\hat{n}+1$, respectively. Let $f:M\to\hat{M}$ be a CR embedding. If (θ,θ^α) is any admissible coframe on M, then in a neighborhood of any point $\hat{p}\in f(M)$ in $\hat{M}$ there exists an admissible coframe $(\hat{\theta},\hat{\theta}^A)$ on $\hat{M}$ with $f^*(\hat{\theta},\hat{\theta}^\alpha,\hat{\theta}^a)=(\theta,\theta^\alpha,0)$. In particular, $\hat{\theta}$ is admissible for the pair $(f(M),\hat{M})$, i.e., the Reeb vector field $\hat{T}$ is tangent to $f(M)$. Also, when the Levi-form of M with respect to the coframe (θ,θ^α) is the identical matrix, then we can also choose $(\hat{\theta},\hat{\theta}^A)$ such that the Levi form of $\widehat{M}$ with respect to $(\hat{\theta},\hat{\theta}^A)$ is also the identical matrix.*

If we fix an admissible coframe (θ,θ^α) on M and let $(\hat{\theta},\hat{\theta}^A)$ be an admissible coframe on $\hat{M}$ near a point $\hat{p}\in f(M)$, we shall say $(\hat{\theta},\hat{\theta}^A)$ is *adapted* to (θ,θ^α) on M if it satisfies the conclusions of the above proposition. We also normalize the Levi-forms with these frames such that they are identical.

Second Fundamental Form Equation (11) implies that when (θ,θ^A) is adapted to M, if the pseudoconformal connection matrix of $(\hat{M},\hat{\theta})$ is $\hat{\omega}_B^A$, then that of (M,θ)

is the pullback of $\hat{\omega}_{\beta}^{\alpha}$. The pulled back torsion $\hat{\tau}^{\alpha}$ is τ^{α}, so omitting the $\hat{}$ over these pullbacks will not cause any ambiguity and we shall do that from now on. By the normalization of the Levi-form, the second equation in (10) reduces to

$$\omega_{B\overline{A}} + \omega_{\overline{A}B} = 0, \tag{13}$$

where as before $\omega_{\overline{A}B} = \overline{\omega_{A\overline{B}}}$.

The matrix of 1-forms (ω_{α}^{b}) pulled back to M defines the *second fundamental form* of the embedding $f : M \to \hat{M}$. Since $\theta^b = 0$ on M, Eq. (11) implies that on M,

$$\omega_{\alpha}^{b} \wedge \theta^{\alpha} + \tau^{b} \wedge \theta = 0, \tag{14}$$

and this implies that

$$\omega_{\alpha}^{b} = \omega_{\alpha\beta}^{b}\theta^{\beta}, \qquad \omega_{\alpha\beta}^{b} = \omega_{\beta\alpha}^{b}, \qquad \tau^{b} = 0. \tag{15}$$

Following [5], we identify the CR-normal space $T_p^{1,0}\hat{M}/T_p^{1,0}M$, also denoted by $N_p^{1,0}\hat{M}$ with $\mathbb{C}^{\hat{n}-n}$ by choosing the equivalence classes of L_a as a basis. Therefore, for fixed $\alpha, \beta = 1, \dots, n$, we view the component vector $(\omega_{\alpha\ \beta}^{\ a})_{a=n+1,\dots,\hat{n}}$ as an element of $\mathbb{C}^{\hat{n}-n}$. We also view the second fundamental form as a section over M of the bundle $T^{1,0}M \otimes N^{1,0}\hat{M} \otimes T^{1,0}M$.

5 The Pseudo-Conformal Geometry

Pseudo-Conformal Geometry We will need the pseudo-conformal connection and structure equations introduced by Chern and Moser in [4]. Let Y be the bundle of coframes $(\omega, \omega^{\alpha}, \omega^{\overline{\alpha}}, \phi)$ on the real ray bundle $\pi_E : E \to M$ of all contact forms defining the same orientation of M, such that $d\omega = i g_{\alpha\overline{\beta}}\omega^{\alpha} \wedge \omega^{\overline{\beta}} + \omega \wedge \phi$ where $\omega^{\alpha} \in \pi_E^*(T'M)$ and ω is the canonical 1-form on E. In [4] it was shown that these forms can be completed to a full set of invariants on Y given by the coframe of 1-forms

$$\left(\omega, \omega^{\alpha}, \omega^{\overline{\alpha}}, \phi, \phi^{\alpha}_{\beta}, \phi^{\overline{\alpha}}, \psi\right) \tag{16}$$

which define the *pseudo-conformal connection* on Y.

$$\begin{aligned}
&\phi_{\alpha\overline{\beta}} + \phi_{\overline{\beta}\alpha} = dg_{\alpha\overline{\beta}},\\
&d\omega = i\omega^{\mu} \wedge \omega_{\mu} + \omega \wedge \phi,\\
&d\omega^{\alpha} = \omega^{\mu} \wedge \phi^{\alpha}_{\mu} + \omega \wedge \phi^{\alpha},\\
&d\phi = i\omega_{\overline{\nu}} \wedge \phi^{\overline{\nu}} + i\phi_{\overline{\nu}} \wedge \omega^{\overline{\nu}} + \omega \wedge \psi,\\
&d\phi^{\alpha}_{\beta} = \phi^{\mu}_{\beta} \wedge \phi^{\alpha}_{\mu} + i\omega_{\beta} \wedge \phi^{\alpha} - i\phi_{\beta} \wedge \omega^{\alpha} - i\delta^{\alpha}_{\beta}\phi_{\mu} \wedge \omega^{\mu} - \frac{\delta^{\alpha}_{\beta}}{2}\psi \wedge \omega + \Phi^{\alpha}_{\beta},\\
&d\phi^{\alpha} = \phi \wedge \phi^{\alpha} + \phi^{\mu} \wedge \phi^{\alpha}_{\mu} - \frac{1}{2}\psi \wedge \omega^{\alpha} + \Phi^{\alpha},\\
&d\psi = \phi \wedge \psi + 2i\phi^{\mu} \wedge \phi_{\mu} + \Psi,
\end{aligned} \tag{17}$$

where the curvature 2-forms Φ_β^α, Φ^α, and Ψ are decomposed as

$$\begin{aligned}\Phi_\beta^\alpha &= S_{\beta\mu\overline{\nu}}^\alpha \wedge \omega^{\overline{\nu}} + V_{\beta\mu}^\alpha \omega^\mu \wedge \omega + V_{\beta\overline{\nu}}^\alpha \omega \wedge \omega^{\overline{\nu}},\\ \Phi^\alpha &= V_{\mu\overline{\nu}}^\alpha \omega^\mu \wedge \omega^{\overline{\nu}} + P_\mu^\alpha \omega^\mu \wedge \omega + Q_{\overline{\nu}}^\alpha \omega^{\overline{\nu}} \wedge \omega,\\ \Psi &= -2i P_{\mu\overline{\nu}} \omega^{\overline{\nu}} + R_\mu \omega^\mu \wedge \omega + R_{\overline{\nu}} \omega^{\overline{\nu}} \wedge \omega,\end{aligned} \tag{18}$$

where the functions $S_{\beta\mu\overline{\nu}}^\alpha$, $V_{\beta\mu}^\alpha$, P_μ^α, $Q_{\overline{\nu}}^\alpha$ together represent the *pseudo-conformal curvature* of M.[1] As in [4] we restrict our attention here to coframes (θ, θ^α) for which the Levi-form $(g_{\alpha\overline{\beta}})$ is constant. The 1-forms ϕ^α, $\phi^{\overline{\alpha}}$, ϕ_β^α, ψ are uniquely determined by requiring the coefficients in (18) to satisfy certain symmetry and trace conditions (see [4] and the appendix), e.g.,

$$S_{\alpha\overline{\beta}\mu\overline{\nu}} = S_{\mu\overline{\beta}\alpha\overline{\nu}} = S_{\mu\overline{\nu}\alpha\overline{\beta}} = S_{\overline{\nu}\mu\overline{\beta}\alpha}, \qquad S_{\mu\ \alpha\overline{\beta}}^{\mu} = V_{\alpha\ \mu}^{\ mu} = P_\mu^\mu = 0.$$

Let us fix a contact form θ that defines a section $M \to E$. Then any admissible coframe (θ, θ^α) for $T^{1,0}M$ defines a unique section $M \to Y$ for which the pullbacks of (ω, ω^α) coincide with (θ, θ^α) and the pullback of ϕ vanishes. As in [23], we shall use the same notation for the pulled back forms on M (which now depend on the choice of the admissible coframe). With this convention, we have

$$\theta = \omega, \qquad \theta^\alpha = \omega^\alpha, \qquad \phi = 0 \tag{19}$$

on M.

Relationship Between Pseudo-Conformal Geometry and Pseudo-Hermitian Geometry In view of Webster [23, (3.8)], the pulled back tangential pseudoconformal curvature tensor $S_{\alpha\ \mu\overline{\nu}}^{\ \beta}$ can be obtained from the tangential pseudo-Hermitian curvature tensor $R_{\alpha\mu\overline{\nu}}^{\beta}$ in (12) by

$$\begin{aligned}S_{\alpha\overline{\beta}\mu\overline{\nu}} = R_{\alpha\overline{\beta}\mu\overline{\nu}} &- \frac{R_{\alpha\overline{\beta}} g_{\mu\overline{\nu}} + R_{\mu\overline{\beta}} g_{\alpha\overline{\nu}} + R_{\alpha\overline{\nu}} g_{\mu\overline{\beta}} + R_{\mu\overline{\nu}} g_{\alpha\overline{\beta}}}{n+2}\\ &+ \frac{R(g_{\alpha\overline{\beta}} g_{\mu\overline{\nu}} + g_{\alpha\overline{\nu}} g_{\mu\overline{\beta}})}{(n+1)(n+2)}\end{aligned} \tag{20}$$

where

$$R_{\alpha\overline{\beta}} := R_{\mu\ \alpha\overline{\beta}}^{\mu} \quad \textit{and} \quad R = R_\mu^\mu$$

are respectively the *pseudo-Hermitian Ricci and scalar curvature* of (M, θ).

[1]The indices of $S_{\beta\mu\overline{\nu}}^\alpha$ here are interchanged comparing to [4] to make them consistent with indices of $R_{\beta\ \mu\overline{\nu}}^{\alpha}$ in (12).

Traceless Component As in [5], we call a tensor $T^{a_1\cdots a_l\overline{b_1}\cdots\overline{b_q}}_{\alpha_1,\ldots,\alpha_r,\overline{\beta_1},\ldots,\overline{\beta_s}}$ *pseudo-conformally equivalent to 0* or *pseudo-conformally flat* if it is a linear combination of tensors with factor $g_{\alpha_i\overline{\beta_j}}$ for $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, s$. Two tensors $T_{\alpha\overline{\beta}\mu\overline{\nu}}$ and $R_{\alpha\overline{\beta}\mu\overline{\nu}}$ are called *conformally equivalent* if $T_{\alpha\overline{\beta}\mu\overline{\nu}} - R_{\alpha\overline{\beta}\mu\overline{\nu}}$ is pseudo-conformally flat. For any tensor $R_{\alpha\overline{\beta}\mu\overline{\nu}}$, its *traceless component* is the unique tensor that is trace zero and that is conformally equivalent to $R_{\alpha\overline{\beta}\mu\overline{\nu}}$. We denote the traceless component by $[R_{\alpha\overline{\beta}\mu\overline{\nu}}]$. Formula (20) expresses the fact that $S_{\alpha\overline{\beta}\mu\overline{\nu}}$ is the "traceless component" of $R_{\alpha\overline{\beta}\mu\overline{\nu}}$ (cf. [5], (5.5)):

$$S_{\alpha\overline{\beta}\mu\overline{\nu}} = [R_{\alpha\overline{\beta}\mu\overline{\nu}}]. \tag{21}$$

6 Real Hypersurfaces of Revolution

Real Hypersurfaces of Revolution As in (1) of the Introduction, let $M = \{(z, w) \mid r = 0\}$ be a real hypersurface of revolution in $\mathbb{C}^{n+1}$ with $n \geq 2$, where

$$r = p(z, \overline{z}) + q(w, \overline{w}), \quad q = \overline{q} \text{ and } p(z, \overline{z}) = h_{\alpha\overline{\beta}} z^\alpha \overline{z}^\beta, \tag{22}$$

where $(h_{\alpha\overline{\beta}})$ is a positive definite Hermitian matrix. Also $d(q) \neq 0$ when $q = 0$, q takes negative values somewhere.

Define $D := \{(z, w) \mid r < 0\}$. As the auxiliary curve and domain in $\mathbb{C}$, we define $C_0 := \{w \mid q(w, \overline{w}) = 0\}$ and $D_0 := \{w \mid q(w, \overline{w}) < 0\}$. M is strictly pseudoconvex if and only if on $D_0 := \{q < 0\}$, $h := -(\log q)_{w\overline{w}} = \frac{q_w q_{\overline{w}} - q q_{w\overline{w}}}{q^2} > 0$. Assume that M is strictly pseudoconvex. Then D_0 admits a Hermitian metric $ds^2 = h dw d\overline{w}$. We denote by K its Gaussian curvature on D_0. It was proved in [24] that for $w \in D_0$ and $(z, w) \in M$ with $n \geq 2$ and $q_w \neq 0$, the fourth-order Chern–Moser tensor $S(z, w) = 0$ if and only if $K(w) = -2$.

The Pseudo-Hermitian Curvature of M By Webster, at a point where $q_w \neq 0$, the pseudo-Hermitian curvature of M is calculated as

$$R_{\beta\overline{\alpha}\rho\overline{\sigma}} = -A(g_{\beta\overline{\alpha}} g_{\rho\overline{\sigma}} + g_{\rho\overline{\alpha}} g_{\beta\overline{\sigma}}) - B p_\beta p_{\overline{\alpha}} p_\rho p_{\overline{\sigma}} \tag{23}$$

where

$$\begin{aligned} &A = -\frac{Q}{1 - Qq}, \qquad g_{\alpha\overline{\beta}} = h_{\alpha\overline{\beta}} + Q p_\alpha p_{\overline{\beta}}, \qquad \theta = -i\partial r, \\ &\theta^\alpha = dz_\alpha - i\eta^\alpha\theta, \qquad \eta^\alpha = g^{\alpha\overline{\beta}} \eta_{\overline{\beta}}, \qquad \eta_\alpha = -Q p_\alpha; \end{aligned} \tag{24}$$

and

$$B = \frac{Q_{w\overline{w}}}{q_w q_{\overline{w}}} + 2Q\left(\frac{Q_w}{q_w} + \frac{Q_{\overline{w}}}{q_{\overline{w}}}\right) + 3Q^3 + \frac{q|(Q_w/q_w) + Q^2|^2}{1 - Qq} \tag{25}$$

where $Q = \frac{q_{w\overline{w}}}{q_w q_{\overline{w}}}$. Notice that the formulas above were slightly modified from those in [24], since we need $(g_{\alpha\overline{\beta}})$ to be positive definite to apply the Gauss–Codazzi equation here.

Here B can also be calculated as

$$B = \frac{(K+2)k^2}{q^3(q_w q_{\overline{w}})^2} \tag{26}$$

where $k = q_w q_{\overline{w}} - q q_{w\overline{w}}$. We notice that B is a real-valued function and $B \le 0$ if and only if $K + 2 \ge 0$.

Umbilic Points of the Fourth-Order Chern–Moser Tensor S Let S be the fourth-order Chern–Moser tensor when $n \ge 2$. (For $n = 1$, it is replaced by the Cartan invariant.) A point $(z, w) \in M$ is called an *umbilic point* if $S(z, w) = 0$.

It was proved by Webster [24] that if $w \in D_0$ and $(z, w) \in M$, then at points where $dq \ne 0$, we have

$$S(z, w) = 0 \quad \text{if and only if} \quad K(w) = -2. \tag{27}$$

If $B \equiv 0$, it implies $K \equiv -2$ by (26).

7 Proof of Theorem 1.1

Assume the notation and assumption in Theorem 1.1. For simplicity of notation, we can simply assume that M_0 can be CR smoothly embedded into $\partial\mathbb{B}^{N+1}$ with $n \le N \le 2n - 1$. Indeed, by the Hopf lemma and shrinking M_0, if needed, we can conclude that F is a CR embedding whenever F is not a constant map. We then need to prove that $F(M)$ must be the CR transversal intersection of an affine subspace with an open piece of the sphere. Notice that $q_w \ne 0$ in a dense open subset of U_0. By passing to the limit, if needed, and by the uniqueness of holomorphic functions, we may simply assume that $q_w \ne 0$ over U_0.

We take an admissible coframe (θ, θ^α) on M as mentioned before with $\theta := -i\partial r$ as the contact form. Fixing any point $p \in M_0$, by Proposition 4.1, there exists a neighborhood $\hat{U}$ of $\hat{p} := F(p)$ in $\partial\mathbb{B}^{N+1}$ and an admissible coframe $(\hat{\theta}, \hat{\theta^A})$ on $\hat{U}$ such that $F^*(\hat{\theta}, \hat{\theta^\alpha}, \hat{\theta^a}) = (\theta, \theta^\alpha, 0)$ on U, where U is a neighborhood of p in M_0 such that $F(U) \subset \hat{U}$, $\{\theta, \theta^\alpha\}$ is an admissible frame over U as defined in the previous section.

Consider the pseudo-conformal Gauss equation (cf. (5.9) in [5])

$$\big[\hat{S}(X, X, X, X)\big] = S(X, X, X, X) + \big[\big\langle II(X, X), II(X, X)\big\rangle\big], \quad \forall X \in T^{1,0}_{\hat{p}} F(M), \tag{28}$$

where S is the pseudo-conformal curvature of $F(M)$, $\widehat{S}$ is the restriction of the pseudo-conformal curvature of $\partial\mathbb{B}^{N+1}$ on $F(M)$, and $II(X, X)$ is the second fundamental form of $F(M) \subset \partial\mathbb{B}^{N+1}$. Here the notation [ ] in (21) is used and we can regard X as a vector in $\mathbb{C}^n$. Locally it can be written as

$$[\hat{S}_{\alpha\overline{\beta}\mu\overline{\nu}}] = S_{\alpha\overline{\beta}\mu\overline{\nu}} + \big[g_{a\overline{b}} \omega^a_{\alpha\mu} \omega^{\overline{b}}_{\overline{\beta}\overline{\nu}}\big] \tag{29}$$

where (ω^b_α) is the second fundamental form of $F(M)$ and $\omega^b_\alpha = \omega^b_{\alpha\beta}\theta^\beta$, and $(g_{a\overline{b}})$ is the (Levi) positive definite Hermitian matrix. Here $\omega^b_{\alpha\beta}$ are functions satisfying

$\omega^b_{\alpha\beta} = \omega^b_{\beta\alpha}$ (cf. [5], (4.3) and (5.6)). Recall the facts that the pseudo-conformal curvature of a sphere vanishes and that we have

$$S_{\alpha\overline{\beta}\mu\overline{\nu}} = [R_{\alpha\overline{\beta}\mu\overline{\nu}}]$$

where $R_{\alpha\overline{\beta}\mu\overline{\nu}}$ is the pseudo-Hermitian curvature induced by the pseudo-Hermitian metric on $F(M)$. Then (29) becomes

$$0 = [R_{\alpha\overline{\beta}\mu\overline{\nu}}] + \left[g_{a\overline{b}}\omega^a_{\alpha\mu}\omega^{\overline{b}}_{\overline{\beta}\,\overline{\nu}}\right]. \tag{30}$$

Since F is a local CR embedding, we can identify the pseudo-Hermitian structure (M, θ) with $(F(M), (F^{-1})^*\theta)$. In other words, we can identify the pseudo-Hermitian curvature $R_{\alpha\overline{\beta}\mu\overline{\nu}}$ on $F(M)$ as the pseudo-Hermitian curvature over M. Then from (23), we have $R_{\alpha\overline{\beta}\mu\overline{\nu}} = -A(g_{\alpha\overline{\beta}}g_{\mu\overline{\nu}} + g_{\mu\overline{\beta}}g_{\alpha\overline{\nu}}) - Bp_\alpha p_{\overline{\beta}} p_\rho p_{\overline{\nu}}$. Since $p(z, \overline{z}) = h_{\beta\overline{\alpha}} z^\beta \overline{z}^\alpha$, we have

$$p_\beta = h_{\beta\overline{\beta}'}\overline{z}^{\beta'}, \qquad p_{\overline{\alpha}} = h_{\alpha'\overline{\alpha}} z^{\alpha'}$$

and thus

$$\begin{aligned}\sum_{\alpha,\beta,\mu,\nu} p_\alpha p_{\overline{\beta}} p_\mu p_{\overline{\nu}} &= \sum_{\alpha,\beta,\mu,\nu,\alpha',\beta',\mu',\nu'} h_{\alpha\overline{\alpha}'}\overline{z}^{\alpha'} h_{\beta'\overline{\beta}} z^{\beta'} h_{\mu\overline{\mu}'}\overline{z}^{\mu'} h_{\nu'\overline{\mu}} z^{\nu'} \\ &= \left|\sum_{\beta,\nu,\beta',\nu'} h_{\beta'\overline{\beta}} z^{\beta'} h_{\nu'\overline{\nu}} z^{\nu'}\right|^2. \end{aligned} \tag{31}$$

Now, as in the proof of Lemma 2.1, we have the following computation:

$$A_{\alpha\overline{\beta}} g_{\mu\overline{\nu}} X^\alpha \overline{X^\beta} X^\mu \overline{X^\nu} = B(X, \overline{X})|X|^2,$$

where $|X|^2 = g_{\alpha\overline{\beta}} X^\alpha \overline{X^\beta}$ and $B(X, \overline{X}) = A_{\alpha\overline{\beta}} X^\alpha \overline{X^\beta}$. We substitute (23) and (31) into (30) to obtain

$$\begin{aligned} 0 &= |X|^2 E(X, \overline{X}) \\ &\quad - B\left|\sum_{\beta,\nu,\beta',\nu'} h_{\beta'\overline{\beta}} z^{\beta'} h_{\nu'\overline{\nu}} z^{\nu'} \overline{X^\beta}\,\overline{X^\nu}\right|^2 \\ &\quad + \sum_{n+1\le a,b\le N} g_{a\overline{b}} \omega_\alpha{}^a{}_\mu X^\alpha X^\mu \omega^{\overline{b}}_{\overline{\beta}\,\overline{\nu}} \overline{X^\beta}\,\overline{X^\nu}, \quad \forall X \in \mathbb{C}^n, \text{ at } \hat{p} \in \hat{U} \end{aligned} \tag{32}$$

for some real analytic function $E(X, \overline{X})$. Since $(N - n) + 1 \le ((2n - 2) - n) + 1 = n - 1$, we apply Lemma 2.1 to yield that

$$-B\left(\sum_{\beta,\nu,\beta',\nu'} h_{\beta'\overline{\beta}} z^{\beta'} h_{\nu'\overline{\nu}} z^{\nu'} \overline{X^\beta}\,\overline{X^\nu}\right)\left(\sum_{\alpha,\mu,\alpha',\mu'} \overline{h_{\alpha\overline{\alpha}'}}\,\overline{z^{\alpha'}}\,\overline{h_{\mu\overline{\mu}'}}\,\overline{z^{\mu'}} X^\alpha X^\mu\right)$$

$$+ \sum_{a,b=n+1}^{N} g_{a\overline{b}} \omega^a_{\alpha\mu} X^\alpha X^\mu \omega^{\overline{b}}_{\overline{\beta}\,\overline{\nu}} \overline{X^\beta}\,\overline{X^\nu} = 0, \quad \forall X \in \mathbb{C}^n. \tag{33}$$

When $B \leq 0$, both terms in the left-hand side of the above equation are nonnegative. Hence, we get that $B \equiv 0$ over U_0 and

$$\sum_{a,b=n+1}^{N} g_{a\overline{b}} \big(\omega^a_{\alpha\mu} X^\alpha X^\mu\big)\big(\omega^{\overline{b}}_{\overline{\beta}\overline{\nu}} \overline{X^\beta}\,\overline{X^\nu}\big) \equiv 0, \quad \forall X \in \mathbb{C}^n.$$

This shows that $K \equiv -2$. Since $(g_{\alpha\overline{\beta}})$ is Hermitian and positive definite, it implies $\omega^a_{\alpha\mu} = 0$, $\forall a, \alpha, \mu$ so that the second fundamental form of $F(M)$ is zero.

Then either by the result of Webster in (27) or by the result in [16], $F(M)$ and M must be spherical. Thus $F(M)$ is in the image $G(\partial\mathbb{B}^{n+1})$ for some linear fractional map $G : \partial\mathbb{B}^{n+1} \to M \subset \partial\mathbb{B}^{N+1}$, by the well-known rigidity result in [11]. The proof of Theorem 1.1 is complete.

Example 7.1 Let $q = |w|^2 + \epsilon|w|^4 + \phi(w, \overline{w}) - 1$ with $\epsilon \in \mathbb{R}$ and $\phi = o(|w|^4)$ being smoothly real-valued. Now $D_0 = \{w \in \mathbb{C} : q < 0\}$. $ds^2 = -(\log q)_{w\overline{w}} dw \otimes d\overline{w}$ defines a Hermitian metric in a neighborhood of $0 \in D_0$. The formula for its Gauss curvature was derived in [24, (15)]:

$$K = -2 + q^3 k^{-3}\big(k q_{ww\overline{w}\overline{w}} + q|q_{ww\overline{w}}|^2 - 2\Re(q_{ww\overline{w}} q_{\overline{w}\overline{w}} q_w) + q_{w\overline{w}}|q_{ww}|^2\big)$$

with $k = q_w q_{\overline{w}} - q q_{w\overline{w}}$. By a direct computation, one sees that $K = -2 - 4\epsilon + o(|w|)$. Hence, for $\epsilon < 0$, we have $K > 2$ in a small neighborhood of 0 in D_0

8 Examples of Pseudo-Conformally Flat Kähler Manifolds

Complex Space Forms A Kähler manifold of constant holomorphic sectional curvature is called a *complex space form*. The universal complex space forms are $\mathbb{C}^n$, $\mathbb{CP}^n$, and $\mathbb{B}^n$ equipped with the Kähler metric

$$h_{ij} = \frac{\delta_{ij}}{1 + \kappa|z|^2} - \frac{\kappa z_i \overline{z}_j}{(1 + \kappa|z|^2)^2}$$

with $\kappa = 0$, 1, and -1, respectively. Also, $z \in \mathbb{C}^n$ in the $\mathbb{C}^n$ and $\mathbb{P}^n$ (local chart in this setting) case; and $|z| < 1$ in the hyperbolic space case. The curvature tensor is given by

$$\Theta_{ij} = \kappa\left(\sum_{k,l=1}^{n} h_{k\overline{l}} dz_k \wedge d\overline{z_l}\right)\delta_{ij} - \kappa \sum_{l=1}^{n} h_{i\overline{l}} d\overline{z_l} \wedge dz_j$$

and

$$R_{i\overline{j}k\overline{l}} = \kappa(h_{i\overline{j}} h_{k\overline{l}} + h_{k\overline{j}} h_{i\overline{l}}).$$

Complex space forms are certainly pseudo-conformally flat.

Bochner–Kähler Manifolds Let (M,ω) be a Kähler manifold. Write $\omega = \sum_{i\bar{j}} g_{i\bar{j}} dz_i \otimes \overline{dz_j}$ in a local holomorphic chart. The Bochner curvature tensor of (M,ω) is defined as the following tensor:

$$B_{\beta\overline{\alpha}\rho\overline{\sigma}} = R_{\beta\overline{\alpha}\rho\overline{\sigma}} - \frac{g_{\beta\overline{\alpha}}R_{\rho\overline{\sigma}} + g_{\rho\overline{\alpha}}R_{\beta\overline{\sigma}} + g_{\beta\overline{\sigma}}R_{\rho\overline{\alpha}} + g_{\rho\overline{\sigma}}R_{\beta\overline{\alpha}}}{n+2} + \frac{R(g_{\beta\overline{\alpha}}g_{\rho\overline{\sigma}} + g_{\rho\overline{\alpha}}g_{\beta\overline{\sigma}})}{(n+1)(n+2)}$$

where $R_{\alpha\bar{b}\gamma\delta}$ is the curvature tensor of (M,ω), $R_{\alpha\overline{\beta}}$ is the Ricci tensor, and R is the scalar curvature of (M,ω). (M,ω) is called a Bochner–Kähler manifold if its Bochner curvature tensor is identically zero. There have been extensive studies on Bochner–Kähler manifolds in the literature, for which we refer the reader to the paper of Bryant [1]. Bochner–Kähler manifolds are pseudo-conformally flat in our definition.

9 Proof of Theorem 1.4

To prove Theorem 1.4, for any point $u_0 \in X$, let $z = (z_1, \ldots, z_n)$ be a holomorphic coordinate system of $f(X)$ at $z_0 = f(u_0)$, and $\hat{z} = (z_1, \ldots, z_n, z_{n+1}, \ldots, z_N)$ an extension of $(z_1, \ldots, z_n)$ to a coordinate system of Y at z_0. We shall fix the following convention for indices: $1 \le i, j, \ldots, \le N$, $1 \le \alpha, \beta, \mu, \nu, \gamma, \delta, \ldots, \le n$, $n+1 \le a, b, A, B, \ldots, \le N$.

Let us denote by $\hat{g}_{i\overline{j}}$ the Hermitian metric of (Y,σ) and $\hat{R}_{i\overline{j}k\overline{l}}$ the curvature tensor of this metric on Y. Let us denote by $g_{\alpha\overline{\beta}}$ the restriction metric of the metric $\hat{g}_{i\overline{j}}$ on $f(X)$ and $R_{\alpha\overline{\beta}\gamma\overline{\sigma}}$ the curvature tensor of this induced metric $g_{\alpha\overline{\beta}}$ on $f(X)$.

By the Gauss–Codazzi equation, we have the following equation of tensors:

$$\hat{R}_{\alpha\overline{\beta}\gamma\overline{\delta}}|_{f(X)} - R_{\alpha\overline{\beta}\gamma\overline{\delta}} = h^A_{\alpha\gamma}\overline{h^B_{\beta\delta}}\hat{g}_{A\overline{B}} \tag{34}$$

where $h^A_{\alpha\gamma} = \hat{g}^{A\overline{j}}\frac{\partial \hat{g}_{\alpha\overline{j}}}{\partial z^\gamma}$ is the second fundamental form of $f(X)$ in Y.

Since (Y,σ) is pseudo-conformally flat, the restriction of the curvature also satisfies

$$\hat{R}_{\alpha\overline{\beta}\gamma\overline{\delta}}|_{f(X)} = \left(G_{\alpha\overline{\beta}}\hat{g}_{\mu\overline{\nu}} + \hat{G}_{\mu\overline{\beta}}\hat{g}_{\alpha\overline{\nu}} + G^*_{\alpha\overline{\nu}}\hat{g}_{\mu\overline{\beta}} + \widetilde{G}_{\mu\overline{\nu}}\hat{g}_{\alpha\overline{\beta}}\right)\Big|_{f(X)} \tag{35}$$

where $G_{\alpha\overline{\beta}}$, $\hat{G}_{\alpha\overline{\nu}}$, $G^*_{\alpha\overline{\nu}}$, $\widetilde{G}_{\mu\overline{\nu}}$ are some Hermitian matrices on $f(X)$.

Since (X,ω) is pseudo-conformally flat, so is $(f(X),(f^{-1})^*(\omega))$. Since f is holomorphic conformal, we have $(f^{-1})^*\omega = k'\sigma|_{f(X)}$ for a certain positive constant $k' > 0$. By the assumption that (X,ω) is pseudo-conformally flat, we conclude that $(f(X),\sigma|_{f(X)})$ is also pseudo-conformally flat. Hence the curvature tensor $R_{\alpha\overline{\beta}\gamma\overline{\delta}}$ is conformally flat on $f(X)$ and it can be written as

$$R_{\alpha\overline{\beta}\gamma\overline{\delta}} = H_{\alpha\overline{\beta}}g_{\mu\overline{\nu}} + \hat{H}_{\mu\overline{\beta}}g_{\alpha\overline{\nu}} + H^*_{\alpha\overline{\nu}}g_{\mu\overline{\beta}} + \widetilde{H}_{\mu\overline{\nu}}g_{\alpha\overline{\beta}} \tag{36}$$

where $H_{\alpha\overline{\beta}}$, $\hat{H}_{\alpha\overline{\nu}}$, $H^*_{\alpha\overline{\nu}}$, $\widetilde{H}_{\mu\overline{\nu}}$ are some Hermitian matrices on $f(X)$.

By (34), (35), and (36), we have

$$\begin{aligned}
&\big(G_{\alpha\overline{\beta}}g_{\mu\overline{\nu}} + \hat{G}_{\mu\overline{\beta}}\hat{g}_{\alpha\overline{\nu}} + G^*_{\alpha\overline{\nu}}\hat{g}_{\mu\overline{\beta}} + \widetilde{G}_{\mu\overline{\nu}}\hat{g}_{\alpha\overline{\beta}}\big)(z_0)X^\alpha\overline{X}^\beta X^\mu\overline{X^\nu} \\
&\quad - \big(H_{\alpha\overline{\beta}}g_{\mu\overline{\nu}} + \hat{H}_{\mu\overline{\beta}}g_{\alpha\overline{\nu}} + H^*_{\alpha\overline{\nu}}g_{\mu\overline{\beta}} + \widetilde{H}_{\mu\overline{\nu}}g_{\alpha\overline{\beta}}\big)(z_0)X^\alpha\overline{X}^\beta X^\mu\overline{X^\nu} \\
&= \big(h^A_{\alpha\mu}\overline{h^B_{\beta\nu}}X^\alpha\overline{X^\beta}X^\mu\overline{X^\nu}, \hat{g}_{A\overline{B}}\big)(z_0)
\end{aligned} \tag{37}$$

for any $X = (X^\alpha) = (X^\beta) = (X^\mu) = (X^\nu) \in \mathbb{C}^n$.

By the same calculation as in (6), the left-hand side of (37) is equal to $|X|^2 A(X, \overline{X})$. Since $N - n \le 2n - 1 - n = n - 1$, we can apply Lemma 2.1 to conclude

$$\sum_{A,B=n+1}^{N} h^A_{\alpha\mu}(z_0)X^\alpha\overline{X^\beta}\,\overline{h^B_{\beta\nu}(z_0)}X^\mu\overline{X^\nu}\hat{g}_{A\overline{B}}(z_0) = 0, \quad \forall X \in \mathbb{C}^n.$$

Since the Hermitian metric $(\hat{g}_{A\overline{B}}(z_0))$ is positive definite, $h^A_{\alpha\mu}(z_0) = 0$ for all α, μ, and A. Since this holds for any point z in X, we have proved that the second fundamental form of $f(X)$ is identically zero, and hence $f(X)$ is totally geodesic in Y, proving Theorem 1.4.

References

1. Bryant, R.: Bochner–Kähler metrics. J. Am. Math. Soc. **14**, 623–715 (2001)
2. Calabi, E.: Isometric imbedding of complex manifolds. Ann. Math. **58**, 1–23 (1953)
3. Cao, H., Mok, N.: Holomorphic immersions between compact hyperbolic space forms. Invent. Math. **100**, 49–61 (1990)
4. Chern, S.S., Moser, J.K.: Real hypersurfaces in complex manifolds. Acta Math. **133**, 219271 (1974)
5. Ebenfelt, P., Huang, X., Zaitsev, D.: Rigidity of CR-immersions into spheres. Commun. Anal. Geom. **12**(3), 631–670 (2004)
6. Ebenfelt, P., Son, D.: On the existence of holomorphic embeddings of strictly pseudoconvex algebraic hypersurfaces into spheres, May 2012. arXiv:1205.1237
7. Faran, J.: The nonimbeddability of real hypersurfaces in spheres. Proc. Am. Math. Soc. **103**(3), 902–904 (1988)
8. Feder, S.: Immersions and embeddings in complex projective spaces. Topology **4**, 143–158 (1965)
9. Forstneric, F.: Embedding strictly pseudoconvex domains into balls. Trans. Am. Math. Soc. **295**(1), 347–368 (1986)
10. Huang, X.: On the mapping problem for algebraic real hypersurfaces in complex spaces of different dimensions. Ann. Inst. Fourier **44**, 433–463 (1994)
11. Huang, X.: On a linearity problem of proper holomorphic mappings between balls in complex spaces of different dimensions. J. Differ. Geom. **51**, 13–33 (1999)
12. Huang, X., Ji, S.: On some rigidity problems Cauchy–Riemann geometry. In: Proceeding on the 10th Anniversary of the Mathematical Institute in the East China Normal University. AMS/IP Studies in Advanced Mathematics, vol. 39, pp. 89–107 (2007)
13. Huang, X., Ji, S., Yin, W.: On the third gap for proper holomorphic maps between balls (2012, submitted)
14. Huang, X., Zaitsev, D.: Non-embeddable real hypersurfaces (2012, preprint)
15. Huang, X., Zhang, Y.: Monotonicity for the Chern–Moser–Weyl curvature tensor and CR embeddings. Sci. China Ser. A **52**(12), 2617–2627 (2009)
16. Ji, S., Yuan, Y.: Flatness of CR submanifolds in a sphere. Sci. China Ser. A **53**(3), 701–718 (2010)

17. Kim, S.Y., Oh, J.W.: Local embeddability of pseudo-Hermitian manifolds into spheres. Math. Ann. **334**(4), 783–807 (2006)
18. Kolar, M., Lamel, B.: Holomorphic equivalence and non-linear symmetries of ruled hypersurfaces in $\mathbb{C}^2$ (preprint)
19. Mok, N., Ng, S.-C.: Germs of measure-preserving holomorphic maps from bounded symmetric domains to their Cartesian products. J. Reine Angew. Math. **669**, 4773 (2012)
20. Mok, N.: Extension of germs of holomorphic isometries up to normalizing constants with respect to the Bergman metric (preprint). http://www.hku.hk/math/imr/
21. Oh, J.W.: Local Cauchy–Riemann embeddability of real hyperboloids into spheres. Proc. Am. Math. Soc. **135**(2), 397–403 (2007)
22. Tanaka, N.: A differential geometric study on strongly pseudo-convex manifolds. In: Lectures in Mathematics, vol. 9, Department of Mathematics, Kyoto University. Kinokuniya Book-Store Co., Tokyo (1975), iv+158 pp.
23. Webster, S.M.: Pseudo-Hermitian structures on a real hypersurface. J. Differ. Geom. **13**(1), 2541 (1978)
24. Webster, S.M.: A remark on the Chern–Moser tensor. Houst. J. Math. **28**(2), 433–435 (2002)
25. Yuan, Y., Zhang, Y.: Rigidity for local holomorphic isometric embeddings from $\mathbb{B}^n$ into $\mathbb{B}^{N_1} \times \cdots \times \mathbb{B}^{N_m}$ up to conformal factors. J. Differ. Geom. **90**(2), 329–349 (2012)
26. Zaitsev, D.: Obstructions to embeddability into hyperquadrics and explicit examples. Math. Ann. **342**(3), 695–726 (2008)