Linearity And Second Fundamental Forms For Proper Holomorphic Maps From $\mathbb{B}^{n+1}$ to $\mathbb{B}^{4n-3}$

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1 Introduction

In CR geometry, by *spherical CR manifold*, we mean a $(2n+1)$-dimension CR manifold M that is locally CR equivalent to a piece of the sphere $\partial\mathbb{B}^{n+1}$ in $\mathbb{C}^{n+1}$. In general, the universal covering space of a spherical CR manifold may not be $\partial\mathbb{B}^{n+1}$ and the fundamental group of M may not be finite. For example, Burns-Schnider [BS76] constructed a compact real analytic CR spherical submanifold of dimension 3 in $\mathbb{C}^3$ with fundamental group of infinite order. However, it is proved by Huang ([H06], corollary 3.3) that any $2n+1$-dimensional compact (Nash) algebraic spherical CR submanifold of $\mathbb{C}^m$, with $n \geq 1$, is CR equivalent to $\partial\mathbb{B}^{n+1}/\Gamma$ where $\Gamma \subset Aut(\mathbb{B}^{n+1})$ is a finite unitary group with the only free points at 0 and $Aut(\mathbb{B}^{n+1})$ is the group of biholomorphisms of $\mathbb{B}^{n+1}$. This implies that if $M \subset \partial\mathbb{B}^{N+1}$ is a compact spherical CR submanifold of dimension $2n+1$, by the argument in [H06], theorem 3.1, M is Nash algebraic if and only if $M = F(\partial\mathbb{B}^{n+1})$ where $F : \mathbb{B}^{n+1} \to \mathbb{B}^{N+1}$ is a proper rational holomorphic map. By Klein's Erlanger program, we should study such submanifolds $M \subset \partial\mathbb{B}^{N+1}$ and the invariant properties under the transitive action of the automorphism group $Aut(\partial\mathbb{B}^{N+1})$ where $Aut(\partial\mathbb{B}^{N+1})$ is the group of CR automorphisms. Elements in both $Aut(\mathbb{B}^{N+1})$ and $Aut(\partial\mathbb{B}^{N+1})$ are linear fractional.

Let us denote by $Prop(\mathbb{B}^{n+1}, \mathbb{B}^{N+1})$ the space of all proper holomorphic maps from the unit ball $\mathbb{B}^{n+1} \subset \mathbb{C}^{n+1}$ to $\mathbb{B}^{N+1}$, and denote by $Prop_k(\mathbb{B}^{n+1}, \mathbb{B}^{N+1})$ the space $Prop(\mathbb{B}^{n+1}, \mathbb{B}^{N+1}) \cap C^k(\overline{\mathbb{B}^{n+1}})$. Write $\mathbb{H}^{n+1} := \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : \text{Im}(w) > |z|^2\}$ for the Siegel upper-half space. Similarly, we can define the space $Prop(\mathbb{H}^{n+1}, \mathbb{H}^{N+1})$ and $Prop_k(\mathbb{H}^{n+1}, \mathbb{H}^{N+1})$. By the Cayley transformation $\rho_{n+1} : \mathbb{H}^{n+1} \to \mathbb{B}^{n+1}$, $\rho_{n+1}(z, w) = (\frac{2z}{1-iw}, \frac{1+iw}{1-iw})$, we can identify a map $F \in Prop_k(\mathbb{B}^{n+1}, \mathbb{B}^{N+1})$ with $\rho_{N+1}^{-1} \circ F \circ \rho_{n+1}$ in the space $Prop_k(\mathbb{H}^{n+1}, \mathbb{H}^{N+1})$. For any map $F \in Prop_2(\mathbb{H}^{n+1}, \mathbb{H}^{N+1})$, the restriction $F : \partial\mathbb{H}^{n+1} \to \partial\mathbb{H}^{N+1}$ is a C^2-smooth CR map.

For $F \in Prop_2(\mathbb{H}^{n+1}, \mathbb{H}^{N+1})$, we denote $M = F(\partial\mathbb{H}^{n+1})$ which is an immersed C^2-smooth CR submanifold. It is known that the following statements are equivalent:

- F is linear fractional.
- The geometric rank of F is zero (cf. [H03], and [HJ01], proposition 2.2).
- The CR second fundamental form $II_M^{CR} \equiv 0$ (cf. [JY10]. Although the smoothness condition was required there, by checking the proof, C^2 smoothness is sufficient. For the definition of II_M^{CR}, also see (33) below).

II_M^{CR} was defined by Cartan's moving frame theory. Again by Cartan's moving frame theory, another second fundamental form II_M can be naturally defined (see the definition in (31) below). We observe that F is linear fractional if and only if $II_M \equiv 0$ (see Corollary 5.2 below).

In this paper, we want to prove the following criterion for linearity.

Theorem 1.1 *Let $F \in Prop_3(\mathbb{H}^{n+1}, \mathbb{H}^{N+1})$ with $4 \le n+1 < N+1 \le 4n-3$. Then F is linear fractional if and only if*

$$II_M - II_M^{CR} \equiv 0. \tag{1}$$

Roughly speaking, by the decomposition $TM = T^{1,0}M \oplus \mathbb{R}\xi$ in (6), we obtain the decomposition $II_M = II_M^{CR} \oplus (II_M - II_M^{CR})$. While $II_M \equiv 0 \Leftrightarrow II_M^{CR} \equiv 0$, the above shows that it is also equivalent to $II_M - II_M^{CR} \equiv 0$. For the definition of $II_M - II_M^{CR}$, see (35). By the condition that $N+1 \le 4n-3$ together with the inequality $N \ge n + \frac{(2n+1-\kappa_0)\kappa_0}{2}$ (cf. Lemma 2.1 (i)), it implies the geometric rank κ_0 of F satisfies $\kappa_0 \le 2$. The condition that $4 \le n+1$ is used to ensure the inequality $\kappa_0 \le n-1$ holds, which allows us to apply the semi-linearity property (cf. [H03]). The conditions $N+1 \le 4n-3$ and $F \in Prop_3(\mathbb{H}^{n+1}, \mathbb{H}^{N+1})$ also imply that F is a rational map ([HJX05], corollary 1.3) so that we indeed deal with real analytic CR manifolds and CR maps in this paper.

The condition $II_M - II_M^{CR} \equiv 0$ indeed means (see (73) below):

$$\frac{\partial^2 \phi_{jl,p}^{***}}{\partial z_k \partial w}|_0 = 0, \quad \forall (j,l) \in \mathcal{S}, \quad 1 \le k \le \kappa_0, \ \forall p \in \partial\mathbb{H}^{n+1}. \tag{2}$$

As an explicit example, we would like to mention a map $F \in Rat(\mathbb{H}^4, \mathbb{H}^9)$ in ([JX04], theorem 6.1) which is not linear, and does not satisfy (2).

The authors conjecture that the coniditon "$N+1 \le 4n-3$" in Theorem 1.1 can be dropped.

2 Preliminaries

On CR mappings between Heisenberg hyperplanes We say that F and $G \in Prop(\mathbb{B}^{n+1}, \mathbb{B}^{N+1})$ are *equivalent* if there are automorphisms $\sigma \in Aut(\mathbb{B}^{n+1})$ and $\tau \in Aut(\mathbb{B}^{N+1})$ such that $F = \tau \circ G \circ \sigma$. We say that F and $G \in Prop(\mathbb{H}^{n+1}, \mathbb{H}^{N+1})$ are *equivalent* if there are automorphisms $\sigma \in Aut(\mathbb{H}^{n+1})$ and $\tau \in Aut(\mathbb{H}^{N+1})$ such that $F = \tau \circ G \circ \sigma$.

We denote by $\partial\mathbb{H}^{n+1} = \{(z,w) \in \mathbb{C}^n \times \mathbb{C} : \ \mathrm{Im}(w) = |z|^2\}$ the Heisenberg hypersurface. For any map $F \in Prop_2(\mathbb{H}^{n+1}, \mathbb{H}^{N+1})$, by restricting to $\partial\mathbb{H}^{n+1}$, we can regard F as a C^2 CR map from $\partial\mathbb{H}^{n+1}$ to $\partial\mathbb{H}^{N+1}$, and we denote it as $F \in CR_2(\partial\mathbb{H}^{n+1}, \partial\mathbb{H}^{N+1})$. We say that F and $G \in CR_2(\partial\mathbb{H}^{n+1}, \partial\mathbb{H}^{N+1})$ are *equivalent* if there are automorphisms $\sigma \in Aut(\partial\mathbb{H}^{n+1}) \simeq Aut(\mathbb{H}^{n+1})$ and $\tau \in Aut(\partial\mathbb{H}^{N+1}) \simeq Aut(\mathbb{H}^{N+1})$ such that $F = \tau \circ G \circ \sigma$.

We can parametrize $\partial\mathbb{H}^{n+1}$ by $(z, \overline{z}, u)$ through the map $(z, \overline{z}, u) \to (z, u + i|z|^2)$. In what follows, we will assign the weight of z and u to be 1 and 2, respectively. For a non-negative integer m, a function $h(z, \overline{z}, u)$ defined over a small ball U of 0 in $\partial\mathbb{H}^{n+1}$ is said to be of quantity $o_{wt}(m)$ if $\frac{h(tz, t\overline{z}, t^2u)}{|t|^m} \to 0$ uniformly for (z,u) on any compact subset of U as $t (\in \mathbb{R}) \to 0$.

Let $F = (f, \phi, g) = (\widetilde{f}, g) = (f_1, \cdots, f_n, \phi_1, \cdots, \phi_{N-n}, g) \in CR_2(\partial\mathbb{H}^{n+1}, \partial\mathbb{H}^{N+1})$ with $F(0) = 0$. For each $p = (z_0, w_0) \in \partial\mathbb{H}^{n+1}$, we write $\sigma_p^0 \in \mathrm{Aut}(\mathbb{H}^{n+1})$ with $\sigma_p^0(0) = p$ and $\tau_p^F \in \mathrm{Aut}(\mathbb{H}^{N+1})$ with $\tau_p^F(F(p)) = 0$ for the maps

$$\sigma_p^0(z,w) = (z + z_0, w + w_0 + 2i\langle z, \overline{z_0}\rangle), \tag{3}$$

$$\tau_p^F(z^*, w^*) = (z^* - \widetilde{f}(z_0,w_0), w^* - \overline{g(z_0,w_0)} - 2i\langle z^*, \overline{\widetilde{f}(z_0,w_0)}\rangle). \tag{4}$$

For each $p \in \partial\mathbb{H}^{n+1}$, there is an automorphism $\tau_p^{**} \in Aut_0(\mathbb{H}^{N+1})$ such that (cf, [HJ01], lemma 2.1) $F_p^{**} := \tau_p^{**} \circ F_p = (f_p^{**}, \phi_p^{**}, g_p^{**})$ satisfies

$$f_p^{**} = z + \frac{i}{2} e_p^{(1)}(z) w + o_{wt}(3), \ \ \phi_p^{**} = \phi_p^{(2)}(z) + o_{wt}(2), \ \ g_p^{**} = w + o_{wt}(4)$$

with $\langle \overline{z}, e_p^{(1)}(z)\rangle |z|^2 = |\phi_p^{(2)}(z)|^2$ where we denote by $h^{(j)}(z)$ a certain weighted holomorphic homogeneous polynomial with weighted degree j.

Let $\mathcal{A}(p) = -2i(\frac{\partial^2 (f_p)_l^{**}}{\partial z_j \partial w}|_0)_{1 \le j,l \le n}$. We call the rank of $\mathcal{A}(p)$, which we denote by $Rk_F(p)$, the *geometric rank* of F at p. $Rk_F(p)$ depends only on p and F, and is a lower semi-continuous function on p. We define the *geometric rank* of F to be $\kappa_0(F) = max_{p \in \partial\mathbb{H}^{n+1}} Rk_F(p)$. Notice that we always have $0 \le \kappa_0 \le n$. We define the geometric rank of $F \in \mathrm{Prop}_2(\mathbb{B}^{n+1}, \mathbb{B}^{N+1})$ to be the one for the map $\rho_N^{-1} \circ F \circ \rho_n \in \mathrm{Prop}_2(\mathbb{H}^{n+1}, \mathbb{H}^{N+1})$.

Lemma 2.1 *([H03], Lemma 3.2 and 3.3) (i) Let F be a C^2-smooth CR map from an open piece $M \subset \partial\mathbb{H}^{n+1}$ into $\partial\mathbb{H}^{N+1}$ with $F(0) = 0$ and $Rk_F(0) = \kappa_0$. Let $P(n, \kappa_0) = \frac{\kappa_0(2n-\kappa_0+1)}{2}$. Then $N \geq n + 1 + P(n, \kappa_0)$ and there are $\sigma \in Aut_0(\partial\mathbb{H}^{n+1})$ and $\tau \in Aut_0(\partial\mathbb{H}^{N+1})$ such that $F_p^{***} = \tau \circ F \circ \sigma := (f, \phi, g)$ satisfies the following normalization conditions:*

$$\begin{cases} f_j = z_j + \dfrac{i\mu_j}{2} z_j w + o_{wt}(3), \quad \dfrac{\partial^2 f_j}{\partial w^2}(0) = 0, \ j = 1 \cdots, \kappa_0, \ \mu_j > 0, \\ f_j = z_j + o_{wt}(3), \quad j = \kappa_0 + 1, \cdots, n \\ \phi_{jl} = \mu_{jl} z_j z_l + o_{wt}(2), \quad \text{with } (j,l) \in \mathcal{S}, \\ g = w + o_{wt}(4), \end{cases} \tag{5}$$

where $\mu_{jl} > 0$ for $(j,l) \in \mathcal{S}_0$, and $\mu_{jl} = 0$ otherwise. More precisely, $\mu_{jl} = \sqrt{\mu_j + \mu_l}$ for $j, l \leq \kappa_0$ $j \neq l$, $\mu_{jl} = \sqrt{\mu_j}$ if $j \leq \kappa_0$ and $l > \kappa_0$ or if $j = l \leq \kappa_0$.

(ii) If, in addition, $F \in Prop_3(\mathbb{B}^{n+1}, \mathbb{B}^{N+1})$ with $0 < \kappa_0 < n$, then

$$\frac{\partial^2 \phi_{jl}}{\partial z_k \partial w}|_0 = 0, \quad \frac{\partial^2 \phi_{jl}}{\partial w^2}|_0 = 0, \quad \forall (j,l) \in \mathcal{S}, \ k > \kappa_0.$$

On CR submanifolds Let M be a smooth strictly pseudoconvex $(2n+1)$-dimensional CR manifold. We denote by $HM \subset TM$ its maximal complex tangent bundle with the complex structure $J : HM \to HM$. Suppose that M is of hypersurface type, i.e., $\dim_{\mathbb{R}} HM = 2n$. Consider the natural extension of J on $HM \otimes \mathbb{C} \subset TM \otimes \mathbb{C}$. The eigenvalues of J in $HM \otimes \mathbb{C}$ is $\pm i$. We denote by $T^{1,0}M$ and $T^{0,1}M$ the eigenspaces of J and have the decomposition $HM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$. All HM, $T^{1,0}M$ and $T^{0,1}M$ are complex vector bundles over M of rank n. There is a $\mathbb{C}$-linear isomorphism: $HM \to T^{1,0}M$, $v \mapsto \frac{1}{2}(v - iJ(v))$.

Let H^0M be the annihilator bundle of HM which is a rank one subbundle. It is known that there exist a real globally defined nowhere zero 1-form $\theta \in \Gamma(M, H^0M)$ such that $Ker(\theta) = HM$. If M is locally defined by a defining function r, then we can take $\theta = i\partial r$. The *Levi-form* L_θ with respect to θ is defined by $L_\theta(X, Y) := -id\theta(X \wedge J(Y)) = i\theta([X, JY])$, $\forall X, Y \in \Gamma(M, HM)$. By $HM \simeq T^{1,0}M$, we have

$$L_\theta(u, v) := -id\theta(u \wedge \overline{v}) = i\theta([u, \overline{v}]), \quad \forall u, v \in T_p^{1,0}(M), \ \forall p \in M.$$

Recall that (M, θ) is *strictly pseudoconvex* if the Levi-form L_θ is positive definite for all $z \in M$. Such real non-vanishing 1-form θ over M is a *contact form* because it satisfies: $\theta \wedge (d\theta)^n \neq 0$. Associated with a contact form θ, there is a unique *Reeb vector field* ξ, defined by the equations: (i) $\theta(\xi) \equiv 1$, (ii) $d\theta(\xi, X) \equiv 0$ for any smooth vector field X over M. We have orthogonal decomposition $TM \simeq HM \oplus \mathbb{R}\xi$, or by $HM \simeq T^{1,0}M$, we have

$$TM \simeq T^{1,0}M \oplus \mathbb{R}\xi. \tag{6}$$

Here $g_\theta|_{HM} = L_\theta$ and $g_\theta(\xi, \xi) = 1$ defines the *Webster metric* associated to θ.

3 Cartan's moving frame theory

Q-frames We consider the real hypersurface Q in $\mathbb{C}^{N+2}$ defined by the homogeneous equation

$$\langle Z, Z\rangle := \sum_A Z^A\overline{Z^A} + \frac{i}{2}(Z^{N+1}\overline{Z^0} - Z^0\overline{Z^{N+1}}) = 0, \tag{7}$$

where $Z = (Z^0, Z^A, Z^{N+1})^t \in \mathbb{C}^{N+2}$. This can be extended to the scalar product

$$\langle Z, Z'\rangle := \sum_A Z^A\overline{Z'^A} + \frac{i}{2}(Z^{N+1}\overline{Z'^0} - Z^0\overline{Z'^{N+1}}), \tag{8}$$

for any $Z = (Z^0, Z^A, Z^{N+1})^t, Z' = (Z'^0, Z'^A, Z'^{N+1})^t \in \mathbb{C}^{N+2}$. This product has the properties: $\langle Z, Z'\rangle$ is linear in Z and anti-linear in Z'; $\overline{\langle Z, Z'\rangle} = \langle Z', Z\rangle$; and Q is defined by $\langle Z, Z\rangle = 0$.

Let $SU(N+1, 1)$ be the group of unimodular linear homogeneous transformations of $\mathbb{C}^{N+2}$ that leave the form $\langle Z, Z\rangle$ invariant (cf. [CM74]). By a unimodular of linear homogeneous transformation, in terms of a matrix A, we mean $det(A) = 1$.

By a *Q-frame* is meant an element $E = (E_0, E_A, E_{N+1}) \in GL(\mathbb{C}^{N+2})$ satisfying (cf. [CM74, (1.10)])

$$\begin{cases} det(E) = 1, \\ \langle E_A, E_B\rangle = \delta_{AB},\ \langle E_0, E_{N+1}\rangle = -\langle E_{N+1}, E_0\rangle = -\frac{i}{2}, \end{cases} \tag{9}$$

while all other products are zero.

There is exactly one transformation of $SU(N + 1, 1)$ which maps a given Q-frame into another. By fixing one Q-frame as reference, the group $SU(N + 1, 1)$ can be identified with the space of all Q-frames. Then $SU(N + 1, 1) \subset GL(\mathbb{C}^{N+2})$ is a subgroup with the composition operation.

The Q-frame bundle over $\mathbb{CP}^{N+1}$ Consider an element $A \in GL(\mathbb{C}^{N+2})$:

$$A = (a_0, ..., a_{N+1}) = \begin{bmatrix} a_0^{(0)} & a_1^{(0)} & ... & a_{N+1}^{(0)} \\ a_0^{(1)} & a_1^{(1)} & ... & a_{N+1}^{(1)} \\ \vdots & \vdots & & \vdots \\ a_0^{(N+1)} & a_1^{(N+1)} & ... & a_{N+1}^{(N+1)} \end{bmatrix}, \tag{10}$$

where each a_j is a column vector in $\mathbb{C}^{N+2}$, $0 \leq j \leq N + 1$. This A is associated to an automorphism $A^\star \in Aut(\mathbb{CP}^{N+1})$ given by

$$A^\star\Big(\big[z_0 : z_1 : ... : z_{N+1}\big]\Big) = \left[A\begin{pmatrix} z_0 \\ \vdots \\ z_{N+1}\end{pmatrix}\right] = \left[\sum_{j=0}^{N+1} a_j^{(0)} z_j : \sum_{j=0}^{N+1} a_j^{(1)} z_j : ... : \sum_{j=0}^{N+1} a_j^{(N+1)} z_j\right]. \tag{11}$$

When $a_0^{(0)} \neq 0$, in terms of the non-homogeneous coordinates $(w_1, ..., w_{N+1})$, $A^\star$ is a linear fractional from $\mathbb{C}^{N+1}$ which is holomorphic near $(0, ..., 0)$:

$$A^\star\big(w_1, ..., w_{N+1}\big) = \left(\frac{\sum_{j=0}^{N+1} a_j^{(1)} w_j}{\sum_{j=0}^{N+1} a_j^{(0)} w_j}, ..., \frac{\sum_{j=0}^{N+1} a_j^{(N+1)} w_j}{\sum_{j=0}^{N+1} a_j^{(0)} w_j}\right), \quad where\ w_j = \frac{z_j}{z_0}. \tag{12}$$

We define a bundle map:

$$\begin{array}{rccc} \pi : & GL(\mathbb{C}^{N+2}) & \to & \mathbb{CP}^{N+1} \\ & A = (a_0, a_1, ..., a_{N+1}) & \mapsto & \pi_0(a_0) \end{array}$$

where

$$\pi_0 : \mathbb{C}^{N+2} - \{0\} \to \mathbb{CP}^{N+1}, \quad (z_0, ..., z_{N+1}) \mapsto [z_0 : ... : z_{N+1}], \tag{13}$$

be the standard projection. By taking restriction, we have the projection

$$\pi : SU(N+1, 1) \to \partial\mathbb{H}^{N+1},\ (Z_0, Z_A, Z_{N+1}) \mapsto span(Z_0). \tag{14}$$

which is called a *Q-frames bundle*. We get $\partial\mathbb{H}^{N+1} \simeq SU(N+1, 1)/P_2$ where P_2 is the isotropy subgroup of $SU(N+1, 1)$. $SU(N+1, 1)$ acts on $\partial\mathbb{H}^{N+1}$ effectively.

The Maurer-Cartan form over $SU(N+1, 1)$ Consider $E = (E_0, E_A, E_{N+1}) \in SU(N+1, 1)$ as a local lift. Then the *Maurer-Cartan form* Θ on $SU(N+1, 1)$ is defined by $dE = (dE_0, dE_A, dE_{N+1}) = E\Theta$, or $\Theta = E^{-1} \cdot dE$, i.e.,

$$d\begin{pmatrix} E_0 & E_A & E_{N+1}\end{pmatrix} = \begin{pmatrix} E_0 & E_B & E_{N+1}\end{pmatrix}\begin{pmatrix} \Theta_0^0 & \Theta_A^0 & \Theta_{N+1}^0 \\ \Theta_0^B & \Theta_A^B & \Theta_{N+1}^B \\ \Theta_0^{N+1} & \Theta_A^{N+1} & \Theta_{N+1}^{N+1}\end{pmatrix}, \tag{15}$$

where Θ_A^B are 1-forms on $SU(N+1, 1)$. By (9) and (15), the Maurer-Cartan form Θ satisfies

$$\begin{array}{l} \Theta_0^0 + \overline{\Theta_{N+1}^{N+1}} = 0,\ \Theta_0^{N+1} = \overline{\Theta_0^{N+1}},\ \Theta_{N+1}^0 = \overline{\Theta_{N+1}^0}, \\ \Theta_A^{N+1} = 2i\overline{\Theta_0^A},\ \Theta_{N+1}^A = -\frac{i}{2}\overline{\Theta_A^0},\ \Theta_B^A + \overline{\Theta_A^B} = 0,\ \Theta_0^0 + \Theta_A^A + \Theta_{N+1}^{N+1} = 0, \end{array} \tag{16}$$

where $1 \leq A, B \leq N$. For example, from $\langle E_A, E_B \rangle = \delta_{AB}$, by taking differentiation, we obtain

$$\langle dE_A, E_B \rangle + \langle E_A, dE_B \rangle = 0.$$

By (15), we have

$$\begin{cases} dE_0 = E_0\Theta_0^0 + \sum_B E_B\Theta_0^B + E_{N+1}\Theta_0^{N+1}, \\ dE_A = E_0\Theta_A^0 + \sum_B E_B\Theta_A^B + E_{N+1}\Theta_A^{N+1}, \\ dE_{N+1} = E_0\Theta_{N+1}^0 + \sum_B E_B\Theta_{N+1}^B + E_{N+1}\Theta_{N+1}^{N+1}. \end{cases}$$

Then

$$\langle E_0\Theta_A^0 + \sum_C E_C\Theta_A^C + E_{N+1}\Theta_A^{N+1},\ E_B \rangle + \langle E_A, E_0\Theta_B^0 + \sum_D E_D\Theta_B^D + E_{N+1}\Theta_B^{N+1} \rangle = 0,$$

which implies $\Theta_A^B + \overline{\Theta_B^A} = 0$. In particular, from (16), $\Theta_A^0 = -2i\overline{\Theta_{N+1}^A}$. Θ satisfies

$$d\Theta = -\Theta \wedge \Theta. \tag{17}$$

CR submanifolds of $\partial\mathbb{H}^{N+1}$ Let $H : M' \to \partial\mathbb{H}^{N+1}$ be a CR smooth embedding where M' is a strictly pseudoconvex smooth real hypersurface in $\mathbb{C}^{n+1}$. We denote $M = H(M')$.

Let $\xi_{M'}$ be the Reeb vector field of M' with respect to a fixed contact form on M'. By (6), we have:

$$TM' \simeq HM' \oplus \mathbb{R}\xi_{M'} \simeq T^{1,0}M' \oplus \mathbb{R}\xi_{M'}. \tag{18}$$

For example, if $M' = \partial\mathbb{H}^{n+1} = \{(z_1, ..., z_n, z_{n+1}) \mid Im(z_{n+1}) = |z|^2\}$, then the above isomorphism is given by

$$\sum_{j=1}^{n}(a_j\frac{\partial}{\partial x_j} + b_j\frac{\partial}{\partial y_j}) + c\xi_{M'} \mapsto \sum_{j=1}^{n}(a_j + ib_j)\frac{\partial}{\partial z_j} + c\xi_{M'}, \quad where\ a_j, b_j, c \in \mathbb{R}. \tag{19}$$

Since H is a CR embedding, we have

$$H_*(T^{1,0}M') = T^{1,0}M \subset T^{1,0}(\partial\mathbb{H}^{N+1}), \tag{20}$$

$$TM \simeq H_*(T^{1,0}M') \oplus H_*(\mathbb{R}\xi_{M'}) \subset T(\partial\mathbb{H}^{N+1}). \tag{21}$$

First-order adapted lifts In order to define more specific lifts, we need to give some relationship between geometry on $\partial\mathbb{H}^{N+1}$ and on $\mathbb{C}^{N+2}$ as follows. For any subset $X \subset \partial\mathbb{H}^{N+1}$, we denote $\hat{X} := \pi_0^{-1}(X)$ where $\pi_0 : \mathbb{C}^{N+2} - \{0\} \to \mathbb{CP}^{N+1}$ is the standard projection map (13). In particular, for any $x \in M$, $\hat{x}$ is a complex line and for the real submanifold M^{2n+1}, the real submanifold $\hat{M}^{2n+3}$ is of dimension $2n+3$.

For any $x \in M$, we take $v \in \hat{x} = \pi_0^{-1}(x) \subset \mathbb{C}^{N+2} - \{0\}$, and we define

$$\hat{T}_x M = T_v \hat{M} \textit{ and } \hat{T}_x^{1,0} M = T_v^{1,0} \hat{M}.$$

These definitions are independent of choice of v. Notice that $\hat{T}_x M = \pi_0^{-1}(T_x M) \cup \{0\}$ and $\hat{T}_x^{1,0} M = \pi_0^{-1}(T_x^{1,0} M) \cup \{0\}$. We denote $\hat{\mathbb{R}\xi}_{M,x} := \pi_0^{-1}(\mathbb{R}\xi_{M,x}) \cup \{0\}$.

Let $M \subset \partial\mathbb{H}^{N+1}$ be the image of $H : M' \to \partial\mathbb{H}^{N+1}$ where $M' \subset \mathbb{C}^{n+1}$ is a CR strictly pseudoconvex smooth hypersurface. Consider the inclusion map $M \hookrightarrow \partial\mathbb{H}^{N+1}$ and a C^2-smooth lift $e = (e_0, e_\alpha, e_\nu, e_{N+1})$ of M where $1 \le \alpha \le n$ and $n+1 \le \nu \le N$

$$\begin{array}{ccc} & & SU(N+1,1) \\ & e \nearrow & \downarrow \pi \\ M & \hookrightarrow & \partial\mathbb{H}^{N+1} \end{array}$$

We call e a *first-order adapted lift* if for any $x \in M$,

$$\begin{cases} \pi_0\big(e_0(x)\big) = x, \\ \mathbb{C} \otimes \{e_0 + \sum_\alpha a_\alpha e_\alpha \mid a_\alpha \in \mathbb{C}\}|_x = \hat{T}_x^{1,0} M, \\ \mathbb{C} \otimes \{e_0 + \sum_\alpha a_\alpha e_\alpha + b e_{N+1} \mid a_\alpha \in \mathbb{C}, b \in \mathbb{R}\}|_x = \hat{T}_x^{1,0} M \oplus \hat{\mathbb{R}\xi}_{M,x}. \end{cases} \tag{22}$$

Locally first-order adapted lifts always exist (cf. [JY10], theorem 7.1). We have the restriction bundle $\mathcal{F}_M^0 := SU(N+1,1)|_M$ over M. The subbundle $\pi : \mathcal{F}_M^1 \to M$ of $\mathcal{F}_M^0$ is defined by

$$\mathcal{F}_M^1 = \{(e_0, e_j, e_\mu, e_{N+1}) \in \mathcal{F}_M^0 \mid \ [e_0] \in M, \ (22) \textit{ are satisfied}\}.$$

Local sections of $\mathcal{F}_M^1$ are exactly all local first-order adapted lifts of M. The fiber of $\pi : \mathcal{F}_M^1 \to M$ over a point is isomorphic to the group

$$G_1 = \left\{ g = \begin{pmatrix} g_0^0 & g_\beta^0 & g_\nu^0 & g_{N+1}^0 \\ 0 & g_\beta^\alpha & g_\nu^\alpha & g_{N+1}^\alpha \\ 0 & 0 & g_\nu^\mu & 0 \\ 0 & 0 & 0 & g_{N+1}^{N+1} \end{pmatrix} \in SU(N+1,1) \right\}, \tag{23}$$

where we use the index range $1 \le \alpha, \beta \le n$ and $n+1 \le \mu, \nu \le N$.

By (9), we have $\langle g_0, g_{N+1} \rangle = -\frac{i}{2}$, it implies $g_0^0 \cdot \overline{g_{N+1}^{N+1}} = 1$ so that $g_{N+1}^{N+1} = \frac{1}{\overline{g_0^0}}$. Since $\langle g_0, g_\mu \rangle = 0$ and $g_0^0 \neq 0$, it implies $g_\mu^{N+1} = 0$. Since $\langle g_\alpha, g_\beta \rangle = \delta_{\alpha\beta}$, it implies that the matrix

(g^β_α) is unitary. Since $det(g) = 1$, it implies $g^0_0 \cdot det(g^\beta_\alpha) \cdot det(g^\nu_\mu) \cdot g^{N+1}_{N+1} = 1$. By (19) and (22), g^{N+1}_{N+1} is a real if $g^0_{N+1} = 0$; g^{N+1}_{N+1}/g^0_{N+1} is real if $g^0_{N+1} \neq 0$.

We pull back the Maurer-Cartan form from $SU(N+1,1)$ to $\mathcal{F}^1_M$ by a first-order adapted lift e of M as

$$\omega = \begin{pmatrix} \omega^0_0 & \omega^0_\beta & \omega^0_\nu & \omega^0_{N+1} \\ \omega^\alpha_0 & \omega^\alpha_\beta & \omega^\alpha_\nu & \omega^\alpha_{N+1} \\ \omega^\mu_0 & \omega^\mu_\beta & \omega^\mu_\nu & \omega^\mu_{N+1} \\ \omega^{N+1}_0 & \omega^{N+1}_\beta & \omega^{N+1}_\nu & \omega^{N+1}_{N+1} \end{pmatrix}.$$

Since $\omega = e^{-1}de$, i.e., $e\omega = de$. Then we have $de_0 = e_0\omega^0_0 + \sum_\alpha e_\alpha\omega^\alpha_0 + \sum_\mu e_\mu\omega^\mu_0 + e_{N+1}\omega^{N+1}_0$. On the other hand, we have (cf.[JY10]) $de_0 = e_0\omega^0_0 + \sum_\alpha e_\alpha\omega^\alpha_0 + e_{N+1}\omega^{N+1}_0$ so that $\omega^\mu_0 = 0, \ \forall\mu$. By the Maurer-Cartan equation $d\omega = -\omega\wedge\omega$, one gets $0 = d\omega^\nu_0 = -\sum_\alpha \omega^\nu_\alpha \wedge \omega^\alpha_0 - \omega^\nu_{N+1}\wedge\omega^{N+1}_0$, i.e., $0 = -\sum_{j\in\{1,2,...,n,N+1\}} \omega^\nu_j \wedge \omega^j_0$. Then by Cartan's lemma,

$$\omega^\nu_k = \sum_j q^\nu_{jk}\omega^j_0, \tag{24}$$

for some functions $q^\nu_{jk} = q^\nu_{kj}$.

Second fundamental form and CR second fundamental form For any first-order adapted lift $s = (e_0, e_j, e_\mu, e_{N+1})$ with $\pi_0(e_0) = x$, we have $e_j \in \hat{T}^{1,0}_x M$. Recall $T_E G(k,V) \simeq E^* \otimes (V/E)$ where $G(k,V)$ is the Grassmannian of k-planes that pass through the origin in a vector space V over $\mathbb{R}$ or $\mathbb{C}$ and $E \in G(k,V)$ ([IL03], p.73). Then $T_xM \simeq (\hat{x})^* \otimes (\hat{T}_xM/\hat{x})$. The vector e_j induces $\underline{e_j} \in T_xM$ by

$$\underline{e}_j = e^0 \otimes \Big(e_j \ mod(e_0)\Big) \in T_{[e_0]}M, \quad \forall j \in \{1,2,...,n,N+1\}$$

where we denote by $(e^0, e^j, e^\mu, e^{N+1})$ the dual basis of $(\mathbb{C}^{N+2})^*$. Similarly, we let

$$\underline{e}_\mu = e^0 \otimes \Big(e_\mu \ mod(\hat{T}_{[e_0]}M)\Big) \in N_{[e_0]}M, \tag{25}$$

where NM is the normal bundle of M defined by $N_xM = T_x(\partial\mathbb{H}^{N+1})/T_xM$.

We claim that

$$\sum_{j,k\in\{1,2,...,n,N+1\},n+1\le\mu\le N} q^\mu_{jk}\omega^j_0\omega^k_0 \otimes \underline{e}_\mu, \text{ is independent of choice of the lift } s. \tag{26}$$

In fact, suppose that s and $\widetilde{s}$ are both such lifts. Then

$$\widetilde{s} = sg = s\begin{pmatrix} g^0_0 & g^0_k & g^0_\mu & g^0_{N+1} \\ 0 & g^j_k & g^j_\mu & g^j_{N+1} \\ 0 & 0 & g^\nu_\mu & 0 \\ 0 & 0 & 0 & g^{N+1}_{N+1} \end{pmatrix} \tag{27}$$

where g is some map from M to $G_1 \subset SU(N+1,1)$. By the general transformation formula $\widetilde{\omega} = g^{-1}\omega g + g^{-1}dg$ (cf. (1.19) in [IL03]), we have

$$\begin{pmatrix} \widetilde{\omega}_0^0 & \widetilde{\omega}_k^0 & \widetilde{\omega}_\nu^0 & \widetilde{\omega}_{N+1}^0 \\ \widetilde{\omega}_0^j & \widetilde{\omega}_k^j & \widetilde{\omega}_\nu^j & \widetilde{\omega}_{N+1}^j \\ 0 & \widetilde{\omega}_k^\mu & \widetilde{\omega}_\nu^\mu & \widetilde{\omega}_{N+1}^\mu \\ \widetilde{\omega}_0^{N+1} & \widetilde{\omega}_k^{N+1} & 0 & \omega_{N+1}^{N+1} \end{pmatrix}$$
$$= \begin{pmatrix} h_0^0 & h_t^0 & h_\kappa^0 & h_{N+1}^0 \\ 0 & h_t^j & h_\kappa^j & h_{N+1}^j \\ 0 & 0 & h_\kappa^\mu & 0 \\ 0 & 0 & 0 & h_{N+1}^{N+1} \end{pmatrix} \begin{pmatrix} \omega_0^0 & \omega_s^0 & \omega_\ell^0 & \omega_{N+1}^0 \\ \omega_0^t & \omega_s^t & \omega_\ell^t & \omega_{N+1}^t \\ 0 & \omega_s^\kappa & \omega_\ell^\kappa & \omega_{N+1}^\kappa \\ \omega_0^{N+1} & \omega_s^{N+1} & 0 & \omega_{N+1}^{N+1} \end{pmatrix} \cdot \begin{pmatrix} g_0^0 & g_k^0 & g_\nu^0 & g_{N+1}^0 \\ 0 & g_k^s & g_\nu^s & g_{N+1}^s \\ 0 & 0 & g_\nu^\ell & 0 \\ 0 & 0 & 0 & g_{N+1}^{N+1} \end{pmatrix}$$
$$+ \begin{pmatrix} h_0^0 & h_t^0 & h_\kappa^0 & h_{N+1}^0 \\ 0 & h_t^j & h_\kappa^j & h_{N+1}^j \\ 0 & 0 & h_\kappa^\nu & 0 \\ 0 & 0 & 0 & h_{N+1}^{N+1} \end{pmatrix} \begin{pmatrix} dg_0^0 & dg_k^0 & dg_\nu^0 & dg_{N+1}^0 \\ 0 & dg_k^t & dg_\nu^t & g_{N+1}^t \\ 0 & 0 & dg_\nu^\kappa & 0 \\ 0 & 0 & 0 & dg_{N+1}^{N+1} \end{pmatrix}$$

where $h = g^{-1}$. Then we find

$$\widetilde{\omega}_0^j = \sum_t g_0^0 h_t^j \omega_0^t, \quad \widetilde{\omega}_k^\mu = \sum_{\kappa,s} h_\kappa^\mu \omega_s^\kappa g_k^s, \quad j,k,t,s \in \{1,2,...,n,N+1\},\ n+1 \le \mu,\kappa \le N. \tag{28}$$

Also, from $\widetilde{s} = s \cdot g$, we obtain

$$\widetilde{e}^0 = h_0^0 e^0, \quad \widetilde{e}_\mu = \sum_{k\in\{1,2,\dots,n,N+1\},n+1\le\nu\le N} (g_\mu^0 e_0 + g_\mu^k e_k + g_\mu^\nu e_\nu).$$

Applying those formulas into $\widetilde{\omega}_k^\mu = \sum_j \widetilde{q}_{jk}^\mu \widetilde{\omega}_0^j$, we obtain $\sum_{\kappa,s} h_\kappa^\mu q_t^\kappa{}_s g_k^s = \sum_{j,t} \widetilde{q}_{jk}^\mu g_0^0 h_t^j$, i.e.,

$$\widetilde{q}_{jk}^\mu = h_0^0 \sum_{\kappa,t,s} h_\kappa^\mu g_k^s g_j^t q_{ts}^\kappa, \tag{29}$$

which implies

$$\sum_{\mu,j,k} \widetilde{q}_{jk}^\mu \widetilde{\omega}_0^j \widetilde{\omega}_0^k \otimes \underline{\widetilde{e}}_\mu = \sum_{\mu,j,k} q_{jk}^\mu \omega_0^j \omega_0^k \otimes \underline{e}_\mu. \tag{30}$$

Thus (26) is proved so that the form

$$II_M = \sum_{j,k\in\{1,2,\dots,n,N+1\},n+1\le\mu\le N} q_{jk}^\mu \omega_0^j \omega_0^k \otimes \underline{e}_\mu \in \Gamma(M, S^2T^*M \otimes NM) \tag{31}$$

is independent of choice of first-order adapted lift s from M into $SU(N+1,1)$. II_M is called the *second fundamental form of* M.

Comparing the identity (30):

$$\sum_{j,k\in\{1,2,...,n,N+1\},n+1\le\mu\le N}\widetilde{q}^{\mu}_{jk}\widetilde{\omega}^{j}_{0}\widetilde{\omega}^{k}_{0}\otimes\underline{\widetilde{e}}_{\mu}=\sum_{j,k\in\{1,2,...,n,N+1\},n+1\le\mu\le N}q^{\mu}_{jk}\omega^{j}_{0}\omega^{k}_{0}\otimes\underline{e}_{\mu},$$

it also holds that

$$\sum_{j,k\in\{1,2,...,n\},n+1\le\mu\le N}\widetilde{q}^{\mu}_{jk}\widetilde{\omega}^{j}_{0}\widetilde{\omega}^{k}_{0}\otimes\underline{\widetilde{e}}_{\mu}=\sum_{j,k\in\{1,2,...,n\},n+1\le\mu\le N}q^{\mu}_{jk}\omega^{j}_{0}\omega^{k}_{0}\otimes\underline{e}_{\mu},\quad mod(\omega^{N+1}_{0}). \tag{32}$$

From this, we define the *CR second fundamental form* II^{CR}_M by moduling ω^{N+1}_0:

$$II^{CR}_M=\sum_{j,k\in\{1,2,...,n\},n+1\le\mu\le N}q^{\mu}_{jk}\omega^{j}_{0}\omega^{k}_{0}\otimes\underline{e}_{\mu}\in\Gamma(M,S^2T^{1,0*}M\otimes NM). \tag{33}$$

Remark

1. The definition of II_M in (31) is similar to the one of the projective second fundamental form for complex submanifolds (cf. [IL03]).

2. The II^{CR}_M defined in (33) was studied in [Wang09] and in [JY10]. It was proved that $II^{CR}_M\equiv 0$ if and only if F is linear fractional [JY10].

3. Let $s,s^{(1)},s^{(2)}$ be three first-order adapted lifts with $II^s_M=\sum_{j,k,\mu}q^{\mu}_{jk}\omega^j_0\omega^k_0\otimes\underline{e}_\mu$, $II^{s^{(1)}}_M=\sum_{j,k,\mu}q^{(1)\mu}_{jk}\omega^j_0\omega^k_0\otimes\underline{e}_\mu$, and $II^{s^{(2)}}_M=\sum_{j,k,\mu}q^{(2)\mu}_{jk}\omega^j_0\omega^k_0\otimes\underline{e}_\mu$. Let $s^{(1)}=sg_1$ and $s^{(2)}=sg_2$ be as in (27). Suppose $g_1(p)=g_2(p)$ holds at one point $p\in M$. Then by (29), we have

$$q^{(1)\mu}_{jk}(p)=q^{(2)\mu}_{jk}(p) \tag{34}$$

   for any $j,k\in\{1,2,...,n,N+1\}$ and $n+1\le\mu\le N$.

By inclusion $T^{1,0*}M\hookrightarrow T^*M\simeq T^{1,0*}M\oplus(\mathbb{R}\xi)^*$, we can regard $II^{CR}M\in\Gamma(M,T^*M\otimes NM)$. Then by (31) and (33), we have defined a section $II_M-II^{CR}_M\in\Gamma(M,T^*M\otimes NM)$, i.e., in terms of local coordinates,

$$II_M-II^{CR}_M=\sum_{1\le j,k\le n,n+1\le\mu\le N}\left(q^{\mu}_{jN+1}\omega^{j}_{0}\omega^{N+1}_{0}+q^{\mu}_{N+1k}\omega^{N+1}_{0}\omega^{k}_{0}+q^{\mu}_{N+1N+1}\omega^{N+1}_{0}\omega^{N+1}_{0}\right)\otimes\underline{e}_{\mu}. \tag{35}$$

Pulling back a lift Let $M \subset \partial\mathbb{H}^{N+1}$ be as above with a point $Q \in M$. Let $A \in SU(N+1,1)$, $A^\star \in Aut(\partial\mathbb{H}^{N+1})$ with $A^\star(Q) = P$ and $\widetilde{M} = A^\star(M)$. Let $\widetilde{s} : \widetilde{M} \to SU(N+1,1)$ be a lift. We claim:

$$s(Q) := (A^{-1} \cdot \widetilde{s})(A^\star(Q)) \tag{36}$$

is also a lift from M into $SU(N+1,1)$. In fact, in order to prove that s is a lift from M into $SU(N+1,1)$, it suffices to prove: $\pi s = Id$. In fact, write $\widetilde{s} = (\widetilde{e}_0, \widetilde{e}_A, \widetilde{e}_{N+1})$ and $s = (e_0, e_A, e_{N+1}) = (A^{-1}\widetilde{e}_0, A^{-1}\widetilde{e}_A, A^{-1}\widetilde{e}_{N+1})$. Here $[\widetilde{e}_0](P) = P$ and $[e_0](Q) = Q$. Then $\pi s(Q) = [A^{-1}\widetilde{e}_0](Q) = [e_0](Q) = Q$ so that our claim is proved.

If, in addition, $\widetilde{s}$ is a first-order adapted lift of $\widetilde{M}$ into $SU(N+1,1)$, s is also a first-order adapted lift of M into $SU(N+1,1)$.

Let Ω be the Maurer-Cartan form over $SU(N+1,1)$. Denote $\omega = s^*\Omega$ and $\widetilde{\omega} = \widetilde{s}^*\Omega$. Since A is a matrix with constant entries, $\omega = (s)^{-1}ds = (A^{-1} \cdot \widetilde{s})^{-1}d(A^{-1}\widetilde{s}) = \widetilde{s}^{-1} \cdot A \cdot A^{-1}d\widetilde{s} = \widetilde{s}^{-1}d\widetilde{s}$, i.e.,

$$\omega = (A^\star)^*\widetilde{\omega} \tag{37}$$

so that $\omega_0^\alpha = (A^\star)^*\widetilde{\omega}_0^\alpha$ and $\omega_\beta^\mu = (A^\star)^*\widetilde{\omega}_\beta^\mu$. The last equality yields

$$q_{\alpha\beta}^\mu = \widetilde{q}_{\alpha\beta}^\mu \circ A^\star. \tag{38}$$

[**Example**] Consider the maps in (3) and (4):

$$\sigma_p^0(z,w) = (z + z_0, w + w_0 + 2i\langle z, \overline{z_0}\rangle),$$
$$\tau_p^F(z^*,w^*) = (z^* - \widetilde{f}(z_0,w_0), w^* - \overline{g(z_0,w_0)} - 2i\langle z^*, \overline{\widetilde{f}(z_0,w_0)}\rangle)$$

where $p = (z_0, w_0)$, $z \in \mathbb{C}^n$, $w = z_{n+1}$, $\sigma_p^0 \in Aut(\partial\mathbb{H}^{n+1})$, and $\tau_p^F \in Aut(\partial\mathbb{H}^{N+1})$.

By (10) and (12), these two maps correspond to two matrices:

$$A_{\sigma_p^0} = \begin{bmatrix} 1 & 0 & ... & 0 & 0 \\ z_{01} & 1 & ... & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_{0n} & 0 & ... & 1 & 0 \\ w_0 & 2i\overline{z_{01}} & ... & 2i\overline{z_{0n}} & 1 \end{bmatrix} \in SU(n+1,1) \tag{39}$$

and

$$A_{\sigma_p^F} = \begin{bmatrix} 1 & 0 & ... & 0 & 0 \\ -\widetilde{f}_{01} & 1 & ... & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\widetilde{f}_{0N} & 0 & ... & 1 & 0 \\ -\overline{g(z_0,w_0)} & -2i\overline{\widetilde{f}_1(z_0,w_0)} & ... & -2i\overline{\widetilde{f}_N(z_0,w_0)} & 1 \end{bmatrix} \in SU(N+1,1) \tag{40}$$

where $z_0 = (z_{01}, ..., z_{0n})$ and $w_0 = z_{0n+1}$.
[**Example**] Consider the map $F_{\lambda,r,\vec{a},U} = (f, g) \in Aut_0(\partial\mathbb{H}^{n+1})$

$$f(z) = \frac{\lambda(z + \vec{a}w)U}{1 - 2i\langle z, \overline{\vec{a}}\rangle - (r + i\|\vec{a}\|^2)w},\ g(z) = \frac{\lambda^2 w}{1 - 2i\langle z, \overline{\vec{a}}\rangle - (r + i\|\vec{a}\|^2)w}$$

where $\lambda > 0, r \in \mathbb{R}, \vec{a} \in \mathbb{C}^n$ and $U = (u_{\alpha\beta})$ is an $(n-1) \times (n-1)$ unitary matrix. By (10) and (12), its corresponding matrix,

$$A_{F_{\lambda,r,\vec{a},U}} = \begin{bmatrix} 1 & -2i\overline{a_1} & ... & -2i\overline{a_n} & -(r + i\|\vec{a}\|^2) \\ 0 & \lambda u_{11} & ... & \lambda u_{1n} & \lambda a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \lambda u_{n1} & ... & \lambda u_{nn} & \lambda a_n \\ 0 & 0 & ... & 0 & \lambda^2 \end{bmatrix}, \tag{41}$$

is not in $SU(n+1, 1)$ in general. In fact, we can write

$$F_{\lambda,r,\vec{a},U} = F_{\lambda,0,0,Id} \circ F_{1,0,0,U} \circ F_{1,r,\vec{a},Id}. \tag{42}$$

or $A_{F_{\lambda,r,\vec{a},U}} = A_{F_{\lambda,0,0,Id}} \cdot A_{F_{1,0,0,U}} \cdot A_{F_{1,r,\vec{a},Id}}$. Here $A_{F_{1,0,0,U}}$ and $A_{F_{1,r,\vec{a},Id}}$ are in $SU(N+1, 1)$; while $A_{F_{\lambda,0,0,Id}}$ is in $SU(N+1, 1)$ if and only if $\lambda = 1$. Therefore

$$A_{F_{\lambda,r,\vec{a},U}} \ is\ in\ SU(n+1, 1)\ if\ and\ only\ if\ \lambda = 1. \tag{43}$$

[**Example**] Let $G \in Aut(\partial\mathbb{H}^{N+1})$. Then G can be written as $G = \sigma^0_{F(0)} \circ F_{\rho,r,\vec{a},U}$ where $F_{\rho,r,\vec{a},U} \in Aut_0(\partial\mathbb{H}^{N+1})$ as in the previous example. By (42), we have

$$G = \sigma^0_{F(0)} \circ F_{\lambda,0,0,Id} \circ F_{1,0,0,U} \circ F_{1,r,\vec{a},Id}. \tag{44}$$

[**Example**] Let $A \in SU(N+1, 1)$. From above, we know $A_{F_{\lambda,0,0,Id}} \cdot A$ may not be in $SU(N+1, 1)$ unless $\lambda = 1$. However, it is possible to modify it so that the modified map is in $SU(N+1, 1)$, namely, for any real number $\lambda \in \mathbb{R}$, we have

$$A_{F_{\lambda,0,0,Id}} \cdot A \cdot {A_{F_{\lambda,0,0,Id}}}^{-1} \in SU(N+1, 1). \tag{45}$$

In fact, we write $A = (A_{ij})$. Then $A_{F_{\lambda,0,0,Id}} \circ A \cdot {A_{F_{\lambda,0,0,Id}}}^{-1} =$

$$\begin{bmatrix} 1 & 0 & ... & 0 & 0 \\ 0 & \lambda & ... & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda & 0 \\ 0 & 0 & \ldots & 0 & \lambda^2 \end{bmatrix} \begin{bmatrix} A_{00} & A_{01} & ... & A_{0N} & A_{0,N+1} \\ A_{10} & A_{11} & ... & A_{1N} & A_{1,N+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{N,0} & A_{N,1} & \ldots & A_{N,N} & A_{N,N+1} \\ A_{N+1,0} & A_{N+1,1} & \ldots & A_{N+1,N} & A_{N+1,N+1} \end{bmatrix} \begin{bmatrix} 1 & 0 & ... & 0 & 0 \\ 0 & \frac{1}{\lambda} & ... & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \frac{1}{\lambda} & 0 \\ 0 & 0 & \ldots & 0 & \frac{1}{\lambda^2} \end{bmatrix}$$

$$= \begin{bmatrix} A_{00} & A_{01} & \dots & A_{0N} & A_{0,N+1} \\ \lambda A_{10} & \lambda A_{11} & \dots & \lambda A_{1N} & \lambda A_{1,N+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda A_{N,0} & \lambda A_{N,1} & \dots & \lambda A_{N,N} & \lambda A_{N,N+1} \\ \lambda^2 A_{N+1,0} & \lambda^2 A_{N+1,1} & \dots & \lambda^2 A_{N+1,N} & \lambda^2 A_{N+1,N+1} \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{\lambda} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{1}{\lambda} & 0 \\ 0 & 0 & \dots & 0 & \frac{1}{\lambda^2} \end{bmatrix}$$

$$= \begin{bmatrix} A_{00} & \frac{1}{\lambda} A_{01} & \dots & \frac{1}{\lambda} A_{0N} & \frac{1}{\lambda^2} A_{0,N+1} \\ \lambda A_{10} & A_{11} & \dots & A_{1N} & \frac{1}{\lambda} A_{1,N+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda A_{N,0} & A_{N,1} & \dots & A_{N,N} & \frac{1}{\lambda} A_{N,N+1} \\ \lambda^2 A_{N+1,0} & \lambda A_{N+1,1} & \dots & \lambda A_{N+1,N} & A_{N+1,N+1} \end{bmatrix} \in SU(N+1,1).$$

If s is a first-order adapted lift, we can define $\widetilde{s} = A_{F_{\lambda,0,0,Id}} \cdot s \cdot A^{-1}_{F_{\lambda,0,0,Id}}$. Recall the pulling back Maurer-Cartan form by s is $\omega = s^{-1}ds$. Since $\widetilde{\omega} = \widetilde{s}^{-1}d\widetilde{s} = (AsA^{-1})^{-1}d(AsA^{-1}) = A \cdot s^{-1}ds \cdot A^{-1} = A \cdot \omega \cdot A^{-1}$. As above, we have

$$\begin{bmatrix} \widetilde{\omega}_0^0 & \widetilde{\omega}_1^0 & \dots & \widetilde{\omega}_N^0 & \widetilde{\omega}_{N+1}^0 \\ \widetilde{\omega}_0^1 & \widetilde{\omega}_1^1 & \dots & \widetilde{\omega}_N^1 & \widetilde{\omega}_{N+1}^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \widetilde{\omega}_0^N & \widetilde{\omega}_1^N & \dots & \widetilde{\omega}_N^N & \widetilde{\omega}_{N+1}^N \\ \widetilde{\omega}_0^{N+1} & \widetilde{\omega}_1^{N+1} & \dots & \widetilde{\omega}_N^{N+1} & \widetilde{\omega}_{N+1}^{N+1} \end{bmatrix} = \begin{bmatrix} \omega_0^0 & \frac{1}{\lambda}\omega_1^0 & \dots & \frac{1}{\lambda}\omega_N^0 & \frac{1}{\lambda^2}\omega_{N+1}^0 \\ \lambda\omega_0^1 & \omega_1^1 & \dots & \omega_N^1 & \frac{1}{\lambda}\omega_{N+1}^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda\omega_0^N & \omega_1^N & \dots & \omega_N^N & \frac{1}{\lambda}\omega_{N+1}^N \\ \lambda^2\omega_0^{N+1} & \lambda\omega_1^{N+1} & \dots & \lambda\omega_N^{N+1} & \omega_{N+1}^{N+1} \end{bmatrix}.$$

4 Geometric Rank, II_M and II_M^{CR}

Lemma 4.1 *(i) ([JY10], theorem 7.1) Let $F \in CR_k(\partial\mathbb{H}^{n+1}, \partial\mathbb{H}^{N+1})$ with $k \geq 2$ and $F(0) = 0$. Then there exists a neighborhood of 0 in $M := F(\partial\mathbb{H}^{n+1})$ and a C^{k-1}-smooth first-order adapted lift $e : U \to SU(N+1,1)$*

$$e = (e_0, e_j, e_b, e_{N+1}) \in SU(N+1,1), \quad 1 \leq j \leq n, \; n+1 \leq b \leq N. \tag{46}$$

(ii) ([JY10], Step 3 of the proof of Theorem 1.1) Let $F = F^{***} = (f, \phi, g)$, the induced first-order adapted lift s, and notation be as in Lemma 2.1. Then

$$h^{\mu}_{j,k}|_0 = \frac{\partial^2 \phi_\mu}{\partial z_j \partial z_k}\Big|_0, \quad j,k \in \{1,2,...,n,N+1\} \tag{47}$$

where h^μ_{jk} are defined in (31) and in (33).

Theorem 4.2 *Let $F \in CR_2(\partial\mathbb{H}^{n+1}, \partial\mathbb{H}^{N+1})$. Then its geometric rank κ_0 equals to*

$$\kappa_0 = \sup_{p\in\partial\mathbb{H}^{n+1}} \left[n - \dim_{\mathbb{C}}\{\nu \mid II^{CR}_{M,F(p)}(\nu,\nu) = 0\} \right]$$

where $II^{CR}_{M,F(p)}$ is the CR second fundamental form of the submanifold M at the point $F(p)$. Here $\{\nu \mid II^{CR}_{M,F(p)}(\nu,\nu) = 0\}$ is a vector space over $\mathbb{C}$.

Let $M \subset \partial\mathbb{H}^{N+1}$ be a CR submanifold which is the image of a smooth CR hypersurface in $\mathbb{C}^{n+1}$ by a C^2-smooth CR map. Fixing one first-order adapted lift s, we write $II^{CR}_M = \sum_{\alpha,\beta,\mu} q^{\mu}_{\alpha\beta}\omega^{\alpha}_0\omega^{\beta}_0 \otimes \underline{e_\mu}$, $\mathrm{mod}(\omega^{N+1}_0)$. Consider the set of vectors in $\mathbb{C}^n$, which is a variety defined by a quadratic polynomial and is called the set of *asymptotic directions*, defined by

$$Baseloc|II^{CR}_{M,x}| := \{v = (v^\alpha) \in \mathbb{C}^n \mid \sum_{\alpha,\beta} q^{\mu}_{\alpha\beta}(x)v^\alpha v^\beta = 0,\ \forall n+1 \le \mu \le N\} \tag{48}$$

which is independent of the choice of the lift s.

Recall from [H99], lemma 5.3, that for any $p \in \partial\mathbb{H}^n$, the induced map $F = F^{**}$ satisfies

$$\langle \overline{z}, e^{(1)}(z)\rangle |z|^2 = |\phi^{(2)}(z)|^2, \quad \forall z \in \partial\mathbb{H}^n. \tag{49}$$

where $e^{(1)}(z) = -2i\sum_j \frac{\partial^2 f}{\partial z_j \partial w}|_0 z_j$.

Then by Lemma 4.1 (ii), any vector $v = (v_1, ..., v_n) \in Baseloc|II^{CR}_{M,F(0)}|$ if and only if $\sum_{i,j} \frac{\partial^2\phi_\mu}{\partial z_i\partial z_j}|_0 v_i v_j = 0,\ \forall\mu$. Then by (49), the statement is equivalent to $\langle\overline{v}, e^{(1)}(v)\rangle = 0$. Since the matrix $(-2i\frac{\partial^2 f}{\partial z_j\partial w}|_0)$ is semi-positive, the statement is equivalent to $e^{(1)}(v) = 0$, i.e.,

$$Baseloc|II^{CR}_{M,0}| = \left\{ v \ :\ -2i\sum_j \frac{\partial^2 f}{\partial z_j \partial w}\bigg|_0 v_j = 0 \right\}, \tag{50}$$

which is a vector space over $\mathbb{C}$, so that it makes sense to define its dimension. Recall $Rk_F(p) = rank(\mathcal{A}(p))$. By the formulas of f_j in Lemma 2.1, we have

$$Rk_F(0) = n - \dim_{\mathbb{C}} Baseloc|II^{CR}_{M,0}|. \tag{51}$$

Proof of Theorem 4.2: **Step 1. The lift s^{***}_p** It suffices to prove

$$Rk_F(p) = n - \dim_{\mathbb{C}} Baseloc|II^{CR}_{M,F(p)}|, \quad \forall p \in \partial\mathbb{H}^{n+1}. \tag{52}$$

The case when $p = 0$ has been proved in (51). Let us consider $p \in \partial\mathbb{H}^{n+1}$ with $P := F(p) \neq 0$.

By the definition,

$$Rk_F(p) = Rk_{F_p^{***}}(0). \tag{53}$$

Here we write $F_p^{***} = G_p \circ \tau_p^F \circ F \circ \sigma_p^0 \circ H_p$ where τ_p^F is as in (4), σ_p^0 is as in (3), $H_p \in Aut_0(\partial\mathbb{H}^{n+1})$ and $G_p \in Aut_0(\partial\mathbb{H}^{N+1})$. Since M is a real analytic hypersurface containing the point $P = F(p)$, $G_p \circ \tau_p^F(M)$ is a real analytic hypersurface containing $0 = \tau_0^F(P)$.

We consider

$$\begin{array}{ccc} (M,P) & \xrightarrow{G_p\circ\tau_p^F} & \big(G_p \circ \tau_p^F(M), 0\big) \\ \uparrow F & & \uparrow F_p^{***} \\ (\partial\mathbb{H}^{n+1}, p) & \xleftarrow{\sigma_p^0\circ H_p} & (\partial\mathbb{H}^{n+1}, 0) \end{array} \tag{54}$$

Now from $F_p^{***} : \partial\mathbb{H}^{n+1} \to G_p \circ \tau_0^F(M)$, we can construct a first-order adapted lift s_p^{***} of $G_p \circ \tau_0^F(M)$ as we constructed s from the map F in (46). Since $F \in Prop_k(\mathbb{H}^{n+1}, \mathbb{H}^{N+1})$, the lift s_p^{***} is C^{k-1} smooth. Write the CR second fundamental form of $G_p \circ \tau_0^F(M)$ with respect to the lift s_p^{***} as

$$II_{M,P}^{CR(s_p^{***})} = q_{ij}^{\mu(s_p^{***})} \omega_0^{i(s_p^{***})} \omega_0^{j(s_p^{***})} \otimes \underline{e}_{\mu}^{(s_p^{***})}. \tag{55}$$

Step 2. Construct the lift s_p

Now we may try to define a first-order adapted lift from M into $SU(N+1,1)$ by (36):

$$s_p = (\tau_p^F)^{-1} \circ {G_p}^{-1} \circ s_p^{***} \circ G_p \circ \tau_p^F. \tag{56}$$

Unfortunately, this lift s_p may not be a lift of M into $SU(N+1,1)$ (See the example in (43)). We have to modify the construction of (56) so that it is a first-order adapted lift of M into $SU(N+1,1)$ as follows.

Since $G_p \in Aut_0(\partial\mathbb{H}^{N+1})$, we can write it as in (44):

$$G_p = F_{\lambda,0,0,Id} \circ F_{1,0,0,U} \circ F_{1,r,\vec{a},Id}. \tag{57}$$

Here $F_{1,0,0,U}, F_{1,r,\vec{a},Id} \in SU(N+1,1)$, but $F_{\lambda,0,0,Id} \in SU(N+1,1)$ if and only if $\lambda = 1$.

Now we begin to modify the s_p in (56).

• **Lift from $F_{\lambda,0,0,Id} \circ F_{1,0,0,U} \circ F_{1,r,\vec{a},Id} \circ \tau_p^F(M)$** For any $P \in G_p \circ \tau_p^F(M)$, the map

$$P \mapsto s_p^{***}|_P \tag{58}$$

is a first-ordered adapted lift from $G \circ \tau_p^F(M)$ into $SU(N+1,1)$.

• **Lift from $F_{1,0,0,U} \circ F_{1,r,\vec{a},Id} \circ \tau_p^F(M)$** Then we consider ${F_{\lambda,0,0,Id}}^{-1} \circ s_p^{**} \circ F_{\lambda,0,0,Id}$: $\forall P \in F_{1,0,0,U} \circ F_{1,r,\vec{a},Id} \circ \tau_p^F(M)$, by a similar formula in (36) and a modification in (45), we define $\big({F_{\lambda,0,0,Id}}^{-1} \circ s_p^{***} \circ F_{\lambda,0,0,Id}\big) \cdot A_{F_{\lambda,0,0,Id}}$; more precisely, $\forall P \in F_{1,0,0,U} \circ F_{1,r,\vec{a},Id} \circ \tau_p^F(M)$,

$$P \mapsto \left({F_{\lambda,0,0,Id}}^{-1} \circ s_p^{***} \circ F_{\lambda,0,0,Id}\right)\Big|_P \cdot \left(A_{F_{\lambda,0,0,Id}}\right)\Big|_P, \tag{59}$$

which is a first-ordered adapted lift from $F_{1,0,0,U} \circ F_{1,r,\vec{a},Id} \circ \tau_p^F(M)$ into $SU(N+1,1)$.

• **Lift from** $F_{1,r,\vec{a},Id} \circ \tau_p^F(M)$ $\quad \forall P \in F_{1,r,\vec{a},Id} \circ \tau_p^F(M)$, by (36), the map

$$P \mapsto \left({F_{1,0,0,U}}^{-1} \circ {F_{\lambda,0,0,Id}}^{-1} \circ s_p^{***} \circ F_{\lambda,0,0,Id} \circ F_{1,0,0,U} \right)\Big|_P \cdot \left(A_{F_{\lambda,0,0,Id}} \right)\Big|_{F_{1,0,0,U}(P)} \tag{60}$$

is a first-ordered adapted lift from $F_{1,r,\vec{a},Id} \circ \tau_p^F(M)$ into $SU(N+1,1)$.

• **Lift from** $\tau_p^F(M)$ $\quad$ Similarly, $\forall P \in \tau_p^F(M)$, by (36), the map

$$P \mapsto \left({F_{1,r,\vec{a},Id}}^{-1} \circ {F_{1,0,0,U}}^{-1} \circ {F_{\lambda,0,0,Id}}^{-1} \circ s_p^{***} \circ F_{\lambda,0,0,Id} \circ F_{1,0,0,U} \circ F_{1,r,\vec{a},Id} \right)\Big|_P \cdot$$

$$\cdot \left(A_{F_{\lambda,0,0,Id}} \right)\Big|_{F_{1,0,0,U} \circ F_{1,r,\vec{a},Id}(P)}$$

is a first-ordered adapted lift from $\tau_p^F(M)$ into $SU(N+1,1)$. In other words,

$$P \mapsto \left({G_p}^{-1} \circ s_p^{***} \circ G_p \right)\Big|_P \cdot \left(A_{F_{\lambda,0,0,Id}} \right)\Big|_{F_{1,0,0,U} \circ F_{1,r,\vec{a},Id}(P)} \tag{61}$$

• **Lift from** M $\quad$ Finally, $\forall P \in M$, by (36), the map

$$P \mapsto \left((\tau_p^F)^{-1} \circ {G_p}^{-1} \circ s_p^{***} \circ G_p \circ \tau_p^F \right)\Big|_P \cdot \left(A_{F_{\lambda,0,0,Id}} \right)\Big|_{F_{1,0,0,U} \circ F_{1,r,\vec{a},Id} \circ \tau_p^F(P)} \tag{62}$$

is a first-ordered adapted lift s_p from M into $SU(N+1,1)$. Without cause confusion, we denote

$$s_p = \left((\tau_p^F)^{-1} \circ {G_p}^{-1} \circ s_p^{***} \circ G_p \circ \tau_p^F \right) \cdot A_{F_{\lambda,0,0,Id}}. \tag{63}$$

Here we recall from §7 that for any $P \in M$,

$$A_{F_{\lambda,0,0,Id}}(P) = \begin{bmatrix} 1 & 0 & ... & 0 & 0 \\ 0 & \lambda & ... & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda & 0 \\ 0 & 0 & \dots & 0 & \lambda^2 \end{bmatrix} (P) \tag{64}$$

where $\lambda = \lambda(P)$ is defined in (57). Since $F \in Prop_k(\partial\mathbb{H}^{n+1}, \partial\mathbb{H}^{N+1})$, by the construction, λ is a C^{k-1}-smooth positive function, and hence the lift s_p is C^{k-1}-smooth.

Step 3. Construct the lift s_p Write the CR second fundamental form of M with respect to the lift s_p as

$$II_{M,P}^{CR(s_p)} = q_{ij}^{\mu(s_p)}\omega_0^{i(s_p)}\omega_0^{j(s_p)} \otimes \underline{e}_\mu^{(s_p)}. \tag{65}$$

Then by (38), for $P = F(p)$ we have

$$q_{ij}^{\mu(s_p)}(P) = q_{ij}^{\mu(s_p^{***})}(0)(G_p \circ \tau_0^F)(0). \tag{66}$$

This implies from (54)

$$\dim_{\mathbb{C}} Baseloc|II_{M,P}^{CR}| = \dim_{\mathbb{C}} Baseloc|II_{G_p\circ\tau_0^F(M),0}^{CR}| = \dim_{\mathbb{C}} Baseloc|II_{F_p^{***}(M),0}^{CR}|. \tag{67}$$

By (53), (67) and (51), we prove (52). $\square$

5 A Lift with Special Property

Theorem 5.1 *Let $F = F^{***} \in Prop_k(\mathbb{H}^{n+1}, \mathbb{H}^{N+1})$ where $k \geq 2$ and $M = F(\partial\mathbb{H}^{n+1})$. For any point of M, there exists a neighborhood U of this point in M and a C^{k-1}-smooth first-order adapted lift s of U into $SU(N+1,1)$ where U is a neighborhood of 0 in M such that the coefficient functions q_{ij}^μ of II_M satisfy*

$$q_{ij}^\mu(P) = \lambda(P)\frac{\partial^2(\phi_p^{***})_\mu}{\partial z_i \partial z_j}\bigg|_0, \qquad i,j \in \{1,2,...,n,N+1\},\ n+1 \leq \mu \leq N, \tag{68}$$

$\forall p \in \partial\mathbb{H}^{n+1}$ with $P = F(p) \in U$, where λ is a positive C^{k-1} smooth function defined on U, and $F_p^{***} = (f_p^{***}, \phi_p^{***}, g_p^{***})$.

Proof of Theorem 5.1 **Step 1. Start with the lift s** Let $s : U \to SU(N+1,1)$ be the C^{k-1}-smooth first-order adapted lift of F defined in Theorem 5.1 where $U \subset M$ is a neighborhood of 0. Since $F(0) = 0$, we can choose small neighborhoods $\widetilde{U}$ of 0 in $\partial\mathbb{H}^{n+1}$ and U of 0 in M such that $F : \widetilde{U} \to U$ is diffeomorphic. Then for any $P \in U$, there is a unique $p \in \widetilde{U}$ with $F(p) = P$.

The second fundamental form with respect to s can be expressed as

$$II_{M,0}^{(s)}(P) = \sum_{j,k} q_{jk}^{(s)\mu}(P)\omega_0^{(s)j}\omega_0^{(s)k} \otimes \underline{e}_\mu^{(s)},$$

Here the coefficient functions $q_{jk}^{(s)\mu}$ satisfy the formulas in Lemma 4.1 above at $P = 0$. In order to prove Theorem 5.1, we need to modify the lift s to construct a new first-order adapted lift $\hat{s}$ of M into $SU(N+1,1)$:

$$\hat{s}(P) = s(P) \cdot \psi(P), \quad \forall P \in U, \tag{69}$$

where $\psi : U \to G_1$ is some C^{k-2}-smooth map where G_1 is defined in (23) such that the coefficients of the second fundamental form with respect to $\hat{s}$ satisfy the formulas in (68) at any $P \in U$.

Step 2. Construct the lift s_p For any point $P \in U$, by Step 2 of the proof of Theorem 4.2, there is a first-order adapted lift s_p defined on a neighborhood U_p of P in M into $SU(N+1,1)$. Then there exists a C^{k-1} smooth map $a_p : U_p \to G_1$ such that

$$s_p = s \cdot a_p \quad on\ U_p \tag{70}$$

In fact $a_p := s^{-1} \cdot s_p$.

Step 3. Construct the lift $\hat{s}$ Now we define C^{k-1}-smooth a first-order adapted lift $\hat{s}$ from a neighborhood U of 0 in M into $SU(N+1,1)$ given by

$$\hat{s}(p) = s(p) \cdot a_p(p), \qquad \forall p \in U \tag{71}$$

where a_p is defined in Step 2. Write the second fundamental form with respect to $\hat{s}$ as

$$II^{(\hat{s})}_{M,\hat{p}} = \sum_{j,k} q^{(\hat{s})\mu}_{jk} \omega^{(\hat{s})j}_0 \omega^{(\hat{s})k}_0 \otimes \underline{e}^{(\hat{s})}{}_{\mu},\ mod(\eta^{N+1}).$$

We claim:

$$q^{(\hat{s})\mu}_{jk}(p) = q^{(s_p)\mu}_{jk}(p), \quad \forall p \in M \tag{72}$$

so that the coefficients $q^{(\hat{s})\mu}_{jk}$ satisfy the formulas in Theorem 5.1. In fact, for any $p_0 \in M$, setting $s_1(q) := a_q(q)$, $\forall q \in M$ and $s_2 := a_{p_0}$. Since $s_1(p_0) = s_2(p_0)$, by (34), we prove Claim (72). □

Corollary 5.2 *Let M and F be as above. $II_M \equiv 0$ if and only if F is linear fractional.*

Proof: In fact, if $II_M \equiv 0$, then $II^{CR}_M \equiv 0$ by the definitions so that F is linear fractional by [JY10]. Conversely, if F is linear fractional, then $\frac{\partial^2 \phi^{***}}{\partial z_i \partial z_j}|_0 = 0$ for $F^{***} = (f^{***}, \phi^{***}, g^{***})$ where we use notation in Lemma 2.1 by standard calculation. Then $\frac{\partial^2 \phi^{***}_p}{\partial z_i \partial z_j}|_0 = 0$ for any F^{***}_p for any $p \in \partial\mathbb{H}^{n+1}$ where we use the notation in Lemma 2.1. We apply Theorem 5.1 to conclude that $q^{\mu}_{ij}(P) = 0$ for any $p \in \partial\mathbb{H}^{n+1}$ with $P = F(p)$, and hence $II_M \equiv 0$. □

Now let $F = F^{***} \in Prop_3(\mathbb{H}^{n+1}, \mathbb{H}^{N+1})$ with $\kappa_0 \le n-1$ and $3 \le n \le N-1$. By (35), for any $P = F(p)$ where $p \in \partial\mathbb{H}^{n+1}$,

$$
\begin{aligned}
&(II_M - II_M^{CR})(P) \\
&= \sum_{1\le j,k\le n, n+1\le\mu\le N} \big(q^{\mu}_{jN+1}\omega_0^j\omega_0^{N+1} + q^{\mu}_{N+1k}\omega_0^{N+1}\omega_0^k + q^{\mu}_{N+1N+1}\omega_0^{N+1}\omega_0^{N+1}\big)\otimes \underline{e}_\mu|_P \\
&= \sum_{1\le j,k\le n, n+1\le\mu\le N} \big(\frac{\partial^2(\phi_p^{***})_\mu}{\partial z_j\partial z_{N+1}}|_0\omega_0^j\omega_0^{N+1} + \frac{\partial^2(\phi_p^{***})_\mu}{\partial z_{N+1}\partial z_k}|_0\omega_0^{N+1}\omega_0^k \\
&\quad + \frac{\partial^2(\phi_p^{***})_\mu}{\partial z_{N+1}\partial z_{N+1}}|_0\omega_0^{N+1}\omega_0^{N+1}\big)\otimes \underline{e}_\mu \qquad (By\ Theorem\ 5.1) \\
&= \sum_{1\le j,k\le \kappa_0, n+1\le\mu\le N} \big(\frac{\partial^2(\phi_p^{***})_\mu}{\partial z_j\partial w}|_0\omega_0^j\omega_0^{N+1} + \frac{\partial^2(\phi_p^{***})_\mu}{\partial w\partial z_k}|_0\omega_0^{N+1}\omega_0^k\big)\otimes \underline{e}_\mu.
\end{aligned}
$$

Here the last equality holds because $\frac{\partial^2(\phi_p^{***})_\mu}{\partial z_j\partial w}|_0 = 0$ for $j \ge \kappa_0$ hold by Lemma 2.1(ii). Then $II_M - II_M^{CR} \equiv 0$ means

$$\frac{\partial^2(\phi_p^{***})_\mu}{\partial z_j\partial w}|_0 = 0, \quad \forall 1\le j\le n, \forall n+1\le\mu\le N,\ \forall p\in\partial\mathbb{H}^{n+1}. \tag{73}$$

6 Maps between balls with rank two

Let $F = F^{***} \in Prop_3(\mathbb{H}^{n+1}, \mathbb{H}^{N+1})$ with $rank(F) = Rk_F(0) = 2$ and $3 \le n$ and $3n \le N+1$. Then we can write $F = (f_1, f_2, f_p, \phi_{p'}, \phi_{n'}, \phi_{p''}, \phi_{(n-1)''}, \phi_b, g)$, where

$f_1 = z_1 + \frac{i\mu_1(0)}{2}z_1w + o_{wt}(3)$,
$f_2 = z_2 + \frac{i\mu_2(0)}{2}z_2w + o_{wt}(3)$,
$f_p = z_p,\ \ 3\le p\le n$,
$\phi_{1p} = \sqrt{\mu_1(0)}z_1z_p + \sum_{q\ge3}0z_qw + o_{wt}(2),\ 3\le p\le n$,
$\phi_{2p} = \sqrt{\mu_2(0)}z_2z_p + \sum_{q\ge3}0z_qw + o_{wt}(2), \quad 3\le p\le n$,
$\phi_{11} = \sqrt{\mu_1(0)}z_1z_1 + \sum_{q\ge3}0z_qw + o_{wt}(2)$,
$\phi_{12} = \sqrt{\mu_1(0)+\mu_2(0)}z_1z_2 + \sum_{q\ge3}0z_qw + o_{wt}(2)$,
$\phi_{22} = \sqrt{\mu_2(0)}z_2z_2 + \sum_{q\ge3}0z_qw + o_{wt}(2)$,
$\{\phi_{33},\ \phi_{34}, ..., \phi_{3,N-3n+3}\} = \{\phi_b\}$
Other $\phi_* = 0 + o_{wt}(2)$,
$g = w$.

In the rest of the paper, we set up the following index ranges:

$$1\le\alpha,\beta,\gamma\le n-2,\quad \alpha' = n+\alpha,\quad \alpha'' = 2n+\alpha,\ n+1\le\mu\le N. \tag{74}$$

When $n \geq 4$, we also denote $3n \leq a, b, c \leq N$. By replacing 1 and 2 with n and $n-1$, we write F as

$F = (f_\alpha, f_{n-1}, f_n, \phi_{\alpha'}, \phi_{\alpha''}, \phi_{n_{11}}, \phi_{n_{22}}, \phi_{n_{12}}, \phi_b, g)$, where

$f_\alpha = z_\alpha + 0 z_\alpha w + o_{wt}(3)$,

$f_{n-1} = z_{n-1} + \frac{i\mu_1(0)}{2} z_{n-1} w + o_{wt}(3)$,

$f_n = z_n + \frac{i\mu_2(0)}{2} z_n w + o_{wt}(3)$,

$\phi_{\alpha'} = \phi_{1\alpha} = \sqrt{\mu_1(0)} z_n z_\alpha + \sum_\sigma 0 z_\sigma w + o_{wt}(2)$,

$\phi_{\alpha''} = \phi_{2\alpha} = \sqrt{\mu_2(0)} z_{n-1} z_\alpha + \sum_\sigma 0 z_\sigma w + o_{wt}(2)$,

$\phi_{n_{11}} = \sqrt{\mu_1(0)} z_n z_n + \sum_\sigma 0 z_\sigma w + o_{wt}(2)$,

$\phi_{n_{22}} = \sqrt{\mu_2(0)} z_{n-1} z_{n-1} + \sum_\sigma 0 z_\sigma w + o_{wt}(2)$,

$\phi_{n_{12}} = \sqrt{\mu_1(0) + \mu_2(0)} z_{n-1} z_n + \sum_\sigma 0 z_\sigma w + o_{wt}(2)$,

$\phi_b = 0 + \sum_\sigma 0 z_\sigma w + o_{wt}(2)$.

Let F be as above. Let $M = F(\partial \mathbb{H}^{n+1})$. Then the following holds in a neighborhood of $0 = F(0)$ in M by Theorem 5.1 :

$h^{\alpha'}_{\beta\gamma} = 0$, $h^{\alpha'}_{\beta\ n} = \lambda \delta_{\alpha\beta} \sqrt{\mu_1}$, $h^{\alpha'}_{\beta\ n-1} = h^{\alpha'}_{n\ n} = h^{\alpha'}_{n-1,\ n-1} = h^{\alpha'}_{n,n-1} = h^{\alpha'}_{\beta,N+1} = h^{\alpha'}_{n-1,N+1} = h^{\alpha'}_{n,N+1} = h^{\alpha'}_{N+1,N+1} = 0$,

$h^{\alpha''}_{\beta\gamma} = h^{\alpha''}_{\beta\ n} = 0$, $h^{\alpha''}_{\beta\ n-1} = \lambda \delta_{\alpha\beta} \sqrt{\mu_2}$, $h^{\alpha''}_{n\ n} = h^{\alpha''}_{n-1,\ n-1} = h^{\alpha''}_{n,n-1} = h^{\alpha''}_{\beta,N+1} = h^{\alpha''}_{n-1,N+1} = h^{\alpha''}_{n,N+1} = h^{\alpha''}_{N+1,N+1} = 0$,

$h^{n_{11}}_{\beta\gamma} = h^{n_{11}}_{\beta\ n} = h^{n_{11}}_{\beta\ n-1} = 0$, $h^{n_{11}}_{n\ n} = 2\lambda\sqrt{\mu_1}$, $h^{n_{11}}_{n-1\ n-1} = h^{n_{11}}_{n\ n-1} = h^{n_{11}}_{\beta\ N+1} = h^{n_{11}}_{n-1\ N+1} = h^{n_{11}}_{n\ N+1} = h^{n_{11}}_{N+1\ N+1} = 0$,

$h^{n_{22}}_{\beta\gamma} = h^{n_{22}}_{\beta\ n} = h^{n_{22}}_{\beta\ n-1} = h^{n_{22}}_{n\ n} = 0$, $h^{n_{22}}_{n-1\ n-1} = 2\lambda\sqrt{\mu_2}$, $h^{n_{22}}_{n\ n-1} = h^{n_{22}}_{\beta\ N+1} = h^{n_{22}}_{n-1\ N+1} = h^{n_{22}}_{n\ N+1} = h^{n_{22}}_{N+1\ N+1} = 0$,

$h^{n_{12}}_{\beta\gamma} = h^{n_{12}}_{\beta\ n} = h^{n_{12}}_{\beta\ n-1} = h^{n_{12}}_{n\ n} = h^{n_{12}}_{n-1\ n-1} = 0$, $h^{n_{12}}_{n\ n-1} = \lambda\sqrt{\mu_1 + \mu_2}$, $h^{n_{12}}_{\beta\ N+1} = h^{n_{12}}_{n-1\ N+1} = h^{n_{12}}_{n\ N+1} = h^{n_{12}}_{N+1\ N+1} = 0$.

$h^b_{\beta\gamma} = h^b_{\beta\ n} = h^b_{\beta\ n-1} = h^b_{n\ n} = h^b_{n-1,\ n-1} = h^b_{n,n-1} = h^b_{\beta,N+1} = h^b_{n-1\ N+1} = h^b_{n\ N+1} = h^b_{N+1,\ N+1} = 0$,

where λ is a positive C^2-smooth function, and μ_1, μ_2 are C^1-smooth functions in the neighborhood of 0 in M.

Recall from (15), any first-order adapted lift $s = (e_0, e_j, e_\mu, e_{N+1}) : M \to SU(N+1, 1)$ of F where $1 \leq i, j \leq n, n+1 \leq \mu, \nu \leq N$, we have $ds = s\theta$ where θ is the pull-back of the Maurer-Cartan form from $SU(N+1, 1)$:

$$d(e_0, e_j, e_\mu, e_{N+1}) = (e_0, e_i, e_\nu, e_{N+1}) \begin{pmatrix} \theta_0^0 & \theta_j^0 & \theta_\mu^0 & \theta_{N+1}^0 \\ \theta_0^i & \theta_j^i & \theta_\mu^i & \theta_{N+1}^i \\ 0 & \theta_j^\nu & \theta_\mu^\nu & \theta_{N+1}^\nu \\ \theta_0^{N+1} & \theta_j^{N+1} & 0 & \theta_{N+1}^{N+1} \end{pmatrix}.$$

Recall $\theta^\mu_j = h^\mu_{ji}\eta^i + h^\mu_{j\ N+1}\eta$ and $\theta^\mu_{N+1} = h^\mu_{N+1\ i}\eta^i + h^\mu_{N+1\ N+1}\eta$. We still use notation in (74) and we write F as $F = (f_\alpha, f_{n-1}, f_n, \phi_{\alpha'}, \phi_{\alpha''}, \phi_{n_{11}}, \phi_{n_{22}}, \phi_{n_{12}}, \phi_b, g)$.

For simplicity, we replace $\lambda\sqrt{\mu_1}$ by $\sqrt{\mu_1}$; replace $\lambda\sqrt{\mu_2}$ by $\sqrt{\mu_2}$; and replace $\lambda\sqrt{\mu_1+\mu_2}$ by $\sqrt{\mu_1+\mu_2}$, by changing notation. Then by the formulas above, we have

$$
\begin{aligned}
&\theta^{\alpha'}_\beta = h^{\alpha'}_{\beta\ \gamma}\eta^\gamma + h^{\alpha'}_{\beta\ n-1}\eta^{n-1} + h^{\alpha'}_{\beta\ n}\eta^n + h^{\alpha'}_{\beta\ N+1}\eta = \delta_{\alpha\beta}\sqrt{\mu_1}\eta^n,\\
&\theta^{\alpha'}_{n-1} = h^{\alpha'}_{n-1\ \gamma}\eta^\gamma + h^{\alpha'}_{n-1\ n-1}\eta^{n-1} + h^{\alpha'}_{n-1\ n}\eta^n + h^{\alpha'}_{n-1\ N+1}\eta = 0,\\
&\theta^{\alpha'}_{n} = h^{\alpha'}_{n\ \gamma}\eta^\gamma + h^{\alpha'}_{n\ n-1}\eta^{n-1} + h^{\alpha'}_{n\ n}\eta^n + h^{\alpha'}_{n\ N+1}\eta = \sqrt{\mu_1}\eta^\alpha,\\
&\theta^{\alpha'}_{N+1} = h^{\alpha'}_{N+1\ \gamma}\eta^\gamma + h^{\alpha'}_{N+1\ n-1}\eta^{n-1} + h^{\alpha'}_{N+1\ n}\eta^n + h^{\alpha'}_{N+1\ N+1}\eta = 0,\\
&\theta^{\alpha''}_\beta = h^{\alpha''}_{\beta\ \gamma}\eta^\gamma + h^{\alpha''}_{\beta\ n-1}\eta^{n-1} + h^{\alpha''}_{\beta\ n}\eta^n + h^{\alpha''}_{\beta\ N+1}\eta = \delta_{\alpha\beta}\sqrt{\mu_2}\eta^{n-1},\\
&\theta^{\alpha''}_{n-1} = h^{\alpha''}_{n-1\ \gamma}\eta^\gamma + h^{\alpha''}_{n-1\ n-1}\eta^{n-1} + h^{\alpha''}_{n-1\ n}\eta^n + h^{\alpha''}_{n-1\ N+1}\eta = \sqrt{\mu_2}\eta^\alpha,\\
&\theta^{\alpha''}_{n} = h^{\alpha''}_{n\ \gamma}\eta^\gamma + h^{\alpha''}_{n\ n-1}\eta^{n-1} + h^{\alpha''}_{n\ n}\eta^n + h^{\alpha''}_{n\ N+1}\eta = 0,\\
&\theta^{\alpha''}_{N+1} = h^{\alpha''}_{N+1\ \gamma}\eta^\gamma + h^{\alpha''}_{N+1\ n-1}\eta^{n-1} + h^{\alpha''}_{N+1\ n}\eta^n + h^{\alpha''}_{N+1\ N+1}\eta = 0,\\
&\theta^{n_{11}}_\beta = h^{n_{11}}_{\beta\ \gamma}\eta^\gamma + h^{n_{11}}_{\beta\ n-1}\eta^{n-1} + h^{n_{11}}_{\beta\ n}\eta^n + h^{n_{11}}_{\beta\ N+1}\eta = 0,\\
&\theta^{n_{11}}_{n-1} = h^{n_{11}}_{n-1\ \gamma}\eta^\gamma + h^{n_{11}}_{n-1\ n-1}\eta^{n-1} + h^{n_{11}}_{n-1\ n}\eta^n + h^{n_{11}}_{n-1\ N+1}\eta = 0,\\
&\theta^{n_{11}}_{n} = h^{n_{11}}_{n\ \gamma}\eta^\gamma + h^{n_{11}}_{n\ n-1}\eta^{n-1} + h^{n_{11}}_{n\ n}\eta^n + h^{n_{11}}_{n\ N+1}\eta = 2\sqrt{\mu_1}\eta^n,\\
&\theta^{n_{11}}_{N+1} = h^{n_{11}}_{N+1\ \gamma}\eta^\gamma + h^{n_{11}}_{N+1\ n-1}\eta^{n-1} + h^{n_{11}}_{N+1\ n}\eta^n + h^{n_{11}}_{N+1\ N+1}\eta = 0,\\
&\theta^{n_{22}}_\beta = h^{n_{22}}_{\beta\ \gamma}\eta^\gamma + h^{n_{22}}_{\beta\ n-1}\eta^{n-1} + h^{n_{22}}_{\beta\ n}\eta^n + h^{n_{22}}_{\beta\ N+1}\eta = 0,\\
&\theta^{n_{22}}_{n-1} = h^{n_{22}}_{n-1\ \gamma}\eta^\gamma + h^{n_{22}}_{n-1\ n-1}\eta^{n-1} + h^{n_{22}}_{n-1\ n}\eta^n + h^{n_{22}}_{n-1\ N+1}\eta = 2\sqrt{\mu_2}\eta^{n-1},\\
&\theta^{n_{22}}_{n} = h^{n_{22}}_{n\ \gamma}\eta^\gamma + h^{n_{22}}_{n\ n-1}\eta^{n-1} + h^{n_{22}}_{n\ n}\eta^n + h^{n_2}_{n\ N+1}\eta = 0,\\
&\theta^{n_{22}}_{N+1} = h^{n_{22}}_{N+1\ \gamma}\eta^\gamma + h^{n_{22}}_{N+1\ n-1}\eta^{n-1} + h^{n_{22}}_{N+1\ n}\eta^n + h^{n_{22}}_{N+1\ N+1}\eta = 0,\\
&\theta^{n_{12}}_\beta = h^{n_{12}}_{\beta\ \gamma}\eta^\gamma + h^{n_{12}}_{\beta\ n-1}\eta^{n-1} + h^{n_{12}}_{\beta\ n}\eta^n + h^{n_{12}}_{\beta\ N+1}\eta = 0,\\
&\theta^{n_{12}}_{n-1} = h^{n_{12}}_{n-1\ \gamma}\eta^\gamma + h^{n_{12}}_{n-1\ n-1}\eta^{n-1} + h^{n_{12}}_{n-1\ n}\eta^n + h^{n_{12}}_{n-1\ N+1}\eta = \sqrt{\mu_1+\mu_2}\eta^n,\\
&\theta^{n_{12}}_{n} = h^{n_{12}}_{n\ \gamma}\eta^\gamma + h^{n_{12}}_{n\ n-1}\eta^{n-1} + h^{n_{12}}_{n\ n}\eta^n + h^{n_{12}}_{n\ N+1}\eta = \sqrt{\mu_1+\mu_2}\eta^{n-1},\\
&\theta^{n_{12}}_{N+1} = h^{n_{12}}_{N+1\ \gamma}\eta^\gamma + h^{n_{12}}_{N+1\ n-1}\eta^{n-1} + h^{n_{12}}_{N+1\ n}\eta^n + h^{n_{12}}_{N+1\ N+1}\eta = 0,\\
&\theta^{b}_\beta = h^{b}_{\beta\ \gamma}\eta^\gamma + h^{b}_{\beta\ n-1}\eta^{n-1} + h^{b}_{\beta\ n}\eta^n + h^{b}_{\beta\ N+1}\eta = 0,\\
&\theta^{b}_{n-1} = h^{b}_{n-1\ \gamma}\eta^\gamma + h^{b}_{n-1\ n-1}\eta^{n-1} + h^{b}_{n-1\ n}\eta^n + h^{b}_{n-1\ N+1}\eta = 0,\\
&\theta^{b}_{n} = h^{b}_{n\ \gamma}\eta^\gamma + h^{b}_{n\ n-1}\eta^{n-1} + h^{b}_{n\ n}\eta^n + h^{b}_{n\ N+1}\eta = 0,\\
&\theta^{b}_{N+1} = h^{b}_{N+1\ \gamma}\eta^\gamma + h^{b}_{N+1\ n-1}\eta^{n-1} + h^{b}_{N+1\ n}\eta^n + h^{b}_{N+1\ N+1}\eta = 0,
\end{aligned}
$$

where μ_1 and μ_2 are C^1-smooth positive functions defined on M.

7 Lemma for mappings of rank 2

Let $F \in CR_2(\partial\mathbb{H}^{n+1}, \partial\mathbb{H}^{N+1})$ with geometric rank $\kappa_0 = 2$. Then by the inequality $N \geq n + \frac{(2n+1-\kappa_0)\kappa_0}{2}$ (cf. Lemma 2.1 (i)), $N \geq n + \frac{(2n+1-\kappa_0)\kappa_0}{2} = 3n-1$, i.e., $N+1 \geq 3n$. In the remaining of the paper, Einstein summation notation is used without mentioning it.

Lemma 7.1 *Let $F \in Prop_3(\partial\mathbb{H}^{n+1}, \partial\mathbb{H}^{N+1})$ with the expression in above section and with $4 \le n+1 \le N+1 \le 4n-3$ and $\kappa_0 = 2$. If $N+1 > 3n$. Then $\theta^{\gamma'}_{n_{12}} = \theta^{\gamma''}_{n_{12}} = \theta^n_\beta = \theta^{\gamma'}_{n_{11}} = \theta^{\gamma'}_{n_{22}} = \theta^{\gamma'}_b = 0$. If $N+1 = 3n$ and $4 \le n$, then $\theta^{\gamma'}_{n_{12}} = \theta^{\gamma''}_{n_{12}} = \theta^n_\beta = \theta^{\gamma'}_{n_{11}} = \theta^{\gamma'}_{n_{22}} = 0$.*

Proof of Lemma: It suffices to prove the case $N+1 > 3n$ for the proof of the case $N+1 = 3$ is similar. We use the notation in the section 6. The facts that $\theta^{\gamma'}_{n_{12}} = \theta^{\gamma''}_{n_{12}} = \theta^n_\beta = \theta^{\gamma'}_{n_{11}} = \theta^{\gamma'}_{n_{22}} = \theta^{\gamma'}_b = 0$ will be proved in Step 2(C), 2(D), 2(A'), 4, 2(C) and 9 below, respectively.

Step 1(A) Differentiating $\theta^{n_{11}}_\beta = 0$, we get $d\theta^{n_{11}}_\beta = 0$. By $d\omega = -\omega \wedge \omega$, we have $-\theta^{n_{11}}_0 \wedge \theta^0_\beta - \theta^{n_{11}}_\alpha \wedge \theta^\alpha_\beta - \theta^{n_{11}}_{n-1} \wedge \theta^{n-1}_\beta - \theta^{n_{11}}_n \wedge \theta^n_\beta - \theta^{n_{11}}_{\alpha'} \wedge \theta^{\alpha'}_\beta - \theta^{n_{11}}_{\alpha''} \wedge \theta^{\alpha''}_\beta - \theta^{n_{11}}_{n_{11}} \wedge \theta^{n_{11}}_\beta - \theta^{n_{11}}_{n_{22}} \wedge \theta^{n_{22}}_\beta - \theta^{n_{11}}_{n_{12}} \wedge \theta^{n_{12}}_\beta - \theta^{n_{11}}_b \wedge \theta^b_\beta - \theta^{n_{11}}_{N+1} \wedge \theta^{N+1}_\beta = 0$, i.e., by §6, $2\sqrt{\mu_1}\theta^n_\beta \wedge \eta^n + \sqrt{\mu_1}\eta^n \wedge \theta^{n_{11}}_{\beta'} + \sqrt{\mu_2}\eta^{n-1} \wedge \theta^{n_{11}}_{\beta''} = 0$, i.e., $\eta^n \wedge \sqrt{\mu_1}(\theta^{n_{11}}_{\beta'} - 2\theta^n_\beta) + \eta^{n-1} \wedge \sqrt{\mu_2}\theta^{n_{11}}_{\beta''} = 0$. By Cartan's lemma, there are some coefficients $A^{(1)}_\beta$, $B^{(1)}_\beta$ and $D^{(1)}_\beta$ such that

$$\begin{pmatrix} \sqrt{\mu_1}(\theta^{n_{11}}_{\beta'} - 2\theta^n_\beta) \\ \sqrt{\mu_2}\theta^{n_{11}}_{\beta''} \end{pmatrix} = \begin{pmatrix} A^{(1)}_\beta & B^{(1)}_\beta \\ B^{(1)}_\beta & D^{(1)}_\beta \end{pmatrix} \begin{pmatrix} \eta^n \\ \eta^{n-1} \end{pmatrix}.$$

Step 1(A′) Differentiating $\theta^{n_{22}}_\alpha = 0$, we get $d\theta^{n_{22}}_\alpha = 0$. Similarly as in Step 1(A), we get

$$\begin{pmatrix} \sqrt{\mu_2}(\theta^{n_{22}}_{\alpha''} - 2\theta^{n-1}_\alpha) \\ \sqrt{\mu_1}\theta^{n_{22}}_{\alpha'} \end{pmatrix} = \begin{pmatrix} A^{(111)}_\alpha & B^{(111)}_\alpha \\ B^{(111)}_\alpha & D^{(111)}_\alpha \end{pmatrix} \begin{pmatrix} \eta^{n-1} \\ \eta^n \end{pmatrix}$$

for some coefficients $A^{(111)}_\alpha$, $B^{(111)}_\alpha$ and $D^{(111)}_\alpha$.

Step 1(B) Differentiating $\theta^b_\beta = 0$, we get $d\theta^b_\beta = 0$. As the calculation in Step 1(A) and §6, this implies with $\sqrt{\mu_1}\eta^n \wedge \theta^b_{\beta'} + \sqrt{\mu_2}\eta^{n-1} \wedge \theta^b_{\beta''} = 0$. By Cartan's lemma, there are some coefficients $C^{(2)b}_\beta, B^{(2)b}_\beta$, and $D^{(2)b}_\beta$ so that

$$\begin{pmatrix} \sqrt{\mu_1}\theta^b_{\beta'} \\ \sqrt{\mu_2}\theta^b_{\beta''} \end{pmatrix} = \begin{pmatrix} 2C^{(2)b}_\beta & B^{(2)b}_\beta \\ B^{(2)b}_\beta & D^{(2)b}_\beta \end{pmatrix} \begin{pmatrix} \eta^n \\ \eta^{n-1} \end{pmatrix}.$$

Step 2(A) Differentiating $\theta^{\alpha'}_\beta = 0$ with $\alpha \ne \beta$, we get $d\theta^{\alpha'}_\beta = 0$. By §6, this implies $\theta^\alpha_\beta \wedge \sqrt{\mu_1}\eta^n + \theta^n_\beta \wedge \sqrt{\mu_1}\eta^\alpha + \sqrt{\mu_1}\eta^n \wedge \theta^{\alpha'}_{\beta'} + \sqrt{\mu_2}\eta^{n-1} \wedge \theta^{\alpha'}_{\beta''} = 0$, i.e., $\sqrt{\mu_1}(\theta^\alpha_\beta - \theta^{\alpha'}_{\beta'}) \wedge \eta^n - \sqrt{\mu_2}\theta^{\alpha'}_{\beta''} \wedge \eta^{n-1} + \sqrt{\mu_1}\theta^n_\beta \wedge \eta^\alpha = 0$. By Cartan's lemma

$$\begin{pmatrix} \sqrt{\mu_1}(\theta^{\alpha'}_{\beta'} - \theta^\alpha_\beta) \\ \sqrt{\mu_2}\theta^{\alpha'}_{\beta''} \\ -\sqrt{\mu_1}\theta^n_\beta \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & F^{(3)\alpha}_\beta & G^{(3)}_\beta \\ 0 & G^{(3)}_\beta & 0 \end{pmatrix} \begin{pmatrix} \eta^n \\ \eta^{n-1} \\ \eta^\alpha \end{pmatrix}, \quad \alpha \ne \beta,$$

for some coefficients $F_\beta^{(3)\alpha}$ and $G_\beta^{(3)}$. Here we use the facts that θ_β^n is independent of α, that $\theta_{\beta'}^{\alpha'} - \theta_\beta^\alpha = -\overline{\theta_{\alpha'}^{\beta'}} + \overline{\theta_\alpha^\beta}$ by (16) and that the matrix is symmetric. So $\theta_{\beta'}^{\alpha'} = \theta_\beta^\alpha$, $\forall\alpha\neq\beta$.
Step 2(A′) Consider $\theta_\alpha^{\beta''}=0$, $\alpha\neq\beta$, and $d\theta_\alpha^{\beta''}=0$. Similarly as in Step 2(A), we get

$$\begin{pmatrix}\sqrt{\mu_2}(\theta_{\alpha''}^{\beta''}-\theta_\alpha^\beta)\\ \sqrt{\mu_1}\theta_{\alpha'}^{\beta''}\\ -\sqrt{\mu_2}\theta_\alpha^{n-1}\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 0 & F_\alpha^{(333)\beta} & G_\alpha^{(333)}\\ 0 & G_\alpha^{(333)} & 0\end{pmatrix}\begin{pmatrix}\eta^{n-1}\\ \eta^n\\ \eta^\beta\end{pmatrix}.$$

for some coefficients $F_\alpha^{(333)\beta}$ and $G_\alpha^{(333)}$. Then $\theta_{\alpha''}^{\beta''}=\theta_\alpha^\beta$, for any $\alpha\neq\beta$. By comparing both formulas for $\theta_{\beta''}^{\alpha'}=-\overline{\theta_{\alpha'}^{\beta''}}$ above and in Step 2(A), we get $F_\beta^{(3)\alpha}=G_\alpha^{(3)}=F_\beta^{(333)\alpha}=G_\beta^{(333)}=0$, $\forall\alpha\neq\beta$. Then $\theta_\beta^{n-1}=\theta_\beta^n=0$. Hence $\theta_{\beta''}^{\alpha'}=0$, $\forall\alpha\neq\beta$.
Step 2(B) Differentiating $\theta_\beta^{n_{12}}=0$, we get $d\theta_\beta^{n_{12}}=0$. Similarly as in Step 1(A), we get

$$\begin{pmatrix}\sqrt{\mu_1}\theta_{\beta'}^{n_{12}}\\ \sqrt{\mu_2}\theta_{\beta''}^{n_{12}}\end{pmatrix}=\begin{pmatrix}A_\beta^{(4)} & B_\beta^{(4)}\\ B_\beta^{(4)} & E_\beta^{(4)}\end{pmatrix}\begin{pmatrix}\eta^n\\ \eta^{n-1}\end{pmatrix}$$

for some coefficients $A_\beta^{(4)}, B_\beta^{(4)}$ and $E_\beta^{(4)}$.
Step 2(C) Differentiating $\theta_{n-1}^{\alpha'}=0$, we get $d\theta_{n-1}^{\alpha'}=0$. By §6 and $\theta_\beta^{n-1}=0$, this implies $\theta_{n-1}^n\wedge\sqrt{\mu_1}\eta^\alpha+\sqrt{\mu_2}\eta^\gamma\wedge\theta_{\gamma''}^{\alpha'}+2\sqrt{\mu_2}\eta^{n-1}\wedge\theta_{n_{22}}^{\alpha'}+\sqrt{\mu_1+\mu_2}\eta^n\wedge\theta_{n_{12}}^{\alpha'}=0$. Recall $\sqrt{\mu_2}\theta_{\gamma''}^{\alpha'}=F_\gamma^{(3)\alpha}\eta^{n-1}+G_\gamma^{(3)}\eta^\alpha=0$ for $\alpha\neq\gamma$. Then $\theta_{n-1}^n\wedge\sqrt{\mu_1}\eta^\alpha+\sqrt{\mu_2}\eta^\alpha\wedge\theta_{\alpha''}^{\alpha'}+2\sqrt{\mu_2}\eta^{n-1}\wedge\theta_{n_{22}}^{\alpha'}+\sqrt{\mu_1+\mu_2}\eta^n\wedge\theta_{n_{12}}^{\alpha'}=0$. In other words, $\eta^n\wedge\sqrt{\mu_1+\mu_2}\theta_{n_{12}}^{\alpha'}+\eta^{n-1}\wedge2\sqrt{\mu_2}\theta_{n_{22}}^{\alpha'}+\eta^\alpha\wedge(-\sqrt{\mu_1}\theta_{n-1}^n+\sqrt{\mu_2}\theta_{\alpha''}^{\alpha'})=0$. By Cartan's lemma, there are coefficients $A^{(5)\alpha}$ etc. so that

$$\begin{pmatrix}\sqrt{\mu_1+\mu_2}\theta_{n_{12}}^{\alpha'}\\ 2\sqrt{\mu_2}\theta_{n_{22}}^{\alpha'}\\ -\sqrt{\mu_1}\theta_{n-1}^n+\sqrt{\mu_2}\theta_{\alpha''}^{\alpha'}\end{pmatrix}=\begin{pmatrix}A^{(5)\alpha} & B^{(5)\alpha} & C^{(5)\alpha}\\ B^{(5)\alpha} & D^{(5)\alpha} & E^{(5)\alpha}\\ C^{(5)\alpha} & E^{(5)\alpha} & F^{(5)}\end{pmatrix}\begin{pmatrix}\eta^n\\ \eta^{n-1}\\ \eta^\alpha\end{pmatrix}.$$

Recall Step 1(A'), $\theta_{\alpha'}^{n_{22}}=\frac{1}{\sqrt{\mu_1}}(B_\alpha^{(111)}\eta^{n-1}+D_\alpha^{(111)}\eta^n)$. Then $\theta_{n_{22}}^{\alpha'}=-\frac{1}{\sqrt{\mu_1}}(\overline{B_\alpha^{(111)}}\overline{\eta^{n-1}}+\overline{D_\alpha^{(111)}}\overline{\eta^n})$ so that, by comparing above, $D^{(5)\alpha}=E^{(5)\alpha}=B_\alpha^{(111)}=D_\alpha^{(111)}=0$. Hence $\theta_{\alpha'}^{n_{22}}=0$.

Recall Step 2(B), $\sqrt{\mu_1}\theta_{n_{12}}^{\alpha'}=-\overline{A_\alpha^{(4)}}\overline{\eta^n}-\overline{B_\alpha^{(4)}}\overline{\eta^{n-1}}$, From above we have $\sqrt{\mu_1+\mu_2}\theta_{n_{12}}^{\alpha'}=A^{(5)\alpha}\eta^n+B^{(5)\alpha}\eta^{n-1}+C^{(5)\alpha}\eta^\alpha$. Then $A_\alpha^{(4)}=B_\alpha^{(4)}=A^{(5)\alpha}=B^{(5)\alpha}=C^{(5)\alpha}=0$ and $\theta_{n_{12}}^{\alpha'}=0$.
Step 2(D) Differentiating $\theta_n^{\beta''}=0$, we get $d\theta_n^{\beta''}=0$. Similarly as in Step Step 2(C), we get

$$\begin{pmatrix}\sqrt{\mu_1+\mu_2}\theta_{n_{12}}^{\beta''}\\ 2\sqrt{\mu_1}\theta_{n_{11}}^{\beta''}\\ \sqrt{\mu_1}\theta_{\beta'}^{\beta''}-\sqrt{\mu_2}\theta_n^{n-1}\end{pmatrix}=\begin{pmatrix}A^{(555)\beta} & B^{(555)\beta} & C^{(555)}\\ B^{(555)\beta} & D^{(555)\beta} & E^{(555)}\\ C^{(555)} & E^{(555)} & F^{(555)}\end{pmatrix}\begin{pmatrix}\eta^{n-1}\\ \eta^n\\ \eta^\beta\end{pmatrix}$$

for some coefficients $A^{(555)\beta}, B^{(555)\beta}, C^{(555)}, D^{(555)\beta}, E^{(555)}$ and $F^{(555)}$. By the formula for $\theta^{n_{11}}_{\beta''}$ in Step 1(A), it implies $B^{(1)}_\beta = B^{(555)\beta} = D^{(555)\beta} = E^{(555)} = 0$, and $\theta^{n_{11}}_{\beta''} = 0$. By the formula for $\theta^{n_{12}}_{\beta''}$ in Step 2(B), it implies $E^{(4)}_\beta = A^{(555)\beta} = C^{(555)\beta} = 0$, and $\theta^{n_{12}}_{\beta''} = 0$.

Step 3(A) Differentiating $\theta^{\alpha'}_\alpha = \sqrt{\mu_1}\eta^n$, we get $d\theta^{\alpha'}_\alpha = d(\sqrt{\mu_1}) \wedge \eta^n + \sqrt{\mu_1}d\eta^n$. By §6 and $\theta^\beta_n = 0$, this implies $\theta^\alpha_\alpha \wedge \sqrt{\mu_1}\eta^n + \sqrt{\mu_1}\eta^n \wedge \theta^{\alpha'}_{\alpha'} + \sqrt{\mu_2}\eta^{n-1} \wedge \theta^{\alpha'}_{\alpha''} = d(\sqrt{\mu_1}) \wedge \eta^n + \sqrt{\mu_1}(\theta^0_0 \wedge \eta^n + \eta^\gamma \wedge \theta^n_\gamma + \eta^{n-1} \wedge \theta^n_{n-1} + \eta^n \wedge \theta^n_n)$, mod$(\eta)$. By writing $\Delta_\alpha := \theta^{\alpha'}_{\alpha'} - \theta^\alpha_\alpha + \theta^0_0 - \theta^n_n$, we have $\eta^n \wedge (\sqrt{\mu_1}\Delta_\alpha + d(\sqrt{\mu_1})) + \eta^{n-1} \wedge (\sqrt{\mu_2}\theta^{\alpha'}_{\alpha''} - \sqrt{\mu_1}\theta^n_{n-1}) = 0$, mod$(\eta)$. By Cartan's lemma,

$$\sqrt{\mu_2}\theta^{\alpha'}_{\alpha''} - \sqrt{\mu_1}\theta^n_{n-1} = B^{(6)\alpha}\eta^{n-1} + C^{(6)\alpha}\eta^n, \ mod(\eta),$$
$$\sqrt{\mu_1}\Delta_\alpha = -d(\sqrt{\mu_1}) + C^{(6)\alpha}\eta^{n-1} + A^{(6)\alpha}\eta^n, \ \ mod(\eta).$$

Recall from Step 2(C) that $\sqrt{\mu_2}\theta^{\alpha'}_{\alpha''} - \sqrt{\mu_1}\theta^n_{n-1} = F^{(5)}\eta^\alpha$. Then $F^{(5)} = B^{(6)\alpha} = C^{(6)\alpha} = 0$. Hence $\sqrt{\mu_1}\theta^{\alpha'}_{\alpha''} = \sqrt{\mu_2}\theta^{n-1}_n$.

Step 3(A′) Differentiating $\theta^{\alpha''}_\alpha = \sqrt{\mu_2}\eta^{n-1}$, we get $d\theta^{\alpha''}_\alpha = d(\sqrt{\mu_2}) \wedge \eta^{n-1} + \sqrt{\mu_2}d\eta^{n-1}$. Similarly as in Step 3(A), there are some coefficeints $A^{(666)\alpha}, B^{(666)\alpha}$ and $E^{(666)\alpha}$ such that

$$\sqrt{\mu_2}(\theta^{\alpha''}_{\alpha''} - \theta^\alpha_\alpha + \theta^0_0 - \theta^{n-1}_{n-1}) = -d(\sqrt{\mu_2}) + A^{(666)\alpha}\eta^{n-1} + B^{(666)\alpha}\eta^n, \ mod(\eta)$$
$$\sqrt{\mu_1}\theta^{\alpha''}_{\alpha'} - \sqrt{\mu_2}\theta^{n-1}_n = B^{(666)\alpha}\eta^{n-1} + E^{(666)\alpha}\eta^n, \ \ mod(\eta).$$

Recall Step 2(D), $\sqrt{\mu_1}\theta^{\beta''}_{\beta'} - \sqrt{\mu_2}\theta^{n-1}_n = F^{(555)}\eta^\beta$. Then, from above, we obtain $F^{(555)} = B^{(666)\alpha} = E^{(666)\alpha} = 0$. Hence $\sqrt{\mu_1}\theta^{\beta''}_{\beta'} = \sqrt{\mu_2}\theta^{n-1}_n$. Recall from Step 3(A) that $\sqrt{\mu_2}\theta^{\alpha''}_{\alpha'} = \sqrt{\mu_1}\theta^{n-1}_n$. It implies either $\mu_1 = \mu_2$, $\theta^{\beta''}_{\beta'} = \theta^{n-1}_n$, or $\theta^{\beta''}_{\beta'} = \theta^{n-1}_n = 0$.

Step 3(B) Differentiating $\theta^{n_{12}}_{n-1} = \sqrt{\mu_1 + \mu_2}\eta^n$, we get $d\theta^{n_{12}}_{n-1} = d(\sqrt{\mu_1 + \mu_2}) \wedge \eta^n + \sqrt{\mu_1 + \mu_2} d\eta^n$. By §6, $\theta^n_\gamma = 0$ and $\theta^{n_{12}}_{\gamma''} = 0$ in Step 2(D), this implies $\theta^{n-1}_{n-1} \wedge \sqrt{\mu_1 + \mu_2}\eta^n + \theta^n_{n-1} \wedge \sqrt{\mu_1 + \mu_2}\eta^{n-1} + 2\sqrt{\mu_2}\eta^{n-1} \wedge \theta^{n_{12}}_{n_{22}} + \sqrt{\mu_1 + \mu_2}\eta^n \wedge \theta^{n_{12}}_{n_{12}} = d(\sqrt{\mu_1 + \mu_2}) \wedge \eta^n + \sqrt{\mu_1 + \mu_2}(\theta^0_0 \wedge \eta^n + \eta^{n-1} \wedge \theta^n_{n-1} + \eta^n \wedge \theta^n_n)$, mod$(\eta)$. Denote $\Delta_{n-1} := \theta^{n_{12}}_{n_{12}} - \theta^{n-1}_{n-1} + \theta^0_0 - \theta^n_n$. Then $\eta^n \wedge (\sqrt{\mu_1 + \mu_2}\Delta_{n-1} + d(\sqrt{\mu_1 + \mu_2})) + \eta^{n-1} \wedge (2\sqrt{\mu_2}\theta^{n_{12}}_{n_{22}} - 2\sqrt{\mu_1 + \mu_2}\theta^n_{n-1}) = 0$, mod$(\eta)$. By Cartan's lemma,

$$\sqrt{\mu_1 + \mu_2}\Delta_{n-1} = -d(\sqrt{\mu_1 + \mu_2}) + A^{(7)}\eta^n + B^{(7)}\eta^{n-1}, \ \ mod(\eta),$$

$$2\sqrt{\mu_2}\theta^{n_{12}}_{n_{22}} - 2\sqrt{\mu_1 + \mu_2}\theta^n_{n-1} = B^{(7)}\eta^n + C^{(7)}\eta^{n-1}, \ \ mod(\eta).$$

Step 4. Differentiating $\theta^{\alpha'}_n = \sqrt{\mu_1}\eta^\alpha$, $d\theta^{\alpha'}_n = d(\sqrt{\mu_1}) \wedge \eta^\alpha + \sqrt{\mu_1}d\eta^\alpha$. By §6 θ^{n-1}_α and $\theta^n_\alpha = 0$ and $\theta^{n_{12}}_{\alpha'} = 0$ in Step 2(C), this implies $\theta^n_n \wedge \sqrt{\mu_1}\eta^\alpha + \sqrt{\mu_1}\eta^\gamma \wedge \theta^{\alpha'}_{\gamma'} + 2\sqrt{\mu_1}\eta^n \wedge \theta^{\alpha'}_{n_{11}} = d(\sqrt{\mu_1}) \wedge \eta^\alpha + \sqrt{\mu_1}(\theta^0_0 \wedge \eta^\alpha + \eta^\gamma \wedge \theta^\alpha_\gamma)$, mod$(\eta)$, i.e., $\eta^\alpha \wedge \left[\sqrt{\mu_1}(\theta^{\alpha'}_{\alpha'} - \theta^\alpha_\alpha + \theta^0_0 - \theta^n_n) + d(\sqrt{\mu_1})\right] + \eta^n \wedge (2\sqrt{\mu_1}\theta^{\alpha'}_{n_{11}}) = 0$, mod$(\eta)$.

$$\sqrt{\mu_1}(\theta_{\alpha'}^{\alpha'} - \theta_\alpha^\alpha + \theta_0^0 - \theta_n^n) = -d(\sqrt{\mu_1}) + A^{(77)\beta}\eta^\beta + B^{(77)\beta}\eta^n, \ mod(\eta),$$
$$2\sqrt{\mu_1}\theta_{n_{11}}^{\alpha'} = B^{(77)\beta}\eta^\beta + E^{(77)\beta}\eta^n, \ mod(\eta).$$

By Step 1(A), $\sqrt{\mu_1}\theta_{\beta'}^{n_{11}} = A_\beta^{(1)}\eta^n$. It implies $A_\beta^{(1)} = B^{(77)\beta} = E^{(77)\beta} = 0$ and $\theta_{n_{11}}^{\alpha'} = 0$.

By Step 3(A), $\sqrt{\mu_1}\Delta_\alpha = -d(\sqrt{\mu_1}) + A^{(6)\alpha}\eta^n, \ \ mod(\eta)$, it implies $A^{(6)\alpha} = 0$.

Step 5 Consider $\theta_\beta^n = 0$. Then $d\theta_\beta^n = 0$. By §6 and $\theta_\beta^{n-1} = \theta_\beta^n = 0$, this implies $\eta^n \wedge (-\theta_\beta^0) - \mu_1\eta^n \wedge \overline{\eta^\beta} + 2i\overline{\eta^\beta} \wedge \theta_{N+1}^n = 0$. Hence $\eta^n \wedge (-\theta_\beta^0 - \mu_1\overline{\eta^\beta}) + \overline{\eta^\beta} \wedge (2i\theta_{N+1}^n) = 0$. Then by Cartan's lemma,

$$-\theta_\beta^0 - \mu_1\overline{\eta^\beta} = A^{(17)\beta}\eta^n + C^{(17)}\overline{\eta^\beta},$$
$$2i\theta_{N+1}^n = C^{(17)}\eta^n + F^{(17)}\overline{\eta^\beta}.$$

Hence $F^{(17)} = 0$. Recalling $\theta_\beta^0 = -2i\overline{\theta_{N+1}^\beta}$, we obtain $-2i\theta_{N+1}^\beta = \overline{A^{(17)\beta}}\overline{\eta^n} + (\mu_1 + \overline{C^{(17)}})\eta^\beta$.

Step 6 From $\theta_{\alpha'}^{\beta''} = 0$ for $\alpha \neq \beta$ by Step 2(A), $d\theta_{\alpha'}^{\beta''} = 0$. By the known formulas, this implies $\theta_{\alpha'}^{\beta'} \wedge \theta_{\beta'}^{\beta''} + \theta_{\alpha'}^{\alpha''} \wedge \theta_{\alpha''}^{\beta''} + \theta_{\alpha'}^b \wedge \theta_b^{\beta''} = 0$. By Step 2(A) and 2(A'), $\theta_{\alpha'}^{\beta'} = \theta_\alpha^\beta$ and $\theta_{\alpha''}^{\beta''} = \theta_\alpha^\beta$, $\forall\alpha \neq \beta$. By Step 1(B), $\frac{1}{\sqrt{\mu_1}}(2C_\alpha^{(2)b}\eta^n + B_\alpha^{(2)b}\eta^{n-1}) \wedge \frac{1}{\sqrt{\mu_2}}(-\overline{B_\beta^{(2)b}}\overline{\eta^n} - \overline{D_\beta^{(2)b}}\overline{\eta^{n-1}}) = 0$. Then

$$C_\alpha^{(2)b}\overline{B_\beta^{(2)b}} = C_\alpha^{(2)b}\overline{D_\beta^{(2)b}} = B_\alpha^{(2)b}\overline{B_\beta^{(2)b}} = B_\alpha^{(2)b}\overline{D_\beta^{(2)b}} = 0, \quad \alpha \neq \beta.$$

Step 7 Consider $\theta_{\alpha'}^{\beta'} = \theta_\alpha^\beta$ where $\alpha \neq \beta$ by Step 2(A). Then $d\theta_{\alpha'}^{\beta'} = d\theta_\alpha^\beta$. By the known formulas, $-\theta_n^{\beta'} \wedge \theta_{\alpha'}^n - \theta_{\gamma'}^{\beta'} \wedge \theta_{\alpha'}^{\gamma'} - \theta_b^{\beta'} \wedge \theta_{\alpha'}^b = -\theta_0^\beta \wedge \theta_\alpha^0 - \theta_\gamma^\beta \wedge \theta_\alpha^\gamma - \theta_{N+1}^\beta \wedge \theta_\alpha^{N+1}$, i.e.,

$-\mu_1\overline{\eta^\alpha} \wedge \eta^\beta + \sum_{\gamma\neq\alpha,\beta}\theta_{\alpha'}^{\gamma'} \wedge \theta_{\gamma'}^{\beta'} + \theta_{\alpha'}^{\alpha'} \wedge \theta_{\alpha'}^{\beta'} + \theta_{\alpha'}^{\beta'} \wedge \theta_{\beta'}^{\beta'} + \frac{1}{\sqrt{\mu_1}}(2C_\alpha^{(2)b}\eta^n + B_\alpha^{(2)b}\eta^{n-1}) \wedge \frac{1}{\sqrt{\mu_1}}(-2\overline{C_\beta^{(2)b}}\overline{\eta^n} - \overline{B_\beta^{(2)b}}\overline{\eta^{n-1}}) = (-\mu_1 - C^{(17)})\overline{\eta^\alpha} \wedge \eta^\beta + \sum_{\gamma\neq\alpha,\beta}\theta_\alpha^\gamma \wedge \theta_\gamma^\beta \wedge +\theta_\alpha^\alpha \wedge \theta_\alpha^\beta + \theta_\alpha^\beta \wedge \theta_\beta^\beta + 2i\overline{\eta^\alpha} \wedge \frac{i}{2}(\mu_1 + \overline{C^{(17)}})\eta^\beta$. Since $\sum_{\gamma\neq\alpha,\beta}\theta_{\alpha'}^{\gamma''} \wedge \theta_{\gamma''}^{\beta'} = \sum_{\gamma\neq\alpha,\beta}\theta_\alpha^\gamma \wedge \theta_\gamma^\beta\wedge$, $\theta_{\alpha'}^{\alpha'} - \theta_\alpha^\alpha = \theta_{\beta'}^{\beta'} - \theta_\beta^\beta$ and $\theta_{\beta''}^{\alpha'} = 0$ $\forall\alpha \neq \beta$, the above identity becomes $-\mu_1\overline{\eta^\alpha} \wedge \eta^\beta + \frac{1}{\sqrt{\mu_1}}(2C_\alpha^{(2)b}\eta^n + B_\alpha^{(2)b}\eta^{n-1}) \wedge \frac{1}{\sqrt{\mu_1}}(-2\overline{C_\beta^{(2)b}}\overline{\eta^n} - \overline{B_\beta^{(2)b}}\overline{\eta^{n-1}}) = (-\mu_1 - C^{(17)})\overline{\eta^\alpha} \wedge \eta^\beta + 2i\overline{\eta^\alpha} \wedge \frac{i}{2}(\mu_1 + \overline{C^{(17)}})\eta^\beta$. Then we obtain $C^{(17)} + \overline{C^{(17)}} = -\mu_1$ again and $\sum_b C_\alpha^{(2)b}\overline{C_\beta^{(2)b}} = 0$, $\forall\alpha \neq \beta$.

Step 8 Notice $\theta_{\alpha'}^{\alpha'} - \theta_\alpha^\alpha = \theta_{\beta'}^{\beta'} - \theta_\beta^\beta$, $\forall\alpha \neq \beta$ (see Step 6). Then $d\theta_{\alpha'}^{\alpha'} - d\theta_\alpha^\alpha = d\theta_{\beta'}^{\beta'} - d\theta_\beta^\beta$. By the known formulas, $d\theta_{\alpha'}^{\alpha'} - d\theta_\alpha^\alpha = -\mu_1\overline{\eta^n} \wedge \eta^n - \mu_1\overline{\eta^\alpha} \wedge \eta^\alpha + \theta_{\alpha'}^{\alpha''} \wedge \theta_{\alpha''}^{\alpha'} + (2C_\alpha^{(2)b}\eta^n + B_\alpha^{(2)b}\eta^{n-1}) \wedge (-2\overline{C_\alpha^{(2)b}}\overline{\eta^n} - \overline{B_\alpha^{(2)b}}\overline{\eta^{n-1}}) - (-\mu_1 - C^{(17)})\overline{\eta^\alpha} \wedge \eta^\alpha + \mu_1\eta^n \wedge \overline{\eta^n} + \mu_2\eta^{n-1} \wedge \overline{\eta^{n-1}} -$

$2i\overline{\eta^\alpha}\wedge(-\frac{i}{2})(-C^{(17)}-\mu_1)\eta^\alpha$. Since $\Delta_\alpha=\theta_{\alpha'}^{\alpha'}-\theta_\alpha^\alpha+\theta_0^0-\theta_n^n$ is independent of α by its formula in Step 3(A), we have $d\theta_{\alpha'}^{\alpha'}-d\theta_\alpha^\alpha=d\theta_{\beta'}^{\beta'}-d\theta_\beta^\beta$, i.e., $-\mu_1\overline{\eta^\alpha}\wedge\eta^\alpha+(2C_\alpha^{(2)b}\eta^n+B_\alpha^{(2)b}\eta^{n-1})\wedge(-2\overline{C_\alpha^{(2)b}}\overline{\eta^n}-\overline{B_\alpha^{(2)b}}\overline{\eta^{n-1}})-(-\mu_1-C^{(17)})\overline{\eta^\alpha}\wedge\eta^\alpha-2i\overline{\eta^\alpha}\wedge\frac{i}{2}(\overline{C^{(17)}}+\mu_1)\eta^\alpha=-\mu_1\overline{\eta^\beta}\wedge\eta^\beta+(2C_\beta^{(2)b}\eta^n+B_\beta^{(2)b}\eta^{n-1})\wedge(-2\overline{C_\beta^{(2)b}}\overline{\eta^n}-\overline{B_\beta^{(2)b}}\overline{\eta^{n-1}})-(-\mu_1-C^{(17)})\overline{\eta^\beta}\wedge\eta^\beta-2i\overline{\eta^\beta}\wedge\frac{i}{2}(\overline{C^{(17)}}+\mu_1)\eta^\beta$. Here we also use the fact that $\theta_{\alpha'}^{\alpha''}=\theta_{n-1}^n$ by Step 2(D). Hence $C^{(17)}+\overline{C^{(17)}}=-\mu_1$ (known) and

$$\sum_b|C_\alpha^{(2)b}|^2=\sum_b|C_\beta^{(2)b}|^2,\quad \sum_b|B_\alpha^{(2)b}|^2=\sum_b|B_\beta^{(2)b}|^2,\quad \alpha\neq\beta$$

It means that $\sum_b|C_\alpha^{(2)b}|^2$ and $\sum_b|B_\alpha^{(2)b}|^2$ are independent of α. Recall $\sum_b B_\alpha^{(2)b}\overline{B_\beta^{(2)b}}=\sum_b C_\alpha^{(2)b}\overline{C_\beta^{(2)b}}=0$ for $\alpha\neq\beta$ in Step 6 and Step 7. Recall $b\in\{3n,3n+1,...,N\}$ and denote $\vec{x}_\alpha:=C_\alpha^{(2)b}$. Then the set of vectors $\{\vec{x}_\alpha\}_{\alpha\in\{1,2,...,n-2\}}\subset\mathbb{C}^{N-3n+1}$ satisfies

$$\langle\vec{x}_\alpha,\vec{x}_\beta\rangle=0,\quad\forall\alpha\neq\beta;\quad\langle\vec{x}_\alpha,\vec{x}_\alpha\rangle=c$$

where c is independent of α. By the hypothesis $N+1\leq 4n-3$, we have $\{\vec{x}_\alpha\}_{\alpha\in\{1,2,...,n-2\}}\subset\mathbb{C}^{(4n-4)-3n+1}=\mathbb{C}^{n-3}$. Since $\#\{1,2,...,n-2\}=n-2$, it implies

$$C_\alpha^{(2)b}=B_\alpha^{(2)b}=0.$$

Step 9 Now $\theta_{n_{12}}^{\gamma'}=0$ by Step 2(C); $\theta_{n_{12}}^{\gamma''}=0$ by Step 2(D); $\theta_\beta^n=0$ by Step 2(A) and $G_\beta^{(3)}=0$ (Step 2(A')); $\theta_{n_{11}}^{\gamma'}=0$ by Step 1(A) and by $\theta_\beta^n=0$ and by $A^{(1)_\beta}=0$ (Step 4) and by $B_\beta^{(1)}=0$ (Step 2(D)); $\theta_{n_{22}}^{\gamma'}=0$ by Step 2(C); and $\theta_b^{\gamma'}=0$ by Step 1(B) and $B_\beta^{(2)b}=C_\beta^{(2)b}=0$ (Step 8). $\square$

8 Proof of Theorem 1.1

Proof of Theorem 1.1: If F is linear fractional, $II_M\equiv 0$ and $II_M^{CR}\equiv 0$ by Corollary 5.2 and [JY10]. Then $II_M-II_M^{CR}\equiv 0$.

Conversely, if $II_M-II_M^{CR}\equiv 0$, we want to show: F is linear fractional. Recall that F is linear fractional if and only if $\kappa_0=0$. Suppose that F is not linear fractional, i.e., $\kappa_0\geq 1$. We seek a contradiction.

Since $N+1\leq 4n-3$, by the inequality $N\geq n+\frac{(2n+1-\kappa_0)\kappa_0}{2}$ (cf. Lemma 2.1 (i)), it implies that the geometric rank κ_0 of F satisfies $\kappa_0\leq 2$. Then its geometric rank $\kappa_0=1$ or 2.

Suppose first that $\kappa_0=2$. Then $N\geq n+\frac{(2n+1-\kappa_0)\kappa_0}{2}=3n-1$, i.e., $N+1\geq 3n$.

If $\kappa_0 = 2$ with $N+1 > 3n$, by Lemma 7.1(i), we have $\theta^{\alpha'}_{n_{12}} = 0$. Differentiating, we obtain $\theta^{\gamma}_{n_{12}} \wedge \theta^{\alpha'}_{\gamma} + \theta^{n-1}_{n_{12}} \wedge \theta^{\alpha'}_{n-1} + \theta^{n}_{n_{12}} \wedge \theta^{\alpha'}_{n} + \theta^{\gamma'}_{n_{12}} \wedge \theta^{\alpha'}_{\gamma'} + \theta^{\gamma''}_{n_{12}} \wedge \theta^{\alpha'}_{\gamma''} + \theta^{n_{11}}_{n_{12}} \wedge \theta^{\alpha'}_{n_{11}} + \theta^{n_{22}}_{n_{12}} \wedge \theta^{\alpha'}_{n_{22}} + \theta^{n_{12}}_{n_{12}} \wedge \theta^{\alpha'}_{n_{12}} + \theta^{b}_{n_{12}} \wedge \theta^{\alpha'}_{b} + \theta^{N+1}_{n_{12}} \wedge \theta^{\alpha'}_{N+1} = 0$. By §6 and Lemma 7.1(i), we obtain $-\sqrt{\mu_1 + \mu_2}\overline{\eta^{n-1}} \wedge \sqrt{\mu_1}\eta^{\alpha} = 0$, but this is a contradiction.

If $\kappa_0 = 2$ with $N+1 = 3n$, by Lemma 7.1(ii), we have $\theta^{\alpha'}_{n_{12}} = 0$, i.e., $\theta^{\gamma}_{n_{12}} \wedge \theta^{\alpha'}_{\gamma} + \theta^{n-1}_{n_{12}} \wedge \theta^{\alpha'}_{n-1} + \theta^{n}_{n_{12}} \wedge \theta^{\alpha'}_{n} + \theta^{\gamma'}_{n_{12}} \wedge \theta^{\alpha'}_{\gamma'} + \theta^{\gamma''}_{n_{12}} \wedge \theta^{\alpha'}_{\gamma''} + \theta^{n_{11}}_{n_{12}} \wedge \theta^{\alpha'}_{n_{11}} + \theta^{n_{22}}_{n_{12}} \wedge \theta^{\alpha'}_{n_{22}} + \theta^{n_{12}}_{n_{12}} \wedge \theta^{\alpha'}_{n_{12}} + \theta^{N+1}_{n_{12}} \wedge \theta^{\alpha'}_{N+1} = 0$. By §6 and Lemma 7.1(ii), we obtain the same contradiction as above.

Next suppose that $\kappa_0 = 1$. By Theorem 3.1 in [HJX06], we can write

$$\begin{cases} f_1 = z_1 f_1^*, \\ f_j = z_j, \quad \forall 2 \le j \le n, \\ \phi_{lk} = \mu_{lk} z_l z_k + z_1 \phi^*_{lk}, \quad \forall (l,k) \in \mathcal{S}_0, \\ \phi_{lk} = z_1 \phi^*_{lk}, \quad \forall (l,k) \in \mathcal{S} \backslash \mathcal{S}_0, \\ g = w \end{cases}$$

where $f_1^* = 1 + \frac{i\mu_1}{2} w + O(|(z,w)|^2)$, and $\phi^*_{lk} = O_{wt}(2)$, $\forall (l,k) \in \mathcal{S}_0$. Since $F(z,w) \in \partial\mathbb{H}^{N+1}$, we have

$$Im(w) = |z_1 f_1^*|^2 + |z_2| + ... + |z_n|^2 + |z_1|^2 \sum_{(l,k)\in\mathcal{S}} |\phi^*_{lk}|^2, \quad \forall Im(w) = |z|^2,$$

i.e.,

$$0 = |f_1^*|^2 - 1 + \sum_{(l,k)\in\mathcal{S}} |\phi^*_{lk}|^2, \quad \forall Im(w) = |z|^2.$$

Then the mapping $(z,w) \mapsto (f_1^*, \phi^*_{lk})$ is a proper holomorphic mapping from $\partial\mathbb{H}^{n+1}$ into $\partial\mathbb{B}^{N-n+1}$. Since $f_1^* = 1 + \frac{i\mu_1}{2} w + O(|(z,w)|^2)$, we conclude that at least one of the components $\{\phi^*_{lk}\}_{(l,k)\in\mathcal{S}}$ must contain a nonzero w term. This is a contradiction with (73). $\square$

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