A survey on the recent progress of some problems in Several Complex Variabes,

Xiaojun Huang*, Shanyu Ji and Wanke Yin

Abstract: We discuss the recent progress on two problems in Several Complex Variables. The first one is on the gap phenomenon for proper holomorphic maps between balls. The second one is on the precise holomorphic structure of Bishop surfaces near a vanishing Bishop invariant.

Key worlds and MSC numbers: Proper rational maps between balls, gap phenomenon, minimum proper maps, Bishop surfaces, exceptional Bishop invariant, Moser-Webster theory, Moser invariant, convergence problem. 32A, 32H, 32V, 53B.

1. Introduction

In this article, we give a survey on the recent studies of two classical problems in several complex variables. The first problem concerns the geometric structure for proper holomorphic mappings between balls in complex spaces of different dimensions . We will be mainly focusing on our investigation on the gap phenomenon carried out in [HJX2] [HJY]. We will also formulate a general conjecture to guide the further study along these lines of research. The second topic to be touched here is on the normal form theory for a Bishop surface with a vanishing Bishop invariant. We will discuss a recent joint work of the first and the third authors [HY]. We will also pose two open problems motivated from the papers of [MW] [HY].

*Supported in part by NSF-0500626

2. Gap phenomenon for proper holomorphic mappings between balls

Write $\mathbb{B}^n$ for the unit ball in the complex space $\mathbb{C}^n$. Write $Prop(\mathbb{B}^n, \mathbb{B}^N)$ for the set of proper holomorphic maps from $\mathbb{B}^n$ into $\mathbb{B}^N$. Namely, $F \in Prop(\mathbb{B}^n, \mathbb{B}^N)$ if F is a holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$ such that for any compact subset $K \subset\subset \mathbb{B}^N$, $F^{-1}(K)$ is also a compact subset of $\mathbb{B}^n$. Roughly speaking, the properness of the map F says that F maps the boundary of $\mathbb{B}^n$ into the boundary of $\mathbb{B}^N$.

Write $Prop_k(\mathbb{B}^n, \mathbb{B}^N)$ for the set of proper holomorphic maps from $\mathbb{B}^n$ into $\mathbb{B}^N$, which are C^k-smooth up to the boundary for a non-negative integer k. We say that F and $G \in Prop(\mathbb{B}^n, \mathbb{B}^N)$ are *equivalent* if there are automorphisms $\sigma \in Aut(\mathbb{B}^n)$ and $\tau \in Aut(\mathbb{B}^N)$ such that $F = \tau \circ G \circ \sigma$. A research field in several complex variables, that has attracted much attention in the past two decades, is to classifiy proper holomorphic mappings between balls under such an equivalence relation. The interested reader may consult the articles of [Fr1] [Hu2] and [HJ] for surveys from various aspects of studies in this direction. In this article, we only address recent studies on the gap property for mappings bewteen balls.

For a proper holomorphic map $F \in Prop(\mathbb{B}^n, \mathbb{B}^N)$, one can always add zero components to F and then compose it with automorphisms from $Aut(\mathbb{B}^N)$ to produce other proper holomorphic maps from $\mathbb{B}^n$ into $\mathbb{B}^{N'}$ with $N' > N$. However, maps obtained in this manner have the same geometric character as that of the original F and should not be regarded as 'different maps'. Motivated by this construction, one gives the following definition (see also [DL] [DLP] for related definitions):

Definition 2.1: *A map $F \in Prop(\mathbb{B}^n, \mathbb{B}^N)$ is said to be minimum if F is not equivalent to a map of the form $(G, 0)$, where G is a proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^{N'}$ with $N' < N$.*

Since a linear fractional transformation in $Aut(\mathbb{B}^m)$ maps an affine complex hyperplane in $\mathbb{C}^m$ to an affine complex hyperplane in $\mathbb{C}^m$, it is not hard for us to see that a map $F \in Prop(\mathbb{B}^n, \mathbb{B}^N)$ is minimum if and only if the image of $\mathbb{B}^n$ under F is not contained in an affine complex hyperplane of $\mathbb{C}^N$. This simple fact will be very useful for us to test whether a map is minimum or not.

A minimum proper holomorphic map can not be constructed with the above mentioned simple method. It is easy to verify that there is no proper holomorphic map from $\mathbb{B}^n$ to $\mathbb{B}^N$ when $N < n$. By definition, $Prop(\mathbb{B}^n, \mathbb{B}^n)$ includes $Aut(\mathbb{B}^n)$ which is isomorphic to $SU(n, 1)$ modifying $\{\pm \mathrm{Id}\}$ and thus is a non-compact Lie group. At this point, we should mention a result of Alexander [Alx] stating that $Prop(\mathbb{B}^n, \mathbb{B}^n) = Aut(\mathbb{B}^n)$ for $n > 1$. In what follows, we will always make the assumption that $N \geq n > 2$. In 1979, S. Webster proved that any proper holomorphic map in $Prop_3(\mathbb{B}^n, \mathbb{B}^N)$ with $N = n + 1 > 4$ is equivalent to the map $(Id, 0)$ and thus is not minimum. After the work of many people (see [Fa] [Fr2], etc), the following was proved by the first author in [Hu4]:

Theorem 2.2 (Huang [Hu4]): *Let $F \in Prop_2(\mathbb{B}^n, \mathbb{B}^N)$ with $n < N \leq 2n - 2$. Then f is equivalent to the 'big circle embedding' $(Id, 0)$.*

In particular, Theorem 2.2 says that there are no minimum proper holomorphic maps which are C^2-smooth up to the boundary from $\mathbb{B}^n$ into $\mathbb{B}^N$ when $n < N \leq 2n - 2$.

Recall that the Whitney map $W_{n,1} := (z', z_n z)$ for $z = (z', z_n) \in \mathbb{C}^n$ is a proper quadratic polynomial map from $\mathbb{B}^n$ into $\mathbb{B}^{2n-1}$. $W_{n,1}$ is minimum, for, otherwise, by Theorem 2.2, $W_{n,1}$ would have degree one. For $\lambda \in [0, 1]$ and a holomorphic map h from $\overline{\mathbb{B}}^n$ into $\mathbb{C}^{N'}$, define $W_{n,1}(z; h, \lambda) := (z', \lambda z_n, \sqrt{1 - \lambda^2} z_n h(z))$. When $h = z$, $W_{n,1}(z; z, \lambda)$ is a proper quadratic polynomial map from $\mathbb{B}^n$ into $\mathbb{B}^{2n}$. When $\lambda \in (0, 1)$, $W_{n,1}(z; z, \lambda)$ is minimum; for, otherwise, there would be complex numbers $\{\mu_j\}_{j=0}^{2n}$, not all zero, such that

$$\sum_{j=1}^{n-1} \mu_j z_j + \mu_n \lambda z_n + \sqrt{1 - \lambda^2} \sum_{j=1}^{n} z_n \mu_{j+n} z_j = \mu_0.$$

Letting $z = 0$, we get $\mu_0 = 0$. Comparing the coefficients of $z_n z_j$ and z_j for $j = 1, \cdots, n$, we get $\mu_j = 0$ for $j = 1, \cdots, 2n$. This is a contradiction.

$W_{n,1}(z; z, \lambda)$ was first constructed by D'Angelo and is also called the D'Angelo family [DA] as λ varies from 0 to 1.

We say that proper holomorphic maps between balls possess the first gap phenomenon when the target dimension N satisfies the property that $n < N < 2n - 1$. In a recent joint paper of the first two authors with Xu, we proved the following:

Theorem 2.3(Huang-Ji-Xu [HJX2]): *Let $F \in Prop_3(\mathbb{B}^n, \mathbb{B}^N)$ with $2n < N < 3n - 3$. Then F is equivalent to a quadratic polynomial map of the form $(W_{n,1}(z; z, \lambda), 0)$ for a certain $\lambda \in [0, 1]$*

Define the generalized Whitney map $W_{n,k}$ for $1 \leq k \leq n$ as follows:

Example 2.4 ([Hu5]): Let

$$
\begin{array}{l}
\psi_1 = (z_1, \sqrt{2}z_2, \cdots, \sqrt{2}z_k, z_{k+1}, \cdots, z_n), \\
\psi_2 = (z_2, \sqrt{2}z_3, \cdots, \sqrt{2}z_k, z_{k+1}, \cdots, z_n), \\
\cdots \\
\psi_{k-1} = (z_{k-1}, \sqrt{2}z_k, z_{k+1}, \cdots, z_n) \\
\psi_k = (z_k, z_{k+1}, \cdots, z_n), \\
\psi_{k+1} = (z_{k+1}, \cdots, z_n).
\end{array}
\tag{2..1}
$$

Let $W_{n,k} = (z_1\psi_1, \cdots, z_k\psi_k, \psi_{k+1})$. Then $W_{n,k}$ is a proper quadratic polynomial map from $\mathbf{B}^n$ into $\mathbf{B}^N$ with $N = Q(n,k)$, where $Q(n,k) = (k+1)n - \frac{k(k+1)}{2}$. As in the case for $W_{n,1}(z; z, \lambda)$, one can similarly verify the following:

Proposition 2.5: *$W_{n,k}$ is minimum.*

Let ψ_j be defined as in Example 2.4. For an integer τ with $1 \le \tau \le k$, positive numbers $\lambda_j \in (0,1)$ with $1 \le j \le \tau$, we define

$$
W_{n,k}(\lambda_1, \cdots, \lambda_\tau) := (z_1\widetilde{\psi_1}, \cdots, z_k\widetilde{\psi_k}, \psi_{k+1}, \lambda_1 z_1, \cdots, \lambda_\tau z_\tau). \tag{2..2}
$$

Here

$$
\begin{array}{l}
\mu_{jl} = \sqrt{1-\lambda_l^2} \text{ for } \ j \le l \le \tau \ \text{ and } \ \mu_{jl} = 1 \text{ for } l > \tau, \\
\widetilde{\psi_1} = (\sqrt{1-\lambda_1^2}z_1, \sqrt{1-\lambda_1^2+\mu_{12}^2}z_2, \cdots, \sqrt{1-\lambda_1^2+\mu_{1k}^2}z_k, \sqrt{1-\lambda_1^2}z_{k+1}, \cdots, \sqrt{1-\lambda_1^2}z_n), \\
\widetilde{\psi_2} = (\sqrt{1-\lambda_2^2}z_2, \sqrt{1-\lambda_2^2+\mu_{23}^2}z_3, \cdots, \sqrt{1-\lambda_2^2+\mu_{2k}^2}z_k, \sqrt{1-\lambda_2^2}z_{k+1}, \cdots, \sqrt{1-\lambda_2^2}z_n), \\
\cdots \\
\widetilde{\psi_\tau} = (\sqrt{1-\lambda_\tau^2}z_\tau, \sqrt{1-\lambda_\tau^2+\mu_{\tau(\tau+1)}^2}z_{\tau+1}, \cdots, \sqrt{1-\lambda_\tau^2+\mu_{\tau k}^2}z_k, \sqrt{1-\lambda_\tau^2}z_{k+1}, \cdots, \\
\sqrt{1-\lambda_\tau^2}z_n) \text{ for } \tau < k \text{ and} \\
\widetilde{\psi_\tau} = (\sqrt{1-\lambda_\tau^2}z_k, \sqrt{1-\lambda_\tau^2}z_{k+1}, \cdots, \sqrt{1-\lambda_\tau^2}z_n) \text{ for } \tau = k, \\
\widetilde{\psi_j} = \psi_j \ \text{ when } \ \tau < k \ \text{ and } \tau < j \le k.
\end{array}
\tag{2..3}
$$

For convenience, we allow $\tau = 0$ in (2..2). In this case, $W_{n,k}(\lambda_1, \cdots, \lambda_\tau)$ is simply defined to be $W_{n,k}$. We then have the following

Proposition 2.6: *For any $0 < \lambda_j < 1$, ($j \le \tau \le k \le n$), $W_{n,k}(\lambda_1, \cdots, \lambda_\tau)$ is a minimum proper monomial map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with*

$$
N = n + (n-1) + \cdots + (n-k) + \tau = Q(n,k) + \tau = (k+1)n - \frac{k(k+1)}{2} + \tau.
$$

Proof of Proposition 2.6: It is straightforward to verify that $W_{n,k}(\lambda_1, \cdots, \lambda_\tau)$ is indeed a proper monomial map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N = Q(n,k) + \tau$. Notice that

$$\widetilde{\psi_j} = 0 \bmod (z_j \cdots, z_n).$$

Suppose that $W_{n,k}(\lambda_1, \cdots, \lambda_\tau)$ is not minimum. Notice that the map preserves the origin. Then, we have a non-zero complex vector $\vec{\mu}$ of length N such that the inner product $< \vec{\mu}, W_{n,k}(\lambda_1, \cdots, \lambda_\tau) > \equiv 0$. Now, comparing the coefficients of terms with $z_1,\ z_2,\ \cdots, z_n$-factor, respectively, as in the case of $W_{n,1}(z; z, \lambda)$, we conclude that $\vec{\mu} = 0$. This is a contradiction. ■

Let F be a holomorphic mapping defined over $\overline{\mathbb{B}}^n$. We can modify the above defined $W_{n,k}(\lambda_1, \cdots, \lambda_\tau)$ to construct a new map, denoted by $W_{n,k}(\lambda_1, \cdots, \lambda_\tau, F)$ by simply changing $\widetilde{\psi_1}$ above as follows, while keeping all the others the same:

$$\widetilde{\psi_1} = (\sqrt{1-\lambda_1^2} z_1 F, \sqrt{1-\lambda_1^2+\mu_{12}^2} z_2, \cdots, \sqrt{1-\lambda_1^2+\mu_{1k}^2} z_k, \sqrt{1-\lambda_1^2} z_{k+1}, \cdots, \sqrt{1-\lambda_1^2} z_n).$$

Then, when F is a proper polynomial minimum map from $\mathbb{B}^n$ into $\mathbb{B}^{N^*}$ with $F(0) = 0$, then

$$W_{n,k}(\lambda_1, \cdots, \lambda_\tau, F) \tag{2..4}$$

can be easily seen to be also a minimum polynomial proper map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N = N^* - 1 + Q(n,k) + \tau$. In the definition of $W_{n,k}(\lambda_1, \cdots, \lambda_\tau, F)$, we also allow $\tau = 0$. In this case, we define $W_{n,k}(\lambda_1, \cdots, \lambda_\tau, F)$ to be constructed through $W_{n,k}$ and F in the same way as for the case of $\tau > 0$. We first notice that there are minimum monomial maps from $\mathbb{B}^n$ into $\mathbb{B}^{ln}$ for any $l \geq 1$. Letting $N^* = (k-k_0)n$ for $k > k_0 > 0$ and replacing k by k_0 in (2..4), we get the following:

Proposition 2.7: *Let F be a minimum proper polynomial map from $\mathbb{B}^n$ into $\mathbb{B}^{n(k-k_0)}$ with $k > k_0 > 0$ and $F(0) = 0$. Then $W_{n,k_0}(\lambda_1, \cdots, \lambda_\tau, F)$ $(0 \leq \tau \leq k_0 \leq n)$ is a proper polynomial minimum map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with*

$$N = (k+1)n - \frac{k_0(k_0+1)}{2} + \tau - 1.$$

Also, there are minimum proper monomial maps from $\mathbb{B}^n$ into $\mathbb{B}^{kn}$ for any $k \geq 1$.

Proof of Proposition 2.7: It suffices for us to construct minimum proper monomial maps from $\mathbb{B}^n$ into $\mathbb{B}^{kn}$ for any $k \geq 1$. We do it by induction. The statement is obvious when

$k = 1$. Suppose we have a proper monomial minimum map F from $\mathbb{B}^n$ into $\mathbb{B}^{(k-1)n}$. Let $\lambda \in (0, 1)$. Then $(\lambda z_n F, \sqrt{1-\lambda^2} z_n, z')$ is easily seen to be a minimum proper monomial map from $\mathbb{B}^n$ into $\mathbb{B}^{kn}$. ∎

Combining Proposition 2.6 with Proposition 2.7, we obtain the following:

Theorem 2.8: *Let $N \geq n > 2$ be such that there does not exist a positive integer k such that $kn < N < (k+1)n - \frac{k(k+1)}{2}$. Then there is a minimum proper monomial map from $\mathbb{B}^n$ into $\mathbb{B}^N$. Equivalently, for $n > 2$, let $K(n) = \max\{m \in \mathbb{Z}^+ : \ m(m+1)/2 < n\}$ and let $\mathcal{I}_k := \{m \in \mathbb{Z}^+ : \ kn < m < (k+1)n - k(k+1)/2\}$ for $1 \leq k \leq K(n)$. Then for any $N \geq n$ with $N \notin \cup_{k=1}^{K(n)} \mathcal{I}_k$, there is a minimum proper monomial map from $\mathbb{B}^n$ into $\mathbb{B}^N$.*

We notice that minimum proper monomial maps from $\mathbb{B}^n$ into $\mathbb{B}^N$ when $N \geq n^2 - 2n + 2$ were also constructed in a recent preprint of D'Angelo and Lebl [DL].

Proof of Theorem 2.8: We need to construct minimum proper monomial map from $\mathbb{B}^n$ into $\mathbb{B}^N$ under the assumption that either $(k+1)n - k(k+1)/2 \leq N \leq (k+1)n$ with $k \leq K(n)$ or $N \geq (K(n)+1)n - \frac{K(n)(K(n)+1)}{2}$. Apparently, $K(n) \leq \sqrt{2n}$.

Let $k \leq n$. By Proposition 2.6, we see the existence of minimum proper monomial maps from $\mathbb{B}^n$ into $\mathbb{B}^N$ when $(k+1)n - k(k+1)/2 \leq N \leq (k+1)n - k(k-1)/2$. If $k - 1 > 0$, applying Proposition 2.7 with $k_0 = k - 1$ and $\tau = 0, \cdots, k-1$, we see the existence of minimum proper monomial maps from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $(k+1)n - k(k-1)/2 \leq N \leq (k+1)n - (k-1)(k-2)/2 - 1$. Again, applying Proposition 2.7 with $k_0 = k - 2$ (if $k - 2 > 0$) and $\tau = 0, \cdots, k-2$, we see the existence of minimum proper monomial maps from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $(k+1)n - (k-1)(k-2)/2 - 1 \leq N \leq (k+1)n - (k-2)(k-3)/2 - 1$. By an inductive use of Proposition 2.7, we see the existence of the required maps for N with $(k+1)n - k(k+1)/2 \leq N \leq (k+1)n$ for $k \leq n$.

Next, letting $k = n + 1$ in Proposition 2.7 and inductively applying Proposition 2.7 with $k_0 = n, n-1, \cdots,$, we conclude the existence of the required maps when $(n+2)n - n(n+1)/2 - 1 \leq N \leq (n+2)n$. In particular, this would give the existence of the required maps when $(n+1)n \leq N \leq (n+2)n$. Applying an induction argument, we easily conclude the existence of the required maps for any $N \geq (n+1)n$. This concludes the proof of the theorem. ∎

Theorem 2.2 shows that there are no minimum proper holomorphic maps from $\mathbb{B}^n$ into $\mathbb{B}^N$ that are C^2-smooth up to the boundary when $N \in \mathcal{I}_1$; and Theorem 2.3 shows that there are no minimum proper holomorphic maps from $\mathbb{B}^n$ into $\mathbb{B}^N$ that are C^3-smooth up to the boundary when $N \in \mathcal{I}_2$. We say that proper holomorphic maps between balls have the second gap when the target dimension $N \in \mathcal{I}_2$. In a recent not-yet published preprint of the authors, we proved the following:

Theorem 2.9 (Huang-Ji-Yin [HJY]): *There are no minimum proper holomorphic maps from $\mathbb{B}^n$ into $\mathbb{B}^N$, that are C^3-smooth up to the boundary, when $N \in \mathcal{I}_3$.*

More generally, let $K(n)$ be defined as above. We conjecture that there are precisely $K(n)$ gaps for proper holomorphic maps between balls with the source dimension n, that are three-times differentiable up to the boundary. More precisely, we pose the following:

Conjecture 2. 10: *Let $K(n)$ be the largest positive integer m such that $n > m(m+1)/2$. Then, there are no minimum proper holomorphic maps from $\mathbb{B}^n$ into $\mathbb{B}^N$, that are three times differentiable, if and only if $N \in \mathcal{I}_k$ for a certain $1 \leq k \leq K(n)$. Here $\mathcal{I}_k$ is the collection of integers m such that $kn < m < (k+1)n - k(k+1)/2$.*

We briefly discuss in the rest of this section another type of gap phenomena for holomorphic maps between the generalized balls in the complex projective spaces, that was motivated from a joint paper of the first author with S. Baouendi [BH]. In [BH], among other things, it is proved that any proper holomorphic map from $\mathbb{B}^n_\ell$ into $\mathbb{B}^N_\ell$ with $N \geq n > 2,\ 0 < \ell < n-1$ must be a linear map and thus is equivalent to a map of the form $(\mathrm{Id}, 0)$. Here, we recall that for $0 \leq \ell < n$, we denote by $\mathbb{B}^n_\ell$ the domain in $\mathbb{CP}^n$ given by

$$\mathbb{B}^n_\ell := \{[z_0, \cdots, z_n] \in \mathbf{CP}^n : |z_0|^2 + \cdots + |z_\ell|^2 > |z_{\ell+1}|^2 + \cdots + |z_n|^2\}.$$

When $\ell = 0$, $\mathbb{B}^n_\ell$ is simply the realization of the unit ball of $\mathbb{C}^n$ in the projective space $\mathbf{CP}^n$. Also, as in the ball case, two proper holomorphic maps from $\mathbb{B}^n_\ell$ into $\mathbb{B}^N_{\ell'}$ are said to be equivalent if there are $\sigma \in Aut(\mathbb{B}^n_\ell)$ and $\tau \in Aut(\mathbb{B}^N_{\ell'})$ such that $F = \tau \circ G \circ \sigma$. In a recent joint paper of the first author with Baouendi and Ebenfelt, we proved the following:

Theorem 2.11 (Baouendi-Ebenfelt-Huang [BEH]): *Let F be a proper holomorphic map from $\mathbb{B}^n_\ell$ to $\mathbb{B}^N_{\ell'}$. Assume that $\ell \leq \ell' < 2\ell$ and $2\ell \leq (n-1),\ 2\ell' \leq (N-1)$. Then, $F(\mathbf{B}^n_\ell)$ is contained in a linear projective subspace of $\mathbf{CP}^N$ of dimension $n + \ell' - \ell$.*

Comparing Theorem 2.11 with Theorems 2.2, 2.3, 2.9, we see that in the generalized ball case, the difference of the signatures plays the role of the codimension in the ball case. We certainly believe that there are other gap phenomena to be explored when $\ell > 0$. However, instead of formulating more conjectures along these lines, we mention here the following elementary lemma that is essential for the statement of Theorem 2.11 to hold. It seems to us that any generalization of Theorem 2.11 should be started with a better formulation this lemma:

Lemma 2.12: *Let $\varphi : (\mathbb{C}^{n-1}, 0) \longrightarrow (\mathbb{C}^{N-n}, 0)$ be the germ of a holomorphic map. Let $A(z, \overline{z})$ be a scalar real analytic function near 0. If $3 < n \leq N$, $\ell < \ell' < 2\ell$, $0 < \ell \leq \frac{n-1}{2}$ and*

$$A(z, \overline{z})|z|_{\ell}^{2} = -\sum_{j=1}^{\tau} |\varphi_j(z)|^2 + \sum_{j=\tau+1}^{N-n} |\varphi_j(z)|^2$$

with $\tau = \ell' - \ell$. Then $A(z, \overline{z}) \equiv 0$ and $(\varphi_{\tau+1}, \cdots, \varphi_{N-n}) = (\varphi_1, \cdots, \varphi_{\tau}) \cdot \mathcal{U}$ with $\mathcal{U} \cdot \overline{\mathcal{U}}^{t} = Id$, where $\mathcal{U}$ is a constant matrix. Here we define $|z|_{\ell} = -\sum_{j \leq \ell} |z_j|^2 + \sum_{j=\ell}^{n-1} |z_j|^2$.

Lemma 2.12 was discovered and proved when the authors of [BEH] were working on Theorem 2.11. It follows from the work [BH] on the study of degeneracy for holomoprhic mappings from hyperquadrics in $\mathbb{C}^n$ into hyperqudrics in $\mathbb{C}^N$ (namely, Lemma 4.1 of [BH] (or Theorem 1.6(ii)) and Lemma 2.1 of [BH]) as follows: First assume that $\phi_j \not\equiv 0$ for some $1 \leq j \leq \tau$, for, otherwise, it following from [§2, BH] that $\varphi \equiv 0$ and thus Lemma I follows. Applying a Cayley transformation in the standard way [see (5.1), pp 396, BH] to the map $\Phi = [\varphi_1, \cdots, \varphi_{N-n}]$ obtained from Lemma 2.12, we immediately have a map $F = (f, \psi, g) := \Psi_{N-n-1} \circ \Phi \circ \Psi_{n-1}^{-1}$ mapping an open piece M of the hyperqudric $\mathbb{H}_{\ell-1}^{n-2}$ into the hyperquadric $\mathbb{H}_{\tau-1}^{N-n-1}$ (see [BH] for the definition). Now, by the assumption that $\tau < \ell$, one concludes from [Lemma 2.1, BH] that the Hopf lemma property can not hold for F at any point in M (namely, the normal component has vanishing normal derivative), for, otherwise, [Lemma 2.1 (b)(c), BH] would imply that $\tau \geq \ell$. Finally, letting $F^{\#} = (f_1, \cdots, f_{\tau}, 0', f_{\tau+1}, \cdots, f_{n-3}, \phi, g)$ with $0'$ a zero vector with $(\ell - \tau)$ components, we can apply [Lemma 4.1, BH] (or even [Theorem 1.6(ii), BH]) with $\ell' = \ell$ to conclude that $F^{\#}$ maps a neighborhood of M in $\mathbb{C}^{n-2}$ into $\mathbb{H}_{\ell-1}^{N-n-1}$, which is equivalent to the conclusion in Lemma 2.12. At this point, we should mention a later generalization of Lemma 2.12 in [BER2] (obtained by studying the degeneracy of mappings from a more general hypersurface in $\mathbb{C}^n$ into a hyperquadric in $\mathbb{C}^N$), which may find nice applications in the further investigation on the gap phenomenon.

3. Bishop surfaces with a vanishing Bishop invariant

In this section, we discuss a recent study, carried out in [HY], on the precise holomorphic structure of a real analytic Bishop surface near a complex tangent point with a vanishing Bishop invariant. A Bishop surface is a generically embedded real surface in the complex space of dimension two. The interesting points on a Bishop surface are points with a non-trivial complex tangent, namely, points with a non-trivial complex tangent space of type $(1,0)$. The study of Bishop surfaces was initiated by Bishop in 1965 in his paper [Bis], where for a point p on a Bishop surface M with a complex tangent, he defined an invariant λ now called the Bishop invariant. Bishop showed that there is a holomorphic change of variables, that maps p to 0, such that M, near $p=0$, is defined in the complex coordinates $(z,w)\in\mathbb{C}^2$ by

$$w = z\overline{z} + \lambda(z^2+\overline{z}^2) + o(|z|^2), \tag{3..5}$$

where $\lambda\in[0,\infty]$. When $\lambda=\infty$, (3..5) is understood as $w = z^2+\overline{z}^2+o(|z|^2)$. It is now a standard terminology to call p a point with an elliptic, hyperbolic or parabolic complex tangent, according to whether $\lambda\in[0,1/2)$, $\lambda\in(1/2,\infty)$ or $\lambda=1/2,\infty$, respectively. When $p\in M$ has an elliptic complex tangent, Bishop proved the existence of a family of holomorphic disks attached to M shrinking down to p. In his famous paper [Bis], he formulated several problems concerning the uniqueness and regularity of the geometric object obtained by taking the union of all locally attached holomorphic disks. These problems, including their higher dimensional cases, were completely answered in the paper of the first author [Hu3], based on the previous work by Kenig-Webster [KW1-KW2], Moser-Webster [MW], Moser [Mos] and Huang-Krantz [HK].

Bishop invariant is a quadratic invariant. The celebrated work of Moser-Webster [MW] first investigated the much more subtle higher order invariants. In [MW], Moser-Webster discovered an intrinsic pair of involutions on the complexification of the surface near a non-exceptional complex tangent, which were related to the higher order holomorphic invariants of M near p. Here, we recall that the Bishop invariant is said to be non-exceptional if $\lambda\neq 0,1/2,\infty$ or if $\lambda\nu^2-\nu+\lambda=0$ has no roots of unity in the variable ν. Moser-Webster proved that, near a non-exceptional complex tangent, M can always be mapped, at least, by a formal transformation to the normal form defined by:

$$w = z\overline{z} + (\lambda+\epsilon w^s)(z^2+\overline{z}^2),\ \epsilon\in\{0,1,-1\},\ s\in\mathbb{Z}^+ \tag{3..6}$$

Moser-Webster also provided a convergence proof of the above mentioned formal transformation for the non-exceptional elliptic case: $0<\lambda<1/2$. However, the elliptic case with $\lambda=0$ has to be excluded from their theory. Instead, Moser in [Mos] carried out a study for $\lambda=0$ from a more formal power series point of view. Moser derived the following formal pseudo-normal form for M with $\lambda=0$:

$$w = z\overline{z} + z^s + \overline{z}^s + 2Re\{\sum_{j\geq s+1} a_j z^j\}. \tag{3..7}$$

Here s is the simplest higher order invariant of M at a complex tangent with the vanishing Bishop invariant, which we call the Moser invariant. Moser showed that when $s = \infty$, M is then holomorphically equivalent to the quadric $M_\infty = \{(z, w) \in \mathbf{C}^2 : w = |z|^2\}$.

Moser's formal pseudo-normal form is still subject to the simplification of a very complicated infinitely dimensional group $aut_0(M_\infty)$, the formal self-transformation group of M_∞. And it was left open from the work of Moser [Mos] to derive any higher order invariant other than s from the Moser pseudo-normal form. Based on his previous work with Webster [MW] and his own work [Mos], Moser posed two basic problems concerning a Bishop surface with a vanishing Bishop invariant. The first one is concerning the analyticity of the geometric object formed by the attached disks up to the complex tangent point. This was answered in the affirmative in [HK]. Hence, the work of [HK], together with that of Moser-Webster [MW], shows that, as far as the analyticity of the local hull of holomorphy is concerned, all elliptic Bishop surfaces are of the same character. The second problem that Moser asked is concerning the higher order invariants. Notice that by the Moser-Webster normal form, an analytic elliptic Bishop surface with $\lambda \neq 0$ is holomorphically equivalent to an algebraic one and possesses at most two more higher order invariants. Moser asked if M with $\lambda = 0$ is of the same character as that for elliptic surfaces with $\lambda \neq 0$. Is the equivalence class of a Bishop surface with $\lambda = 0$ determined by an algebraic surface obtained by truncating the Taylor expansion of its defining equation at a sufficiently higher order level? Gong showed in [Gon2] that under the equivalence relation of a smaller class of transformation group, called the group of holomorphic symplectic transformations, M with $\lambda = 0$ does have an infinite set of invariants. However, under this equivalence relation, elliptic surfaces with non-vanishing invariants also have infinitely many invariants. Gong's work later on (see, for example, [Gon2-3]) demonstrates that as far as many dynamical properties are concerned, exceptional and non-exceptional hyperbolic complex tangents are not much different from each other.

In [HY], a joint paper of the first and the third authors, we derived a complete formal normal form for a Bishop surface near a vanishing Bishop invariant. We obtained a complete set of invariants under the action of the formal transformation group. We showed, in particular, that the modular space for Bishop surfaces with a vanishing Bishop invariant and with a fixed (finite) Moser invariant s is an infinitely dimensional manifold in a Frèchet space. This then provides an answer, in the negative, to Moser's problem concerning the determination of a Bishop surface with a vanishing Bishop invariant from a finite truncation of its Taylor expansion. Furthermore, it was also used to show that most Bishop surfaces with $\lambda = 0$, $s \neq \infty$ are not holomorphically equivalent to algebraic surfaces. Hence, one sees a striking difference of an elliptic Bishop surface with a vanishing Bishop invariant from elliptic Bishop surfaces with non-vanishing Bishop invariants:

Theorem 3.1 (Huang-Yin [HY]): *Let M be a formal Bishop surface in $\mathbb{C}^2$ with an elliptic complex tangent at 0, whose Bishop invariant $\lambda = 0$ and whose Moser invariant $s < \infty$ ($s \geq 3$). Namely, let M be defined by $w = |z|^2 + z^s + \overline{z^s} + o(|z|^s)$ with $s < \infty$. Then There exists a formal transformation,*

$$(z', w') = F(z, w) = (\widetilde{f}(z, w), \widetilde{g}(z, w)), \quad F(0, 0) = (0, 0),$$

such that in the (z', w') coordinates, $M' = F(M)$ is represented near the origin by a formal equation of the following normal form:

$$w' = z'\bar{z}' + z'^s + \bar{z}'^s + \varphi(z') + \overline{\varphi(z')}$$

where

$$\varphi(z') = \sum_{k=1}^{\infty} \sum_{j=2}^{s-1} a_{ks+j} z'^{ks+j}.$$

Such a formal transform is unique up to a composition from the left with a rotation of the form:

$$z'' = e^{i\theta} z', \ w'' = w', \quad \text{where } \theta \text{ is a constant with } \ e^{is\theta} = 1.$$

When the M in Theorem 3.1 is real analytic, namely, the defining equation of M is given by a convergent power series near 0, one would expect that its unique normal form (up to a rotation) is also convergent. However, we were not able to answer such a problem at the moment. Namely, the following conjecture remains unknown:

Conjecture 3.2: *Under the same notation and assumption as in Theorem 3.1. Assume that M is real analytic. Then its formal normal form is also convergent. More precisely,*

$$w' = z'\bar{z}' + z'^s + \bar{z}'^s + \varphi(z') + \overline{\varphi(z')}$$

with

$$\varphi(z') = \sum_{k=1}^{\infty} \sum_{j=2}^{s-1} a_{ks+j} z'^{ks+j}.$$

being given by a convergent power series.

Concerning Conjecture 3.2, we were able to show in [HY] that if the formal normal form is convergent, then the map transforming the surface to its normal form must be convergent in case the Moser invariant $s \neq \infty$. Remark that there are many non-convergent formal maps transforming real analytic Bishop surfaces with a vanishing Bishop invariant and with $s = \infty$ to the model surface M_∞ defined before. (See [MW] [Mos] [Hu1]). This result may also be compared with many recent studies concerning convergence of formal CR maps between not too degenerate real analytic CR manifolds, though our method for proving such a result is quite different from what is used in the CR setting. Indeed, the main idea in [HY] for dealing with such a problem is to find a new hyperbolic geometry associated with surfaces from the Bishop geometry. We refer the reader for the work done in the CR setting to the papers of Baouendi-Ebenfelt-Rothschild [BER1], Baouendi-Mir-Rothschild [BMR], Meylan-Mir-Zaitsev [MMZ], and the references therein.

As an application of Theorem 3.1, we also derived in [HY] the following:

Theorem 3.3: *A generic real analytic Bishop surface with a vanishing Bishop invariant and $s \neq \infty$ is not holomorphically equivalent to an algebraic surface in $\mathbf{C}^2$.*

For a Bishop surface M with a non-exceptional hyperbolic complex tangent, Moser-Webster [MW] showed that it must be formally equivalent to the model $M_{\lambda,\epsilon,s} = \{(z,w) : w = z\overline{z} + (\lambda + \epsilon w^s)(z^2 + \overline{z^2})\}$, where s is a positive integer and $\epsilon \in \{\pm 1, 0\}$. Moser-Webster and Gong [Gon3] also constructed various examples showing that the formal process is divergent in general. A natural question is then the following:

Problem 3.4: *Construct the modular space for germs of non-exceptional hyperbolic Bishop surfaces, which are formally equivalent to $M_{\lambda,\epsilon,s}$.*

It seems reasonable to conjecture that such a modular space is of infinite dimension modeled over a Banach space whose basis is uncountable. One may compare this with the well-known modular space problem for germs of holomorphic maps of $(\mathbf{C}, 0)$ to itself with the identity as their linear term (see [Vor]).

References

[Alx] H. Alexander, *Proper holomorphic maps in $\mathbf{C}^n$*, *Indiana Univ. Math. Journal* 26, 137-146, 1977.

[BER1] S. Baouendi, P. Ebenfelt and L. Rothschild, *Local geometric properties of real submanifolds in complex space*, *Bull. Amer. Math. Soc.* (N.S.) 37 (2000), no. 3, 309–33.

[BER2] . Baouendi, P. Ebenfelt and L. Rothschild, *Transversality of holomorphic mappings between real hypersurfaces in different dimensions*, preprint, 2007.

[BEH] S. Baouendi, P. Ebenefelt and X. Huang, *Holomorphic mappings between hyperquadrics with small signature difference*, preprint, 2007.

[BH] S. Baouendi and X. Huang, *Super-rigidity for holomorphic mappings between hyperquadrics with positive signatures*, *Jour. of Diff. Geom.* Vol. 69, 379-398, 2005.

[BMR] S. Baouendi, N. Mir and L. Rothschild, *Reflection ideals and mappings between generic submanifolds in complex space*, *J. Geom. Anal. 12* (2002), 543-580.

[Bis] E. Bishop, *Differentiable manifolds in complex Euclidean space*, *Duke Math. J.* (32), 1-21, 1965.

[Car] É. Cartan, *Sur les variétés pseudo-conformal des hypersurfaces de l'espace de deux variables complexes*, *Ann. Mat. Pura Appl. (4) 11*, 17-90(1932).

[CD] D. Catlin and J. D'Angelo, *A stabilization theorem for Hermitian forms and applications to holomorphic mappings*, Math Research Letters, **3**(1996), 149-166.

[CM] : S. S. Chern and J. K. Moser, *Real hypersurfaces in complex manifolds*, *Acta Math. 133*, 219-271(1974).

[DA] J. P. D'Angelo, *Several Complex Variables and the Geometry of Real Hypersurfaces*, CRC Press, Boca Raton, 1993.

[DL] J. P. D'Angelo and Jiri Lebl, *Complexity results for CR mappings between spheres*, preprint, 2007. (arXiv:0708.3232)

[DLP] . P. D'Angelo, J. Lebl and H. Peters, *Degree estimates for polynomials constant on hyperplanes*, *Mich. Math. Jour.* , 2007. (to appear)

[Fa] J. Faran, *The linearity of proper holomorphic maps between balls in the low codimension case*, J. Diff. Geom. **24**(1986), 15-17.

[Fr1] F. Forstneric, *A survey on proper holomorphic mappings*, Proceeding of Year in SCVs at Mittag-Leffler Institute, Math. Notes 38, Princeton, NJ: Princeton University Press , 1992.

[Fr2] F. Forstneric, *Extending proper holomorphic mappings of positive codimension*, *Invent. Math. 95*, 31-62, 1989.

[For3] F. Forstneric, *Most real analytic Cauchy-Riemann manifolds are nonalgebraizable*, *Manuscripta Math. 115* (2004), 489–494.

[Gon1] X. Gong, *On the convergence of normalizations of real analytic surfaces near hyperbolic complex tangents*, *Comment. Math. Helv.* 69 (1994), no. 4, 549–574.

[Gon2] X. Gong, *Normal forms of real surfaces under unimodular transformations near elliptic complex tangents*, *Duke Math. J. 74* (1994), no. 1, 145–157.

[Gon3] X. Gong, *Existence of real analytic surfaces with hyperbolic complex tangent that are formally but not holomorphically equivalent to quadrics*, *Indiana Univ. Math. J. 53* (2004), no. 1, 83–95.

[Grov] M. Gromov, *Pseudo holomorphic curves in symplectic geometry*, *Invent Math. Vol. 82*, 1985, 307-347.

[Hu1] X. Huang, *Local Equivalence Problems for Real Submanifolds in Complex Spaces, Lecture Notes in Mathematics 1848 (C.I.M.E. series)*, Springer-Verlag, pp 109-161, Berlin-Heidelberg-New York, 2004.

[Hu2] X. Huang, *On some problems in several complex variables and Cauchy-Riemann Geometry*, Proceedings of ICCM (edited by L. Yang and S. T. Yau), *AMS/IP Stud. Adv. Math. 20*, 383-396, 2001.

[Hu3] X. Huang, *On an n-manifold in* $\mathbf{C}^n$ *near an elliptic complex tangent*, *J. Amer. Math. Soc.* (11), 669–692, 1998.

[Hu4] X. Huang, *On a linearity problem of proper holomorphic mappings between balls in complex spaces of different dimensions*, J. Diff. Geom. **51**(1999), 13-33.

[Hu5] X. Huang, *On a semi-linearity property for holomorphic maps*, *Asian Jour. of Math.* Vol 7 (No. 4), 463-492, 2003. (A special issue in honor of Professor Y-T Siu's 60th birthday).

[HJ] X. Huang and S. Ji, *On some rigidity problems in Cauchy-Riemann analysis*, *AMS/IP Studies in Advanced Mathematics* Vol. 39, 89-107, 2007.

[HJX1] X. Huang, S. Ji and D. Xu, *Several results for holomorphic mappings from* $\mathbf{B}^n$ *into* $\mathbf{B}^N$, *Contemporary Math.*, a special issue in honor of F. Treves, 267-293, 2005.

[HJX2] X. Huang, S. Ji and D. Xu, *A new gap phenomenon for proper holomorphic mappings from* $\mathbb{B}^n$ *to* $\mathbb{B}^N$, Math Research Letter, **3**(2006), no. 4, 509-523.

[HK] X. Huang and S. Krantz, *On a problem of Moser*, *Duke Math. J.* (78), 213-228, 1995.

[HJY] X. Huang, S. Ji and W. Yin, *On the third gap for proper holomorphic maps between balls*, preprint, 2007.

[HY] X. Huang and W. Yin, *A Bishop surface with a vanishing Bishop invariant*, preprint, 2007.

[KW1] C. Kenig and S. Webster, *The local hull of holomorphy of a surface in the space of two complex variables*, *Invent. Math. 67*, 1-21, 1982.

[KW2] C. Kenig and S. Webster, *On the hull of holomorphy of an n-manifold in* $\mathbb{C}^n$, *Annali Scoula Norm. Sup. de Pisa IV Vol. 11 (No. 2)*, 261-280, 1984

[MMZ] F. Meylan, N. Mir and D. Zaitsev, Approximation and convergence of formal CR-mappings, International Mathematics Research Notices 2003, no. 4, 211-242.

[Mir] N. Mir, *On the convergence of formal mappings*, *Comm. Anal. Geom. 10* (2002), 23–59.

[Mos] J. Moser, *Analytic surfaces in* $\mathbb{C}^2$ *and their local hull of holomorphy*, *Annales Aca -demiæFennicae Series A.I. Mathematica* (10), 397-410, 1985.

[MW] J. Moser and S. Webster, *Normal forms for real surfaces in* $\mathbb{C}^2$ *near complex tangents and hyperbolic surface transformations*, *Acta Math.* (150), 255-296, 1983.

[Po] H. Poincaré, *Les fonctions analytiques de deux variables et la représentation conforme*, *Ren. Cire. Mat. Palermo, II. Ser. 23*, 185-220, 1907.

[Vor] S. M. Voronin, *Analytic classification of germs of conformal mappings* $(\mathbf{C}, 0)$ *to* $(\mathbf{C}, 0)$, *Funct. Anal. Appl.* 15 (1), 1-13, 1981.

Xiaojun Huang (huangx@math.rutgers.edu), Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA.

Shanyu Ji (shanyuji@math.uh.edu), Department of Mathematics, University of Houston, Houston, TX 77204, USA.

Wanke Yin, School of Mathematics, Wuhan University, Hubei 430072, China.