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Non-abelian cohomology of abelian Anosov actions

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Abstract. We develop a new technique for calculating the first cohomology of certain classes of actions of higher-rank abelian groups ($\mathbb{Z}^k$ and $\mathbb{R}^k$, $k \geq 2$) with values in a linear Lie group. In this paper we consider the discrete-time case. Our results apply to cocycles of different regularity, from Hölder to smooth and real-analytic. The main conclusion is that the corresponding cohomology trivializes, i.e. that any cocycle from a given class is cohomologous to a constant cocycle. The principal novel feature of our method is its geometric character; no global information about the action based on harmonic analysis is used. The method can be developed to apply to cocycles with values in certain infinite dimensional groups and to rigidity problems.

1. *Introduction*

1.1. *Basic definitions.* Let G be a group acting on a compact boundaryless Riemannian manifold M by $\alpha : G \times M \to M$, $(g, x) \mapsto \alpha_g(x) \equiv gx$. Let Γ be some topological group. A cocycle β over the action α is a continuous function $\beta : G \times M \to \Gamma$ such that

$$\beta(g_1 g_2, x) = \beta(g_1, g_2 x)\beta(g_2, x), \tag{1.1}$$

for all $g_1, g_2 \in G$, $x \in M$.

A geometric interpretation of a cocycle is the following: consider the trivial principal Γ-bundle $E = M \times \Gamma$ over M. Then the cocycle β described above corresponds to a lift of the action α to an action $\tilde{\alpha} : G \times E \to E$ by principal bundle maps. Namely,

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$g \in G$ induces the map $\tilde{\alpha}_g : E \to E$ given by $(x, h) \mapsto (\alpha_g(x), \beta(g, x)h)$. The cocycle equation (1.1) is equivalent to the fact that $\tilde{\alpha}$ is an action, i.e. $\tilde{\alpha}_{g_1}\tilde{\alpha}_{g_2} = \tilde{\alpha}_{g_1 g_2}$.

If $\Gamma = \mathrm{Aut}(F)$ for some space F, then a cocycle $\beta : G \times M \to \Gamma$ also corresponds to a lift of α to an action by bundle maps on the trivial bundle $M \times F$. In this case $g \in G$ acts by $(x, \xi) \mapsto (\alpha_g(x), \beta(g, x)(\xi))$. Here 'Aut(·)' has the meaning appropriate for the structure of F. It can be GL(·) for F a linear space, or Diff(·) for F a manifold.

The natural equivalence relation for cocycles is the cohomology. Two cocycles β_1 and β_2 are called cohomologous if there exists a continuous map $P : M \to \Gamma$ such that

$$\beta_1(g, x) = P(gx)\beta_2(g, x)P(x)^{-1}, \tag{1.2}$$

for all $g \in G$, $x \in M$. Such a map P is called a *transfer map*.

A cocycle β is cohomologous to a constant cocycle if there exists a continuous function $P : M \to \Gamma$ and a homomorphism $\pi : G \to \Gamma$ such that

$$\beta(g, x) = P(gx)\pi(g)P(x)^{-1}.$$

In particular, if π is the trivial homomorphism, β is said to be cohomologous to the trivial cocycle. In order for a cocycle β to be cohomologically trivial, it has to satisfy the *closing conditions*: $\beta(g, x) = \mathrm{Id}_\Gamma$ for all $g \in G$ and $x \in M$ such that $gx = x$.

One of the central questions in studying cocycles over group actions is: *When is a cocycle cohomologous to a constant (trivial) cocycle?*

Remark. For brevity, we use the term *small* for a cocycle whose values are close to the identity, on a compact generating set in the group whose action we consider.

We recall the definition of a partially hyperbolic diffeomorphism.

Let M be a compact manifold. A C^1 diffeomorphism $T : M \to M$ is called *partially hyperbolic* if there is a continuous invariant splitting of the tangent bundle $TM = E^s(T) \oplus E^0(T) \oplus E^u(T)$ and constants $C = C(T)$, $\lambda_\pm = \lambda_\pm(T)$, $\widetilde{\lambda}_\pm = \widetilde{\lambda}_\pm(T)$, $C > 0$, $0 < \lambda_- < \widetilde{\lambda}_- \le \widetilde{\lambda}_+ < \lambda_+$, $\lambda_- < 1 < \lambda_+$, such that for $n \in \mathbb{Z}$, $n \ge 0$:

$$\begin{aligned}
\|DT^n v^s\| &\le C\lambda_-^n \|v^s\|, \quad v^s \in E^s(T),\\
\|DT^{-n} v^u\| &\le C\lambda_+^{-n} \|v^u\|, \quad v^u \in E^u(T),\\
\|DT^{-n} v^0\| &\le C\widetilde{\lambda}_-^{-n} \|v^0\|, \quad v^0 \in E^s(T),\\
\|DT^n v^0\| &\le C\widetilde{\lambda}_+^n \|v^0\|, \quad v^0 \in E^s(T).
\end{aligned}$$

If $E^0 = \{0\}$ then the diffeomorphism T is called *Anosov*.

The sub-bundles $E^s(T)$ and $E^u(T)$ are called the *stable* and, respectively, *unstable*, distributions. These distributions are integrable. We denote by $W^s(x; T)$ and $W^u(x; T)$, respectively, the stable and unstable manifolds of the point $x \in M$. The stable and unstable foliations are Hölder foliations. If the diffeomorphism $T \in C^K(M)$, then the leaves of the stable and unstable foliations are C^K too.

1.2. *Historic remarks and outline.* The study of cocycles over (transitive) Anosov diffeomorphisms and flows (i.e. actions of $\mathbb{Z}$ and $\mathbb{R}$, respectively) was started in two papers by Livsic (more appropriately spelled Livshits) [**Li1, Li2**], which became very influential and generated extensive literature.

Livsic proved that a real-valued Hölder cocycle which satisfies the closing conditions is cohomologous to the trivial cocycle [**Li1**]. He also proved a similar result for small cocycles with values in a finite-dimensional Lie group [**Li2**]. In the same paper he claims a global result for cocycles with values in arbitrary Lie groups. His argument works for solvable groups but it is mistaken for the general case. This question is still open.

Some early applications of Livsic's results appeared in [**LS**], where, in particular, a necessary and sufficient condition for the existence of an absolutely continuous invariant measure for an Anosov system is given.

Several major developments followed the work of Livsic. One direction is concerned with the regularity of the (essentially unique, if it exists) solution P of the cohomological equation

$$\beta(g, x) = P(gx)P(x)^{-1}.$$

Livsic showed that if a real-valued cocycle β over an Anosov system is C^1, then the transfer map P is also C^1 [**Li1**]. For some linear actions on a torus he also showed that if the cocycle is C^∞, respectively C^ω, then so is the solution [**Li2**]; this was obtained by studying the decay of the Fourier coefficients.

Later Guillemin and Kazhdan [**GK1, GK2**] showed the C^∞ regularity of the solutions in the case of geodesic flows on negatively curved surfaces. Collet, Epstein and Gallavotti [**CEG**] proved a C^ω version for geodesic flows on surfaces of constant negative curvature.

The complete solution for the C^∞ case appears in the paper by de la Llave, Marco and Moriyon [**LMM**]. They showed that if a real-valued cocycle over a C^∞ Anosov system is cohomologically trivial and C^∞, then the transfer map is C^∞. This follows from a general theorem from harmonic analysis which asserts that if a function is smooth along two transverse foliations which are absolutely continuous and whose Jacobians have some regularity properties, then it is smooth globally. This theorem was proved in [**LMM**] using properties of elliptic operators. Later a more general result was proved by Journé [**J**], relying mainly on Taylor expansions and the estimate of the error: if a function is $C^{K+\alpha}$ along the leaves of two transverse foliations with uniformly smooth leaves, then the function is $C^{K+\alpha}$, $(0 < \alpha < 1,\ K = 1, 2, \ldots, \infty)$. Another approach is presented by Hurder and Katok [**HK**], based on an unpublished idea of C. Toll, in which the decay of the Fourier coefficients is used to characterize smoothness. The method can be applied for spanning families of foliations which have the same property as those used in [**LMM**]. Note that foliations arising from Anosov diffeomorphisms have this property. Using the approach in [**HK**], de la Llave proved the analytic case in [**Ll1**].

In [**NT1**] the second and third authors proved that a small cocycle with values in the diffeomorphism group of a compact manifold with trivial tangent bundle is cohomologous to the trivial cocycle, provided the closing conditions hold. The regularity results were extended to cocycles with values in Diff and Lie groups in [**NT2, NT3**]. See Theorem 5.5 later for such a statement. The results in [**NT3**] are optimal, as far as the initial regularity of the transfer map is concerned. Several improvements of [**NT1**] are presented in [**Ll2**], as well as a different treatment of Livsic's results.

For Anosov actions of groups other than $\mathbb{Z}$ and $\mathbb{R}$ the situation may be quite different. While in the above cases the closing conditions imply infinitely many independent obstructions to trivialization, for actions of many other groups various rigidity phenomena appear. For 'large' groups, such as lattices in higher-rank Lie groups, this is related to the super-rigidity theorem of Zimmer [**Z**] and are not a consequence of hyperbolicity. For other groups (e.g. free non-abelian groups) rigidity does not take place. Note, however, that for *generic* actions of any group that contain a transitive Anosov element, the closing conditions still imply triviality of the cocycle (see [**NT1**, §5]).

On the other hand, for actions of higher-rank abelian groups (e.g. $\mathbb{Z}^k$ and $\mathbb{R}^k$ for $k \geq 2$), cocycle rigidity appears in connection with hyperbolic behavior.

Nevertheless, the proofs of these rigidity results relied on harmonic analysis (abelian and non-abelian), more specifically on the exponential decay of Fourier coefficients for smooth functions on a torus and exponential decay of matrix coefficients for irreducible representations of semisimple Lie groups. Using these methods, Katok and Spatzier showed in [**KSp1, KSp2, KSp3**] that real-valued cocycles over certain Anosov $\mathbb{R}^k$ and $\mathbb{Z}^k$ actions, $k \geq 2$, are cohomologous to constant cocycles. Related results for expansive $\mathbb{Z}^k$ actions by automorphisms of compact abelian groups were found by Katok and Schmidt [**KSch**], and for higher-dimensional shifts of finite type were found by Schmidt [**Sch1, Sch2**]. Katok and Katok proved in [**KK**] similar results for higher-order cohomologies.

A different approach was suggested by Katok in the spring of 1994, based on the notion of TNS (i.e. *totally non-symplectic*) $\mathbb{Z}^k$ action. His original argument provided a geometric (i.e. independent of harmonic analysis) proof for some of the results in [**KSp1, KSp2**]. This method does not require algebraicity of the action, but assumes a special structure of the stable and unstable manifolds of various elements of the action. Using the notion of TNS actions, Niţică and Török proved cocycle rigidity for some Diff- and Lie-valued cocycles.

The current paper represents an account of these developments. We restrict ourselves to the case of $\mathbb{R}$ and Lie-group valued cocycles. Our results give a partial answer to a question asked by Katok and Spatzier in the introduction of [**KSp1**] about the generalization of their rigidity results to cocycles with values in non-abelian groups. The results for cocycles with values in diffeomorphism groups are be presented elsewhere [**NT4**]. These are used in the study of partially hyperbolic actions of higher-rank abelian groups, and to prove local rigidity of some partially hyperbolic actions of lattices in higher-rank Lie groups.

We describe the necessary notions and formulate the results in §2. In §3 we consider in detail the case of real-valued C^∞ cocycles. This emphasizes the main geometric idea of the method, namely that the expected solution of the cohomological equation is first constructed as a differential 1-form, the TNS condition implying that the form is closed. The fact that this form is exact follows from the hyperbolicity of the induced action on homology. This method can be extended to some situations where the TNS condition does not hold, e.g. Weyl chamber flows [**FK**]. In that case the constant cocycles do not correspond to closed forms anymore, however, their exterior derivatives are of a particular form, and one can show that for an arbitrary sufficiently smooth cocycle the exterior derivative of the corresponding form is also of this special form.

In §4 we gathered some general results, independent of the TNS property, which are used for the case of Lie-group valued or Hölder cocycles. As a substitute for the differential 1-form, one constructs an invariant foliation by putting together the 'stable' and 'unstable' foliations of the generators of the skew-product action determined by the cocycle. In §5 we complete the proofs of the theorems given in §2. Due to certain technical difficulties, for Hölder cocycles we restrict ourselves to the case of an action on a torus. The results for smooth cocycles are proven for actions on infranilmanifolds.

In §6 we use the main result to show that the derivative cocycle of a small C^1 perturbation of a linear TNS $\mathbb{Z}^k$ action on a torus is cohomologous to a constant cocycle via a Hölder transfer map. The derivative cocycle of such an action is not a small cocycle, but one can reduce it to that case by considering the splitting into Lyapunov spaces. Some examples and related questions are included in §7.

2. *The main results*

The only manifolds which are known to admit Anosov diffeomorphisms are tori, nilmanifolds and infranilmanifolds. It is an outstanding conjecture that these are the only ones supporting Anosov diffeomorphisms (see [**F2**]).

A nilmanifold is the quotient of a connected, simply connected nilpotent Lie group N by a lattice Γ. All such lattices are cocompact, torsion free and finitely generated (see [**Ra**, Theorems 2.1 and 2.18]). An infranilmanifold is finitely covered by a nilmanifold. More precisely, let N be a connected, simply connected nilpotent Lie group and C a compact group of automorphisms of N. Let Γ be a torsion-free cocompact discrete subgroup of the semi-direct product NC. Recall that an element (x, c) of NC (where $x \in N$ and $c \in C$) acts on N by first applying c and then left translating by x. By a result of Auslander (see [**A**]), $\Gamma \cap N$ is a cocompact discrete subgroup of N and $\Gamma \cap N$ has finite index in Γ. The quotient space N/Γ is a compact manifold called an infranilmanifold.

Anosov diffeomorphisms on nilmanifolds and infranilmanifolds were introduced in [**Sm, F2, Sh**]. Let $\bar{f} : NC \to NC$ be an automorphism for which $\bar{f}(\Gamma) = \Gamma$, $\bar{f}(N) = N$. Then $\bar{f}$ induces a diffeomorphism $f : N/\Gamma \to N/\Gamma$, called an *infranilmanifold automorphism*. If the derivative $D\bar{f}|_N$ at the identity is hyperbolic, i.e. has all the eigenvalues of absolute value different from one, then f is an Anosov diffeomorphism. Note that in this case the stable and unstable distributions are smooth.

In the sequel we consider $\mathbb{Z}^k$ actions only on infranilmanifolds.

Definition. We call an action *linear* if it is given by infranilmanifold automorphisms.

Recall the Franks–Manning classification of Anosov diffeomorphisms on infranilmanifolds (see [**F1, Man**] for the case of a $\mathbb{Z}$ action and [**H1**, proof of Proposition 2.18] for the case of a $\mathbb{Z}^k$-action). Let M be an infranilmanifold and $\alpha : \mathbb{Z}^k \times M \to M$ an abelian C^1 action containing an Anosov diffeomorphism. Assume that α has a fixed point x_0. Then the action α is Hölder conjugate to the linear $\mathbb{Z}^k$ action $\bar{\alpha} : \mathbb{Z}^k \times M \to M$ given by automorphisms induced by the map in homotopy $\alpha_* : \mathbb{Z}^k \times \pi_1(M, x_0) \to \pi_1(M, x_0)$. Note that the action always has a periodic point. In general, the action is Hölder conjugate to an affine action, whose restriction to a subgroup of $\mathbb{Z}^k$ of finite index is an action by linear automorphisms. Recall that Hurder constructed in [**H2**] abelian Anosov actions on the torus by affine maps without fixed points.

Let $\alpha : \mathbb{Z}^k \times M \to M$ be an abelian C^K action. View α as a homomorphism from $\mathbb{Z}^k$ into $\mathrm{Diff}^K(M)$ and denote by $\mathcal{A} \subset \mathrm{Diff}^K(M)$ its image.

In order to obtain the rigidity results about cocycles over $\mathbb{Z}^k$ actions, we introduce the following.

Definition. We say that an action α is TNS, if there is a family S of partially hyperbolic elements in $\mathcal{A}$ and a continuous splitting of the tangent bundle $TM = \oplus_{i=1}^m E_i$ into $\mathcal{A}$-invariant distributions such that:

(i) the stable and unstable distributions of any element in S are direct sums of a subfamily of the E_i's;

(ii) any two distributions E_i and E_j, $1 \le i, j \le m$, are included in the stable distribution of some element in S.

If, moreover, the action α is C^∞ and each distribution E_i is smooth, we say that the action is *smoothly-TNS*.

Remarks.

(1) It is easy to see that given a TNS action, one can assume that S consists only of Anosov elements.

(2) Given a TNS action described by $S \subset \mathcal{A}$ with all elements of S Anosov and a splitting $TM = \oplus_{i=1}^m E_i$, one can replace the distributions $\{E_i\}$ by the non-zero intersections $\bigcap_{a \in S} E^{\sigma(a)}(a)$, where $\sigma(a) \in \{u, s\}$. Indeed, denote the new splitting by $TM = \oplus_{i=1}^k F_i$. It obviously satisfies (i), and (ii) can be checked as follows: given F_i and F_j, there are $1 \le i', j' \le m$ such that $E_{i'} \subset F_i$ and $E_{j'} \subset F_j$ and $a \in S$ such that $E_{i'}, E_{j'} \subset E^s(a)$; then $F_i, F_j \subset E^s(a)$, by the choice of the new splitting.

If the original splitting was smooth, so will be the new one.

(3) In view of the above, one can always assume that the distributions E_i are integrable.

(4) By Remarks (1) and (2), any linear TNS action on an infranilmanifold is actually smoothly-TNS. If the linear action is on a torus, one can assume that the distributions E_i are constant (i.e. given by translates of some fixed vector subspaces).

(5) Consider a TNS $\mathbb{Z}^k$ action α on an infranilmanifold M. Since it contains Anosov elements, there is a subgroup $\Gamma \subset \mathbb{Z}^k$ of finite index acting with a fixed point, say x_0, and by the Franks–Manning classification $\alpha|_\Gamma$ is conjugated to the linear action $\bar{\alpha} := (\alpha|_\Gamma)_*$ induced on $\pi_1(M, x_0)$. Using Remark (1) and the fact that the elements of $S \subset \mathbb{Z}^k$ can be replaced by their powers, one can assume that $S \subset \Gamma$ and S consists of Anosov elements only. Then, by Remark (2), the action $\bar{\alpha}$ is TNS as well, because the TNS property can be described in terms of the intersections of the stable and unstable foliations of the elements of S.

Let $G \subset \mathrm{GL}(d, \mathbb{R})$ be a closed subgroup, with the metric induced by the matrix norm on $\mathrm{GL}(d, \mathbb{R})$.

The following theorems apply for G-valued cocycles that are small. However, the smallness assumption is not necessary in the proof if $G = \mathbb{R}$. Since any $\mathbb{R}$-valued cocycle can be made arbitrarily small by multiplying it by some non-zero number, we will not make this distinction in the sequel.

For Hölder cocycles, our result is as follows.

THEOREM 2.1. *Let M be a torus and $\alpha : \mathbb{Z}^k \times M \to M$ a TNS action. Let $\beta : \mathbb{Z}^k \times M \to G$ be a small δ-Hölder cocycle over α. Then β is cohomologous to a constant cocycle, i.e. there is a δ-Hölder function $P : M \to G$ and a representation $\pi : \mathbb{Z}^k \to G$ such that*

$$\beta(a, x) = P(ax)^{-1}\pi(a)P(x).$$

Moreover, if α and β are C^K, $K = 1, 2, \ldots, \infty, \omega$, then P is $C^{K-\varepsilon}$, for any small $\varepsilon > 0$. ($K - \varepsilon = K$ for $K \in \{1, \infty, \omega\}$).

The main part of the proof is to deal with Hölder cocycles over a linear action.

THEOREM 2.2. *Let M be a torus and $\alpha : \mathbb{Z}^k \times M \to M$ a linear TNS action. Let $\beta : \mathbb{Z}^k \times M \to G$ be a small δ-Hölder cocycle over α. Then β is cohomologous to a constant cocycle through a δ-Hölder transfer map.*

The reduction to this case essentially involves the Franks–Manning classification and previous regularity results.

For C^∞ cocycles we do not have to require that the manifold be a torus.

THEOREM 2.3. *Let M be an infranilmanifold and $\alpha : \mathbb{Z}^k \times M \to M$ a smoothly-TNS action. Let $\beta : \mathbb{Z}^k \times M \to G$ be a small C^∞ cocycle over α. Then β is cohomologous to a constant cocycle through a C^∞ transfer map.*

Moreover, if α and β are C^ω, then the transfer map is C^ω.

Remark. As can be seen from the proof, a similar result holds for cocycles that are only finitely smooth. In that case there is a loss of regularity for the transfer map.

We now introduce some notation which will be used in the sequel.

Let a be a partially hyperbolic diffeomorphism. We denote by $\lambda_\pm(a)$ the contraction and expansion coefficients of a, defined by

$$\begin{aligned} \lambda_-(a) &:= \lim_{n\to\infty} \|D(na)|_{E^s(a)}\|^{1/n}, \\ \lambda_+(a) &:= \lim_{n\to\infty} \|D(na)^{-1}|_{E^u(a)}\|^{-1/n}. \end{aligned} \tag{2.1}$$

Let $\beta : \mathbb{Z}^k \times M \to \mathrm{GL}(d, \mathbb{R})$ be a cocycle and $a \in \mathbb{Z}^k$. We denote by $\mu_\pm(a) = \mu_\pm(\beta, a)$ the contraction and expansion coefficients for $\beta|_{\langle a\rangle}$, defined by

$$\begin{aligned} \mu_-(a) &:= \lim_{n\to\infty} \inf_{x\in M} \|\beta(na, x)^{-1}\|^{-1/n}, \\ \mu_+(a) &:= \lim_{n\to\infty} \sup_{x\in M} \|\beta(na, x)\|^{1/n}. \end{aligned} \tag{2.2}$$

Note that $\inf_{x\in M} \|\beta(a, x)^{-1}\|^{-1} \le \mu_-(a) \le \mu_+(a) \le \sup_{x\in M} \|\beta(a, x)\|$.

If the cocycle takes values in $\mathbb{R}$ (which we see as the additive group) then $\mu_\pm = 1$ because β acts by translations.

If W is a foliation of M and $x \in M$, denote by $W_{\mathrm{loc}}(x)$ the path-connected component of $\{y \in W(x) \mid \mathrm{dist}_M(x, y) < \delta_0\}$ which contains x, where $\delta_0 > 0$ is small and fixed. The constant δ_0 is called the *size* of the local foliation.

3. *Proof for real-valued cocycles*

We prove here a special case of Theorem 2.3 in order to illustrate the main geometric idea of the method (as mentioned in the introduction).

THEOREM 3.1. *Let M be an infranilmanifold and $\alpha : \mathbb{Z}^k \times M \to M$ a linear TNS $\mathbb{Z}^k$ action. Let $\beta : \mathbb{Z}^k \times M \to \mathbb{R}$ be a C^∞ cocycle over α. Then β is cohomologous to a constant cocycle through a C^∞ transfer map.*

By Remark (4) in §2, we can assume that the distributions E_i are smooth.

The proof will follow from a sequence of lemmas. The TNS property is required only for Lemma 3.3.

Assume that $\beta : \mathbb{Z}^k \times M \to \mathbb{R}$ is a real-valued cocycle over the TNS linear action α, i.e.

$$\beta(a_1 + a_2, x) = \beta(a_1, a_2 x) + \beta(a_2, x), \quad \text{for all } a_1, a_2 \in \mathbb{Z}^k,\ x \in M.$$

We want to show that, under certain regularity conditions, β is cohomologous to a constant cocycle, i.e. there is a function $P : M \to \mathbb{R}$ and a homomorphism $\pi : \mathbb{Z}^k \to \mathbb{R}$ such that

$$\beta(a, x) = P(ax) + \pi(a) - P(x).$$

The idea of the proof is to construct a C^∞ 1-form on M which is closed and determines a $\mathbb{Z}^k$-invariant class in cohomology. Since the action induced in cohomology is hyperbolic, the above form has to be exact. This allows us to recover the homomorphism π and the transfer map P.

We also mention a second argument, which will be developed in detail for the case of Lie-group valued cocycles. Namely, since the form is closed, it describes a foliation of $M \times \mathbb{R}$ with leaves of dimension $m = \dim M$. Considering the holonomy of this foliation, one can show that the leaves are closed and cover M simply (i.e. the form is actually exact).

Let $a \in \mathbb{Z}^k$ be a hyperbolic element. Assume that $x \in M$ and y is in the stable leaf of a through x, $W^s(x; a)$. Then the following sum is convergent in C^∞ (see, for example, [**LMM**, proof of Lemma 2.2]; note that if β is only Hölder then the sum still converges in C^0)

$$P_a^-(y; x) := -\sum_{n=0}^{\infty} [\beta(a, (na)y) - \beta(a, (na)x)],$$

and we can define a 1-form ω_a^- on $E_x^s(a)$ by taking the differential of P_a^- in the y-variable along the stable leaf. Actually, the differential of $P_a^-(\cdot; x)|_{W^s(x;a)}$ defines the form on the whole $TW^s(x; a)$ and it does not depend on the point x chosen on the stable leaf.

Similarly, for $x \in M$ and $z \in W^u(x; a) = W^s(x; -a)$, let the 1-form ω_a^+ on $E_x^u(a)$ be defined as the z-differential of $P_{-a}^-(z; x)$ along the unstable leaf of a. Consider the form $\omega_a = \omega_a^+ \oplus \omega_a^-$ on $T_x M = E_x^u(a) \oplus E_x^s(a)$.

We will show that for a large set of hyperbolic elements in $\mathbb{Z}^k$ the above construction leads to the same form. Moreover, this form is smooth and closed. We introduce first the notions of the Lyapunov exponent and the Weyl chamber, which we use only in this section (see [**KSp4**] for more details).

The action of the derivative α_* of the action α on the tangent bundle of the universal cover of M is determined by commuting invertible matrices. There are linear functionals

$L_j : \mathbb{Z}^k \to \mathbb{R}$, called Lyapunov exponents, whose values for each $a \in \mathbb{Z}^k$ are given by the logarithms of the absolute values of the eigenvalues of the matrix corresponding to the derivative of $\alpha(a)$. Each Lyapunov exponent can be extended to a linear map $L_j : \mathbb{R}^k \to \mathbb{R}$, also called the Lyapunov exponent. There is a splitting of the tangent bundle into $\mathbb{Z}^k$-invariant sub-bundles $TM = \oplus_j F_j$ such that the Lyapunov exponent of $v \in F_j$ with respect to $\alpha(a)$ is given by $L_j(a)$. We call F_j a *Lyapunov space* or *Lyapunov distribution* for the action. The kernel of each Lyapunov exponent is a hyperplane $\mathcal{H}_j$ in $\mathbb{R}^k$. We denote by $\mathcal{H}_j^-$ the half-space where L_j is negative. The connected components of $\mathbb{R}^k - \cup\mathcal{H}_j$ are called Weyl chambers.

Note that using Lyapunov exponents, the TNS property can be characterized by

$$L_j = cL_i \text{ for some constant } c \implies c > 0.$$

LEMMA 3.2. *Consider a linear $\mathbb{Z}^k$ action α which contains an Anosov element. Then there is a subset $S \subset \mathbb{Z}^k$ of hyperbolic generators of $\mathbb{Z}^k$ which contains elements from each Weyl chamber, and with the property that if $a, b \in S$ then*

$$\omega_a = \omega_b.$$

Proof. Let $\lambda_j := \exp \circ L_j : \mathbb{Z}^k \to [0, \infty)$, and denote by $\mathcal{F}_j$ the foliation corresponding to F_j.

Assume first that $a, b \in \mathbb{Z}^k$ are partially hyperbolic, $F_j \subset E^s(a) \cap E^s(b)$ and $\lambda_j(b)$, the contraction coefficient along F_j, is smaller than the inverse of the Lipschitz norm of $\alpha(a - b)$. Let $z \in \mathcal{F}_j(x) \subset W^s(x; a) \cap W^s(x; b)$. Using the cocycle relation we find that

$$\sum_{k=0}^{n-1} \beta(a, (ka)z) = \beta(na, z),$$

$$\beta(na, z) - \beta(nb, z) = \beta(n(a - b), (nb)z),$$

and similarly for x instead of z. Therefore, in order to show that $P_a^-(z; x) = P_b^-(z; x)$, and consequently that $\omega_a|_{F_j} = \omega_b|_{F_j}$, it is enough to show that

$$\lim_{n\to\infty} (\beta(n(a - b), (nb)z) - \beta(n(a - b), (nb)x)) = 0.$$

But

$$\begin{aligned}
&|\beta(n(a - b), (nb)z) - \beta(n(a - b), (nb)x)| \\
&\quad = \left| \sum_{k=0}^{n-1} (\beta(a - b, [nb + k(a - b)]z) - \beta(a - b, [nb + k(a - b)]x)) \right| \\
&\quad \le \|\beta(a - b, \cdot)\|_{\text{Hölder}} \left[\sum_{k=0}^{n-1} \text{dist}_M(\alpha(nb + k(a - b))(z), \alpha(nb + k(a - b))(x))^\delta \right] \\
&\quad \le \|\beta(a - b, \cdot)\|_{\text{Hölder}} \cdot \lambda_j(b)^{n\delta} \cdot C \cdot (\text{dist}_M(z, x))^\delta \sum_{k=0}^{n-1} \|\alpha(a - b)\|_{\text{Lip}}^{k\delta},
\end{aligned}$$

where C is a constant that is independent of n. Since $\lambda_j(b) < 1$ and $\lambda_j(b) \cdot \|\alpha(a - b)\|_{\text{Lip}} < 1$, the conclusion follows. □

We now construct the set $S \subset \mathbb{Z}^k$. Consider first a finite set F of elements in $\mathbb{Z}^k$ close to the origin, which contains a $\mathbb{Z}$ basis of $\mathbb{Z}^k$. There is a constant $M > 1$ such that $\|\alpha(c)\|_{\mathrm{Lip}} \le M$, for all $c \in F$. Let $L_j : \mathbb{R}^k \to \mathbb{R}$ be the j's Lyapunov exponent and $\mathcal{H}_j$ the hyperplane in $\mathbb{R}^k$ determined by the kernel of λ_j. Then there exist a ball B around the origin and cones $C(\mathcal{H}_j) \subset \mathcal{H}_j^-$ intersecting all Weyl chambers in $\mathcal{H}_j^-$, such that for each j and any element $b \in C(\mathcal{H}_j) \cap (\mathbb{Z}^k - B)$ we have

$$L_j < -\log M,$$

and therefore

$$\lambda_j(b) < M^{-1}. \tag{3.1}$$

Consider two elements $a, b \in C(\mathcal{H}_j) \cap (\mathbb{Z}^k - B)$. We can join a and b by a sequence of elements in $C(\mathcal{H}_j) \cap (\mathbb{Z}^k - B)$ adding at each step an element from F. Formula (3.1) allows us to apply the first part of the proof repeatedly and deduce that

$$\omega_a|_{F_j} = \omega_b|_{F_j}. \tag{3.2}$$

By the construction of the 1-form, (3.2) still holds if a and b are in the union of $C(\mathcal{H}_j)$ with the opposite cone, $-C(\mathcal{H}_j)$.

Define the set S to be

$$S = \left[\bigcap_{j=1}^{m} (C(\mathcal{H}_j) \cup (-C(\mathcal{H}_j)))\right] \cap (\mathbb{Z}^k - B).$$

LEMMA 3.3. *If the linear $\mathbb{Z}^k$ action is TNS then the form $\omega \equiv \omega_a$, $a \in S$ constructed above is smooth and closed.*

Proof. Denote $m = \dim M$. Let $U \subset M$ be a small-enough open set.

Since the distributions E_i are smooth, one can find a frame of smooth vector fields $\{X_j\}_{j=1,m}$ over U such that each field X_j is contained in some E_i. Let $\{\eta_j\}_{j=1,m}$ be the dual frame of 1-forms over U, and write

$$\omega|_U = \sum_{j=1}^{m} f_j \eta_j, \quad \text{where } f_j = \omega(X_j).$$

We will show that each function f_j is smooth along all the distributions E_i and the derivatives are continuous on U. However, this implies that each f_j is smooth on U (in some cases one can use the characterization of smoothness via a Fourier transform, or the theorem of Journé; in general, one needs [**HK**, Theorem 2.6]).

Indeed, in order to show that f_j is smooth along E_i, pick an Anosov element $a \in S$ such that $X_i, E_j \subset E^s(a)$. Since $P_a^-(\cdot, x)$ is smooth along $W^s(x; a)$ and varies continuously in the C^∞ topology with $x \in M$, one concludes that $\omega_a^-|_{E^s(a)}$ is continuously C^∞ along $W^s(a)$. By Lemma 3.2, this proves our assertion.

To show that ω is closed, use again the TNS condition and Lemma 3.2. Clearly $\omega_a^-|_{W^s(x;a)}$ is exact, hence, using the fact that pull-back and exterior differentiation commute,

$$(d\omega)|_{W^s(x;a)} = d(\omega|_{W^s(x;a)}) = 0 \quad \text{for } a \in S.$$

Since over U any two directions X_i and X_j are included in the stable subspace of some hyperbolic element $a \in S$, we obtain that $(d\omega)|_U = 0$. $\square$

LEMMA 3.4. *The cohomology class of ω in $\mathrm{H}^1(M,\mathbb{R})$ is $\mathbb{Z}^k$ invariant, hence it has to be zero, i.e. ω is exact.*

Proof. Let $a \in \mathbb{Z}^k$ be hyperbolic and $\omega_a = \omega_a^+ \oplus \omega_a^-$ on $TM = E^u(a) \oplus E^s(a)$ (as defined at the beginning of this section). Then

$$b^*\omega_a = \omega_a + d\beta(b,\cdot) \tag{3.3}$$

for any diffeomorphism $b \in \mathbb{Z}^k$. Indeed, since $ab = ba$, the cocycle relation (1.1) implies that

$$\beta(a, bt) = \beta(a,t) + \beta(b, at) - \beta(b,t),$$

and therefore

$$\begin{aligned} P_a^-(by, bx) &= P_a^-(y,x) + [\beta(b,y) - \beta(b,x)], \\ P_{-a}^-(bz, bx) &= P_{-a}^-(z,x) + [\beta(b,z) - \beta(b,x)] \end{aligned}$$

for $y \in W^s(x;a)$ and $z \in W^u(x;a)$. Hence, for $\xi \in E_x^s(a)$,

$$\begin{aligned} (b^*\omega_a^-)_x(\xi) &= (\omega_a^-)_{bx}(Db(\xi)) \\ &= d_- P_a^-(\cdot, bx)(Db(\xi)) = d_- P_a^-(b\cdot, bx)(\xi) \\ &= d_-[P_a^-(\cdot, x) + \beta(b,\cdot) - \beta(b,x)](\xi) = \omega_a^-(\xi) + d_-\beta(b,\cdot)(\xi), \end{aligned}$$

where d_- denotes the differential along $E^s(a)$. A similar computation for ω_a^+ completes the proof of (3.3). This shows that the class $\bar\omega \in \mathrm{H}^1(M,\mathbb{R})$ corresponding to ω is $\mathbb{Z}^k$ invariant.

That ω is exact (i.e. that $\bar\omega = 0$) now follows from the fact that any linear hyperbolic automorphism of an infranilmanifold induces a hyperbolic map of the first cohomology group, and therefore the only invariant class is the trivial one.

Indeed, let the infranilmanifold be $M = N/\Gamma$ where $\Gamma \subset NC$ is a lattice, and let $\bar A : NC \to NC$ be an automorphism which leaves invariant both N and Γ, is hyperbolic on N and induces the infranilmanifold automorphism $A : M \to M$ (we use the notations introduced at the beginning of §2). Then $\pi_1(M) = \Gamma$, $\mathrm{H}_1(M,\mathbb{Z}) = \Gamma/[\Gamma,\Gamma]$, $\mathrm{H}_1(M,\mathbb{R}) = \mathrm{H}_1(M,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathrm{H}^1(M,\mathbb{R})$ is the dual of $\mathrm{H}_1(M,\mathbb{R})$ in a natural way, where $[\Gamma,\Gamma]$ is the commutator subgroup of Γ. Note that $\bar A$ invariates $[\Gamma,\Gamma]$, hence it defines a map on $\Gamma/[\Gamma,\Gamma]$, which induces the action of A on $\mathrm{H}_1(M,\mathbb{R})$.

Let $\Gamma_0 := \Gamma \cap N$, which has finite index in Γ and is an $\bar A$-invariant lattice of N. Recall that a lattice in a simply connected nilpotent Lie group and any subgroup of such a lattice is finitely generated [**Ra**, Theorems 2.10 and 2.7].

Since $\Gamma_0/(\Gamma_0 \cap [\Gamma,\Gamma]) \hookrightarrow \Gamma/[\Gamma,\Gamma]$ is of finite index and both are finitely generated abelian groups, $(\Gamma_0/(\Gamma_0 \cap [\Gamma,\Gamma])) \otimes_{\mathbb{Z}} \mathbb{R} \cong (\Gamma/[\Gamma,\Gamma]) \otimes_{\mathbb{Z}} \mathbb{R}$ in a way that identifies the natural actions of $\bar A$. Therefore, it is enough to show that the action of $\bar A$ on $(\Gamma_0/(\Gamma_0 \cap [\Gamma,\Gamma])) \otimes_{\mathbb{Z}} \mathbb{R}$ is hyperbolic. Since $[\Gamma_0,\Gamma_0] \subset \Gamma_0 \cap [\Gamma,\Gamma]$, the above statement follows once we show that $\bar A$ acts hyperbolically on $(\Gamma_0/[\Gamma_0,\Gamma_0]) \otimes_{\mathbb{Z}} \mathbb{R}$, because

$$\Gamma_0/(\Gamma_0 \cap [\Gamma,\Gamma]) \cong \frac{\Gamma_0/[\Gamma_0,\Gamma_0]}{(\Gamma_0 \cap [\Gamma,\Gamma])/[\Gamma_0,\Gamma_0]},$$

and all of the above quotient groups are finitely generated abelian.

Consider the short exact sequence of finitely generated abelian groups

$$\{1\} \to (\Gamma_0 \cap [N,N])/[\Gamma_0,\Gamma_0] \to \Gamma_0/[\Gamma_0,\Gamma_0] \to \Gamma_0/(\Gamma_0 \cap [N,N]) \to \{1\}.$$

Since both $\Gamma_0 \cap [N,N]$ and $[\Gamma_0,\Gamma_0]$ are cocompact in $[N,N]$ [**Ra**, Corollary 1 of Theorem 2.3 and proof of Theorem 2.1], the left-hand group in the above sequence is finite. On the other hand, $\Gamma_0/(\Gamma_0 \cap [N,N]) \hookrightarrow N/[N,N]$ and the action of $\bar{A}$ on the abelian group $N/[N,N]$ is hyperbolic because the derivative of $\bar{A}$ is hyperbolic at the origin of N. These two observations complete the proof of the fact that A acts hyperbolically on $\mathrm{H}_1(M,\mathbb{R})$, hence on $\mathrm{H}^1(M,\mathbb{R})$ as well. □

Once we know that ω is exact, the conclusion of Theorem 3.1 follows easily. Let $P : M \to \mathbb{R}$ be a C^∞ function such that $\omega = dP$ (P can be chosen C^∞ because ω is smooth). From (3.3) we obtain that

$$d[\beta(b,\cdot) - P \circ b(\cdot) + P(\cdot)] = 0$$

for each $b \in \mathbb{Z}^k$. We are done, because this means that the cocycle cohomologous to β given by $\tilde{\beta}(b,\cdot) := \beta(b,\cdot) - P \circ b(\cdot) + P(\cdot) : M \to \mathbb{R}$ is constant for all $b \in \mathbb{Z}^k$. □

4. *Some general results*

We describe in this section a few lemmas and constructions that will be used for the proof of Theorems 2.1, 2.2 and 2.3. The results of this section are independent of the TNS property.

Consider a $\mathbb{Z}^k$ action α on M and a small δ-Hölder cocycle $\beta : \mathbb{Z}^k \times M \to G \subset \mathrm{GL}(d,\mathbb{R})$ over it. The smallness of the cocycle is specified by the conditions given after Lemma 4.2 and by Lemma 4.5.

We can see the cocycle as taking values in $\mathrm{GL}(d,\mathbb{R})$. Moreover, since G was assumed closed and the construction of the transfer map P and of the representation π will involve only limits of products of the cocycle values, it is enough to deal with the case $G = \mathrm{GL}(d,\mathbb{R})$.

Define the extended action $\widetilde{\alpha} : \mathbb{Z}^k \times (M \times \mathrm{GL}(d,\mathbb{R})) \to M \times \mathrm{GL}(d,\mathbb{R})$ by

$$\widetilde{\alpha}(a)(x,g) = (ax, \alpha(a)g).$$

The main step in the proof of the theorems is to construct an $\widetilde{\alpha}$-invariant (topological) foliation $\mathcal{F}_\beta$ of $M \times \mathrm{GL}(d,\mathbb{R})$ with leaves of dimension equal to $\dim M$. It is here where the TNS property plays a role. Then, using a holonomy argument and the hyperbolicity of the action, we show that all the leaves of the foliation are closed manifolds, which cover M simply. This fact and the invariance of the foliation allow us to find the representation π and the transfer map P.

We begin with some results about Hölder cocycles over a partially hyperbolic action.

The following lemma gives a family of invariant foliations for a Hölder cocycle over a partially hyperbolic diffeomorphism.

LEMMA 4.1. *Let a be a partially hyperbolic diffeomorphism of M, β a cocycle over a and $\{W(x)\}_{x\in M}$ an a-invariant foliation of M whose leaves are included in the stable foliation of a.*

(i) *Assume that $\beta(a, \cdot)$ is δ-Hölder and*

$$\lambda_-(a)^\delta < \mu_+(a)^{-1} \cdot \mu_-(a).$$

(Note that this condition is automatically satisfied if the range of the cocycle is a compact Lie group.)
Then, for any $x \in M$, there is a δ-Hölder function $\gamma_x^{a,W} : W(x) \to \mathrm{GL}(d, \mathbb{R})$ such that:
(1) $\gamma_x^{a,W}(x) = I$;
(2) *the family of 'graphs' $\mathbf{W}(x; g) := \{(t, \gamma_x^{a,W}(t)g) \mid t \in W(x)\}$, $x \in M$, $g \in \mathrm{GL}(d, \mathbb{R})$, gives an $\widetilde{\alpha}(a)$-invariant foliation of $M \times \mathrm{GL}(d, \mathbb{R})$.*

These functions are defined by the formula

$$\gamma_x^{a,W}(t) = \lim_{n\to\infty} \beta(na, t)^{-1}\beta(na, x), \quad t \in W(x), \tag{4.1}$$

and depend continuously on the point $x \in M$. Moreover, these are the only functions that are uniformly δ-Hölder on W_{loc} and satisfy conditions (1) and (2).

(ii) *If, moreover, the cocycle β is C^∞ and the foliation $\{W(x)\}$ has smooth leaves varying continuously in the C^∞ topology, then each function $\gamma_x^{a,W}$ is smooth along $W(x)$, with derivatives varying continuously on M.*

Remarks. 1. If the foliation $\{W(x)\}$ is the stable foliation of a then we denote $\gamma_x^{a,W}$ by γ_x^a. By the last statement of the lemma, the functions $\gamma_x^{a,W}$ are the restrictions of γ_x^a to $W(x)$.

2. The proof of part (ii), with $\delta = 1$, is essentially contained in [**NT3**, Theorem 6.1]. The Hölder case is proved along the same lines. One can also prove these results by the methods of [**HPS**, Chapter 5].

Proof of Lemma 4.1(i). In order to simplify the notation, we write γ_x for $\gamma_x^{a,W}$.

The invariance property of the family $\{\mathbf{W}(x; g)\}_{x,g}$ is equivalent to the relation

$$\beta(a, t)\gamma_x(t) = \gamma_{ax}(at)\beta(a, x), \quad t \in W(x), \tag{4.2}$$

or

$$\gamma_x(t) = \beta(a, t)^{-1}\gamma_{ax}(at)\beta(a, x). \tag{4.3}$$

Iterating (4.3) we obtain that

$$\gamma_x(t) = \beta(na, t)^{-1}\gamma_{na(x)}(na(t))\beta(na, x). \tag{4.4}$$

Since $\gamma_{na(x)}(na(t))$ should approach the identity as $n \to \infty$, formula (4.4) suggests the definition (4.1) of γ_x.

Note that it is enough to construct each γ_x on $W_{\mathrm{loc}}(x)$, and then extend them using (4.3).

We prove first the uniqueness of the functions $\{\gamma_x\}_{x\in M}$. Assume $\{\gamma_x\}$ and $\{\tilde{\gamma}_x\}$ are two families that both satisfy the conditions (1) and (2) given in Lemma 4.1. Let $R_x(t) := \tilde{\gamma}_x(t)^{-1}\gamma_x(t)$. Then $R_x(x) = I$ and, by (4.4),

$$R_x(t) = \beta(na, x)^{-1}R_{na(x)}(na(t))\beta(na, x).$$

By restricting W_{loc}, we may assume that the δ-Hölder norm of $R_x|_{W_{\mathrm{loc}}(x)}$ is bounded by some constant $C_* < \infty$, uniformly with respect to $x \in M$.

Choose $\kappa_- > \lambda_-(a)$, $0 < \nu_- < \mu_-(a)$ and $\nu_+ > \mu_+(a)$ such that $\nu_+ \cdot \nu_-^{-1} \cdot \kappa_-^{\delta} < 1$. There is a constant $C > 0$ such that for $n \geq 0$ and $t \in W^s_{\mathrm{loc}}(x; a)$

$$\begin{aligned} \|D(na)|_{E^s(a)}\| &\leq C\kappa_-^n, \\ \mathrm{dist}_M(na(t), na(x)) &\leq C\kappa_-^n \, \mathrm{dist}_M(t, x), \\ \sup_{y \in M} \|\beta(na, y)^{-1}\| &\leq C\nu_-^{-n}, \\ \sup_{y \in M} \|\beta(na, y)\| &\leq C\nu_+^n. \end{aligned}$$

Then, for $t \in W_{\mathrm{loc}}(x)$ and $n \geq 0$,

$$\begin{aligned} \|R_x(t) - I\| &= \|\beta(na, x)^{-1}[R_{na(x)}(na(t)) - R_{na(x)}(na(x))]\beta(na, x)\| \\ &\leq \|\beta(na, x)^{-1}\| \cdot C_* \cdot \mathrm{dist}_M(na(x), na(t))^{\delta} \cdot \|\beta(na, x)\| \\ &\leq C^{2+\delta} C_* (\nu_-^{-1}\nu_+\kappa_-^{\delta})^n \, \mathrm{dist}_M(t, x)^{\delta}, \end{aligned}$$

hence $R_x(\cdot) \equiv I$.

We prove now the existence of the family $\{\gamma_x\}$. Denote $\beta(a, \cdot)$ by $\beta_a(\cdot)$. Consider the functions $\gamma_{x,n} : W(x) \to \mathrm{GL}(n, \mathbb{R})$ given by

$$\gamma_{x,n}(t) := \beta(na, t)^{-1}\beta(na, x).$$

We show that the sequence $\{\gamma_{x,n}\}$ is uniformly Cauchy on $W_{\mathrm{loc}}(x)$. In particular, there is a constant $C_2 > 0$ such that $\sup_{x \in M}\{\|\gamma_x(t)\| \mid t \in W_{\mathrm{loc}}(x)\} < C_2$. Indeed, let $m > n$ be positive integers and $t \in W_{\mathrm{loc}}(x)$. Then:

$$\begin{aligned} \|\gamma_{x,m}(t) - \gamma_{x,n}(t)\| &\leq \sum_{k=n}^{m-1} \|\gamma_{x,k+1}(t) - \gamma_{x,k}(t)\| \\ &= \sum_{k=n}^{m-1} \|\beta(ka, t)^{-1}\beta_a((k+1)a(t))^{-1}\beta_a((k+1)a(x))\beta(ka, x) \\ &\quad - \beta(ka, t)^{-1}\beta_a((k+1)a(t))^{-1}\beta_a((k+1)a(t))\beta(ka, x)\| \\ &\leq \sum_{k=n}^{m-1} C^2 \cdot \nu_-^{-k-1}\nu_+^k \cdot \|\beta_a((k+1)a(x)) - \beta_a((k+1)a(t))\| \\ &\leq C^{2+\delta}\nu_+^{-1} \sum_{k=n}^{m-1} (\nu_-^{-1}\nu_+\kappa_-^{\delta})^{k+1} \|\beta_a\|_{\text{Hölder}} \, \mathrm{dist}_M(t, x)^{\delta} \\ &\leq C_1(\nu_-^{-1}\nu_+\kappa_-^{\delta})^n, \end{aligned}$$

where the constant C_1 does not depend on m, n, x or t.

We show next that the functions $\gamma_x|_{W_{\mathrm{loc}}(x)}$ are δ-Hölder, and their Hölder norm is bounded by some constant C_3, independently of $x \in M$. Let $t, t' \in W_{\mathrm{loc}}(x)$ and $n > 0$.

Then:

$$
\begin{aligned}
&\|\gamma_{x,n+1}(t)-\gamma_{x,n+1}(t')\| \\
&\quad=\|\beta_a^{-1}(t)\cdots\beta_a^{-1}(na(t))\beta_a(na(x))\cdots\beta_a(x) \\
&\qquad-\beta_a^{-1}(t')\cdots\beta_a^{-1}(na(t'))\beta_a(na(x))\cdots\beta_a(x)\| \\
&\quad\le\sum_{k=0}^{n}\|\beta_a^{-1}(t')\cdots\beta_a^{-1}((k-1)a(t'))\beta_a^{-1}(ka(t)) \\
&\qquad\cdot\beta_a^{-1}((k+1)a(t))\cdots\beta_a^{-1}(na(t))\beta_a(na(x))\cdots\beta_a(x) \\
&\qquad-\beta_a^{-1}(t')\cdots\beta_a^{-1}((k-1)a(t))\beta_a^{-1}(ka(t')) \\
&\qquad\cdot\beta_a^{-1}((k+1)a(t))\cdots\beta_a^{-1}(na(t))\beta_a(na(x))\cdots\beta_a(x)\| \\
&\quad\le\sum_{k=0}^{n}\|\beta(ka,t')^{-1}\|\cdot\|\beta_a^{-1}(ka(t'))-\beta_a^{-1}(ka(t))\| \\
&\qquad\cdot\|\gamma_{(k+1)a(x),n-k}((k+1)a(t))\|\cdot\|\beta((k+1)a,x)\| \\
&\quad\le\sum_{k=0}^{n}(C\nu_-^{-k})\cdot(\|\beta_a^{-1}\|_{\text{Hölder}}(C\kappa_-^k\,\mathrm{dist}_M(t',t))^{\delta})\cdot C_2\cdot(C\nu_+^{k+1}) \\
&\quad\le C_3\,\mathrm{dist}_M(t,t')^{\delta},
\end{aligned}
$$

where C_3 does not depend on n, x, t or t'. Now take the limit as $n\to\infty$.

In particular, since $\gamma_x(x)=I$ and it is uniformly δ-Hölder on the local leaves, $\gamma_x(t)$ is an invertible matrix for $t\in W_{\mathrm{loc}}(x)$, $\mathrm{dist}_M(x,t)<C_3^{-1/\delta}$.

The remaining claims follow from the identities $\gamma_{x,n}(t)=\gamma_{x',n}(t)\gamma_{x,n}(x')$ and $\gamma_{x,n+1}(t)=\beta(a,t)^{-1}\gamma_{a(x),n}(a(t))\beta(a,x)$. □

In the following lemma we prove some properties of γ_x^a.

LEMMA 4.2. *Let a and b be two commuting diffeomorphisms which generate the abelian group $\langle a,b\rangle$ in $\mathrm{Diff}^1(M)$. Let $\beta:\langle a,b\rangle\times M\to GL(d,\mathbb{R})$ be a δ-Hölder cocycle. Assume that a is partially hyperbolic and $\lambda_-(a)^\delta<\mu_-(a)\mu_+(a)^{-1}$.*

(i) *If b is partially hyperbolic and $\lambda_-(b)^\delta<\mu_-(b)\mu_+(b)^{-1}$, then*

$$\gamma_x^a|_{W^s(x;a)\cap W^s(x;b)}=\gamma_x^b|_{W^s(x;a)\cap W^s(x;b)};$$

(ii) *$\beta(b,t)\gamma_x^a(t)=\gamma_{bx}^a(bt)\beta(b,x)$, for $t\in W^s(x;a)$;*

(iii) *$\gamma_{x_1}^a(x_n)=\gamma_{x_{n-1}}^a(x_n)\ldots\gamma_{x_k}^a(x_{k+1})\ldots\gamma_{x_1}^a(x_2)$, for $x_1,x_2,\ldots,x_n\in W^s(x;a)$.*

Proof. We derive first (ii). Consider the family $\tilde\gamma_x:W^s(x;a)\to GL(d,\mathbb{R})$ given by

$$\tilde\gamma_x(t):=\beta(b,t)^{-1}\gamma_{bx}^a(bt)\beta(b,x)$$

(since b commutes with a, it invariates the stable foliation of a). Clearly $\tilde\gamma_x(x)=I$. We will show that $\tilde\gamma_x$ satisfies (4.2) and then the uniqueness part of Lemma 4.1 implies that $\tilde\gamma_x=\gamma_x^a$, i.e. (ii).

Indeed, since $ab = ba$, the cocycle equation (1.1) gives $\beta(b, ax)\beta(a, x) = \beta(a, bx)\beta(b, x)$. Together with (4.2), this yields:

$$\begin{aligned}\beta(a,t)\tilde{\gamma}_x(t) &= [\beta(a,t)\beta(b,t)^{-1}]\gamma^a_{bx}(bt)\beta(b,x)\\ &= \beta(b,at)^{-1}[\beta(a,bt)\gamma^a_{bx}(bt)]\beta(b,x)\\ &= \beta(b,at)^{-1}\gamma^a_{ab(x)}(ab(t))[\beta(a,bx)\beta(b,x)]\\ &= [\beta(b,at)^{-1}\gamma^a_{ab(x)}(ab(t))\beta(b,ax)]\beta(a,x)\\ &= \tilde{\gamma}_{ax}(at)\beta(a,x),\end{aligned}$$

as claimed.

To prove (i), notice that γ^a_x satisfies condition (2) (i.e. equation (4.2)) in the characterization of γ^b_x: indeed, this is exactly (ii). Therefore, in view of the Remark following Lemma 4.1, we obtain the equality (i) by applying again the uniqueness part of Lemma 4.1 for b, the b-invariant foliation $W := W^s(x; a) \cap W^s(x; b) \subset W^s(x; b)$ and $\gamma^a_x|_W$.

Finally, (iii) follows from formula (4.1), the definition of γ^a_x. □

After these preliminaries, we describe the construction of the foliation mentioned at the beginning of this section. More precisely, we will construct a family of plaques. This construction requires only one Anosov diffeomorphism; the TNS condition is used to show that the result is indeed a foliation.

Consider a δ-Hölder cocycle β which is small enough for $\lambda_-(a)^\delta < \mu_-(a)\mu_+(a)^{-1}$ to hold for all $a \in S \cup (-S)$. Further smallness requirements will be imposed by Lemma 4.5. As mentioned in Remark (1) of §2, we may assume that S contains an Anosov element, say a. Due to the product structure of the stable and unstable foliations of a, the following holds:

(P0) there are $K_0 > 0$ and a size δ_0 of the local foliation such that if $x, y \in M$ and $\mathrm{dist}_M(x, y) < \delta_0$, then $W^s_{loc}(x; a) \bigcap W^u_{loc}(x; a)$ contains a unique point, and its distance to both x and y is at most $K_0\, \mathrm{dist}_M(x, y)$.

We want to obtain a continuous $\mathcal{A}$-invariant foliation $\mathcal{F}_\beta$ of $M \times \mathrm{GL}(d, \mathbb{R})$. The leaves are determined locally by graphs of functions $\{F_{U,x}\}_{x\in U}$ to be introduced as follows.

Let $U \subset M$ be a open set of diameter less than δ_0; U is foliated by the local (un)stable manifolds of a. By (P0) for any $x \in U$, $W^s_{\mathrm{loc}}(x; a)$ intersects any local unstable manifold foliating U (not necessarily at a point in U). Then the function $F_{U,x} : U \to \mathrm{GL}(d, \mathbb{R})$ is defined by: if $z \in U$, let u be the unique point in $W^s_{\mathrm{loc}}(x; a) \cap W^u_{\mathrm{loc}}(z; a)$ and set

$$F_{U,x}(z) := \gamma^{-a}_u(z)\gamma^a_x(u)$$

(this function should be denoted $F^a_{U,x}$, but no confusion will arise from the simplification). Note that $F_{U,x}(x) = I$ and $F_{U,x}$ is continuous.

Consider the foliation chart whose plaques (local leaves) are given by the graphs of the functions $F_{U,x}(\cdot)h$ where $h \in \mathrm{GL}(d, \mathbb{R})$. The local leaves can be extended to a global foliation if the standard cocycle condition is satisfied by the foliation charts (see [**Re**]). In our case this is equivalent to the following fact: let $U \subset M$ be as above, $x, y \in U$ and $g_1, g_2 \in \mathrm{GL}(d, \mathbb{R})$; if the graphs of the functions $F_{U,x} \cdot g_1$ and $F_{U,y} \cdot g_2$ have a common point, then the two functions coincide on U.

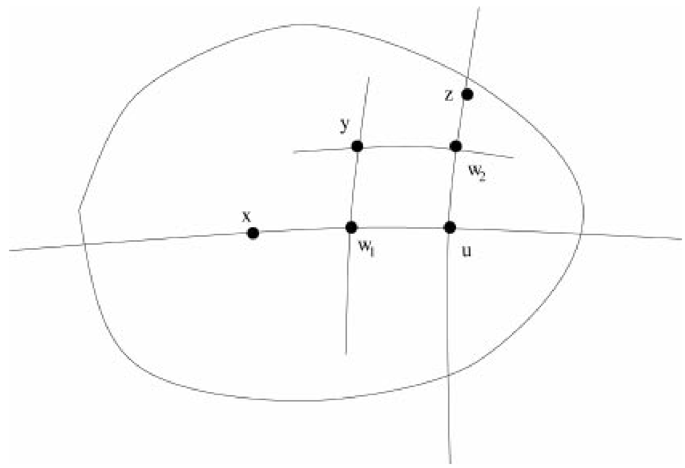

FIGURE 1.

In order to prove this, it is enough to consider the case when $g_1 = I$ and the common point is the center of one of the plaques. Assume therefore that the common point is $(y, F_{U,x}(y))$; then $g_2 = F_{U,x}(y)$, and one has to show that $F_{U,x}(z) = F_{U,y}(z)F_{U,x}(y)$ for $z \in U$.

Let $z \in U$. Denote $u := W^s_{\text{loc}}(x; a) \cap W^u_{\text{loc}}(z; a)$, $w_1 := W^u_{\text{loc}}(y; a) \cap W^s_{\text{loc}}(x; a)$ and $w_2 := W^s_{\text{loc}}(y; a) \cap W^u_{\text{loc}}(z; a)$ (see Figure 1). Using Lemma 4.2(iii),

$$F_{U,x}(z) = \gamma_u^{-a}(z)\gamma_x^a(u) = \gamma_{w_2}^{-a}(z)\gamma_u^{-a}(w_2)\gamma_{w1}^a(u)\gamma_x^a(w_1)$$

and

$$F_{U,x}(y) = \gamma_{w_1}^{-a}(y)\gamma_x^a(w_1), \quad F_{U,y}(z) = \gamma_{w_2}^{-a}(z)\gamma_x^a(w_2).$$

Hence the identity $F_{U,x}(z) = F_{U,y}(z)F_{U,x}(y)$ is equivalent to

$$\gamma_u^{-a}(w_2)\gamma_{w_1}^a(u) = \gamma_y^a(w_2)\gamma_{w_1}^{-a}(y). \tag{4.5}$$

Remark. Our goal is to obtain a foliation by 'integrating' the foliations described by Lemma 4.1 for a and $-a$ (these can be seen, respectively, as the stable and unstable foliations of $\widetilde{\alpha}(a)$). The functions $F_{U,x}$ describe plaques obtained by stacking the unstable leaves along one stable leaf. Equation (4.5) is the standard condition for two foliations to commute, and hence to span together a new foliation.

We postpone the proof of (4.5), respective to the fact that the above construction yields a foliation, to §5 (see Lemmas 5.4 and 5.6). This is where the TNS property is used. We continue with the other results necessary for the proof of the main theorems.

For the rest of this section *we assume that (4.5) holds*, for $\text{diam}_M(U)$ small enough. We denote the obtained foliation by $\mathcal{F}_\beta$.

Once we obtained the foliation, we want to deduce that the cocycle β is constant. The first observation is the following.

LEMMA 4.3. *Assume that (4.5) holds. Then the foliation $\mathcal{F}_\beta$ is $\mathcal{A}$ invariant and has δ-Hölder local leaves.*

Proof. The invariance of $\mathcal{F}_\beta$ is the consequence of the fact that it is obtained by integrating two $\mathcal{A}$-invariant foliations. Indeed, let $b \in \mathcal{A}$. Using Lemma 4.2(ii) for $-a$ and b, respectively a and b, we have:

$$\begin{aligned}\widetilde{\alpha}(b)(z, F_{U,x}(z)h) &= (bz, \beta(b,z)\gamma_u^{-a}(z)\gamma_x^a(u)h) \\ &= (bz, \gamma_{bu}^{-a}(bz)\beta(b,u)\gamma_x^a(u)h) \\ &= (bz, \gamma_{bu}^{-a}(bz)\gamma_{bx}^a(bu)\beta(b,x)h) \\ &= (bz, F_{bU,bx}(bz)\beta(b,x)h)\end{aligned}$$

where $z \in U$, $u = W^s_{\mathrm{loc}}(x;a) \cap W^u_{\mathrm{loc}}(z;a)$ and $h \in \mathrm{GL}(d,\mathbb{R})$. Therefore the local leaves are carried by $\widetilde{\alpha}(b)$ into local leaves, which shows that $\mathcal{F}_\beta$ is $\mathcal{A}$ invariant.

The remaining statement follows from the fact that $F_{U,x}$ is δ-Hölder. To see this, in view of (P0), it is enough to show that $F_{U,x}$ is Hölder when restricted to either $W^s_{\mathrm{loc}}(z;a)$ or $W^u_{\mathrm{loc}}(z;a)$, for any $z \in U$. For the restriction to $W^u_{\mathrm{loc}}(z;a)$ use the definition of $F_{U,x}$ and the fact that γ_u^{-a} is Hölder (see Lemma 4.1). For the restriction to $W^s_{\mathrm{loc}}(z;a)$ use the commutation relation (4.5) to write $F_{U,x}(z) = \gamma_v^a(z)\gamma_x^{-a}(v)$ where $v := W^u_{\mathrm{loc}}(x;a) \cap W^s_{\mathrm{loc}}(z;a)$ and then apply the same argument. □

The next step is to show that $\mathcal{F}_\beta$ has closed leaves. Moreover, these leaves cover simply M under the projection $M \times \mathrm{GL}(d,\mathbb{R}) \to M$.

A leaf is a component of $M \times \mathrm{GL}(d,\mathbb{R})$ in the leaf topology, i.e. the topology induced by the topology of the local leaves. Pick a point $x_0 \in M$ which is fixed by some hyperbolic element of $\mathcal{A}$. Due to the way the foliation $\mathcal{F}_\beta$ was constructed, it is clear that each leaf is a covering space of M. Therefore one can define a group homomorphism $H : \pi_1(M, x_0) \to \mathrm{Maps}(\mathrm{GL}(d,\mathbb{R})_{x_0}, \mathrm{GL}(d,\mathbb{R})_{x_0})$, where $\mathrm{GL}(d,\mathbb{R})_{x_0}$ stands for the fiber over x_0. This map is obtained by associating to a loop $\gamma \in \Omega(M, x_0)$ and $h \in \mathrm{GL}(d,\mathbb{R})_{x_0}$ the endpoint of the lift of γ in $\mathcal{F}_\beta(h)$ starting at h. Since $\mathcal{F}_\beta$ is invariant under right multiplication by $\mathrm{GL}(d,\mathbb{R})$, the range of the above map is actually in $\{\phi : \mathrm{GL}(d,\mathbb{R}) \to \mathrm{GL}(d,\mathbb{R}) \mid \phi(h) = \phi(I)h\} \cong \mathrm{GL}(d,\mathbb{R})$. Hence there is a well defined holonomy map $H : \pi_1(M, x_0) \to \mathrm{GL}(d,\mathbb{R})$. Our next goal is to show that H is the trivial homomorphism, in view of the following lemma.

LEMMA 4.4. *The cocycle β is cohomologous to a constant cocycle via a δ-Hölder transfer map if and only if (4.5) holds and the holonomy of the foliation $\mathcal{F}_\beta$ is trivial.*

If the foliation $\mathcal{F}_\beta$ has smooth leaves, then the transfer map is also smooth.

Proof. Assume first that β is cohomologous to a constant cocycle via a δ-Hölder transfer map. This gives an invariant Hölder foliation which, by the uniqueness result of Lemma 4.1, has to coincide with $\mathcal{F}_\beta$. The statement about the holonomy follows.

For the converse implication, assume that the holonomy H is trivial. Then one can find a global horizontal section $F : M \to \mathrm{GL}(d,\mathbb{R})$ of $\mathcal{F}_\beta$, given by a Hölder function. If the leaves of the foliation are smooth, then F will be smooth too. Note that up to right multiplication by appropriate elements of $\mathrm{GL}(d,\mathbb{R})$, $F|_U$ coincides with $F_{U,x}$ for any $x \in U \subset M$. This F will be the transfer map P. The desired conclusion follows from the invariance of $\mathcal{F}_\beta$ under the action $\widetilde{\alpha}$.

Indeed, let $a \in \mathcal{A}$ and $x, y \in M$. Since $(x, F(x))$ and $(y, F(y))$ are in the same leaf of $\mathcal{F}_\beta$, so are their images under $\widetilde{\alpha}(a)$; i.e. there is some $t \in \mathrm{GL}(d, \mathbb{R})$ such that

$$\widetilde{\alpha}(a)(x, F(x)) = (ax, \beta(a, x)F(x)) = (ax, F(ax)t)$$

and

$$\widetilde{\alpha}(a)(y, F(y)) = (ay, \beta(a, y)F(y)) = (ay, F(ay)t),$$

which shows that

$$F(ax)^{-1}\beta(a, x)F(x) = F(ay)^{-1}\beta(a, y)F(y).$$

Therefore $\pi : \mathcal{A} \to \mathrm{GL}(d, \mathbb{R})$ defined by

$$\pi(a) := F(ax)^{-1}\beta(a, x)F(x)$$

does not depend on x and satisfies $\beta(a, x) = F(ax)\pi(a)F(x)^{-1}$. □

Since $\mathcal{F}_\beta$ is $\widetilde{\alpha}$ invariant, the holonomy map is equivariant in the sense that

$$H(\alpha(a')_*\gamma) = \beta(a', x_0)H(\gamma)\beta(a', x_0)^{-1},$$

for any $\gamma \in \pi_1(M, x_0)$ and $a' \in \mathcal{A}_{x_0} := \{a \in \mathcal{A} \mid a(x_0) = x_0\}$.

Moreover, it is clear from the construction of the holonomy map and the Hölder estimates on the foliation $\mathcal{F}_\beta$ that by requiring the cocycle β to be close enough to the identity one can obtain that a set of generators of $\pi_1(M)$ be mapped by H into an arbitrarily small neighborhood of the identity in $\mathrm{GL}(d, \mathbb{R})$.

These last two properties of the holonomy map imply that H has to be trivial for β small. We prove this in three steps, first for M a torus, then for a nilmanifold and finally for any infranilmanifold. Note that $\mathcal{A}_{x_0}$ does not have to be of rank higher than one.

Although we need the next lemma only for $\mathrm{GL}(d, \mathbb{R})$, we state it for a general finite-dimensional Lie group. We apply this lemma for $H : \pi_1(M, x_0) \to \mathrm{GL}(d, \mathbb{R})$ the holonomy of $\mathcal{F}_\beta$, ρ the action induced by some Anosov element $a \in \mathcal{A}_{x_0}$ on $\pi_1(M, x_0)$ and $\bar{g} = \beta(a, x_0)$. We denote an inner automorphism of a group by $\mathrm{Int}_g : h \mapsto ghg^{-1}$.

LEMMA 4.5. *Let M be an infranilmanifold and $a \in \mathrm{Diff}(M)$ an Anosov diffeomorphism which fixes a point $x_0 \in M$. Fix a set T of generators of $\pi_1(M, x_0)$. Consider the automorphism $\rho \in \mathrm{Aut}(\pi_1(M, x_0))$ induced by a (by the Franks–Manning classification, ρ is 'hyperbolic').*

Given a finite-dimensional Lie group G, there is a neighborhood U of the identity in G with the following property: if $\tilde{\rho} := \mathrm{Int}_{\bar{g}} \in \mathrm{Aut}(G)$ with $\bar{g} \in U$ and $H : \pi_1(M, x_0) \to G$ is a ρ–$\tilde{\rho}$ equivariant homomorphism (i.e. $H \circ \rho = \tilde{\rho} \circ H$) which maps T into U, then H is the trivial homomorphism.

Proof. Case 1: $M = \mathbb{T}^n$. Note that it is enough to prove the conclusion for the canonical set of generators of $\mathbb{Z}^n \cong \pi_1(\mathbb{T}^n)$. We denote it by $T = \{f_i\}_{i=1,\dots,n}$, and let $A = (a_{ij})_{i,j}$ be the hyperbolic matrix $\rho \in \mathrm{Aut}(\pi_1(\mathbb{T}^n)) \cong \mathrm{GL}(n, \mathbb{Z})$.

Denote by $\mathfrak{g}$ the Lie algebra of G. By Ado's theorem [**P**, Lecture 10], we may assume that $\mathfrak{g}$ is the Lie algebra of a matrix Lie group to which G is locally isomorphic. Choose a neighborhood $\mathfrak{U}_0 \subset \mathfrak{g}$ of O, the origin in $\mathfrak{g}$, such that the exponential map $\exp : \mathfrak{U}_0 \subset \mathfrak{g} \to G$ is a diffeomorphism onto its image and its inverse, $\log := \exp^{-1} : \exp(\mathfrak{U}_0) \to \mathfrak{U}_0$, admits a power series expansion on $\exp(\mathfrak{U}_0)$ (see [**P**]). Let $\mathfrak{U}_1 \subset \frac{1}{2}\mathfrak{U}_0$ be a neighborhood of O such that

$$X_i \in \mathfrak{U}_1 \text{ for all } i = 1, \dots, n \implies \sum_{i=1}^{n} a_{ij} X_i \in \mathfrak{U}_0 \text{ for all } j = 1, \dots, n,$$

and set $U := \exp(\mathfrak{U}_1) \cap \{g \in G \mid \operatorname{spec}(\mathrm{Ad}_g) \cap \operatorname{spec}(A) = \emptyset, \mathrm{Ad}_g(\mathfrak{U}_1) \subset \mathfrak{U}_0\}$, where $\mathrm{Ad}_g \in \mathrm{Aut}(\mathfrak{g})$ denotes the differential of Int_g.

Assume now that $H(T) \subset U$ and $\bar{g} \in U$. Let $\mathfrak{f}_i := \log(H(f_i)) \in \mathfrak{U}_1$ and define a linear map $\mathfrak{H} : \mathbb{Z}^n \to \mathfrak{g}$ by $\mathfrak{H}(f_i) = \mathfrak{f}_i$. Since log is given by a power series, $\{\mathfrak{f}_i\}_i \subset \mathfrak{g}$ is a commutative family, hence $\exp \circ \mathfrak{H} = H$. Using the identity $\mathrm{Int}_g(\exp X) = \exp(\mathrm{Ad}_g(X))$ for $g \in G$, $X \in \mathfrak{g}$, the equivariance property of H yields

$$\begin{aligned} \exp(\mathfrak{H}(Af_j)) = H(Af_j) &= \mathrm{Int}_{\bar{g}}(Hf_j) = \mathrm{Int}_{\bar{g}}(\exp(\mathfrak{f}_j)) \\ &= \exp(\mathrm{Ad}_{\bar{g}}(\mathfrak{f}_j)) = \exp(\mathrm{Ad}_{\bar{g}}(\mathfrak{H} f_j)), \end{aligned}$$

for $j = 1, \dots, n$. By our choice of U this implies that $\mathfrak{H} \circ A = \mathrm{Ad}_{\bar{g}} \circ \mathfrak{H}$ (note that $\mathfrak{H}(Af_j) = \sum_{i=1}^{n} a_{ij}\mathfrak{f}_i$). However, this is possible only if $\mathfrak{H} = 0$ because $\mathfrak{H}$ intertwines the linear mappings A and $\mathrm{Ad}_{\bar{g}}$ that have disjoint spectra.

Indeed, assume that the linear maps $A \in \mathrm{End}(E)$, $B \in \mathrm{End}(F)$, $C : E \to F$ satisfy $CA = BC$. Consider the induced maps $\hat{A} \in \mathrm{End}(E/\operatorname{Ker} C)$, $\hat{B} \in \mathrm{End}(\operatorname{Im} C)$ and $\hat{C} : E/\operatorname{Ker} C \to \operatorname{Im} C$. If $C \neq 0$ then $\hat{C}$ is invertible, therefore $\hat{C}\hat{A} = \hat{B}\hat{C}$ implies that $\operatorname{spec}(\hat{A}) = \operatorname{spec}(\hat{B})$, and clearly $\operatorname{spec}(\hat{A}) \subset \operatorname{spec}(A)$, $\operatorname{spec}(\hat{B}) \subset \operatorname{spec}(B)$.

Case 2: M is a nilmanifold. Assume that $M = N/\Gamma$, where N is a connected, simply connected nilpotent Lie group and Γ is a lattice in N.

Then $\pi_1(M) \cong \Gamma$ and $\rho \in \mathrm{Aut}(\Gamma)$ is the restriction to Γ of some automorphism of N whose differential at the identity has no eigenvalues of absolute value one. We denote this automorphism by A.

Via the exponential map we can identify N with its Lie algebra $\mathfrak{n}$; in this identification the automorphism A becomes a linear hyperbolic mapping.

Consider the upper central sequence of normal subgroups

$$N_{m+1} = \{0\} \subsetneqq N_m \subsetneqq \cdots \subsetneqq N_1 = N,$$

where $N_k = [N, N_{k-1}]$. These correspond to Lie subalgebras in $\mathfrak{n}$; A invariates the subgroups N_k and induces hyperbolic automorphisms on both N_k and N/N_k.

We proceed by induction on the depth m of N, assuming only that Γ is a finitely-generated subgroup of a connected, simply connected nilpotent Lie group N and is invariant under a hyperbolic automorphism of N. In particular, such a group Γ is torsion free, because N is (see the proof of Theorem 2.18 in [**Ra**]).

If $m = 1$ then N is abelian and we are done by Case 1. Let $\Gamma_m := \Gamma \cap N_m$, which is normal in Γ. If $\Gamma_m = \{0\}$, then we can reduce the problem to one about $\Gamma \cong \Gamma/N_m \subset N/N_m$.

Assume therefore that $m > 1$ and $\Gamma_m \neq \{0\}$. Then Γ_m is free abelian and finitely generated (see [**Ra**], Theorem 2.7: every subgroup of a finitely generated nilpotent group is finitely generated) and we can invoke Case 1 to obtain that H is trivial on Γ_m for U chosen correspondingly. However, then there is a well defined map $\hat{H} : \Gamma/\Gamma_m \to G$ and the problem is reduced to one about $\Gamma/\Gamma_m \cong \Gamma/N_m \subset N/N_m$.

Case 3: M is an infranilmanifold. Recall that an infranilmanifold is a quotient N/Γ, where N is a connected, simply connected nilpotent Lie group, and Γ is a lattice in the semi-direct product NC of N by a compact group of automorphisms C. Hence $\pi_1(M) \cong \Gamma$. From the previous discussion it follows that $\Gamma \cap N$ is included in the kernel of H, provided U is chosen correspondingly. Consider then $\hat{H} : \Gamma/\Gamma \cap N \to G$, which is a homomorphism of a finite group into a Lie group. Since a Lie group has no small subgroups, we conclude that H is trivial for the appropriate choice of U. □

5. *Proofs for Lie-group valued cocycles*

According to §4, what remains to be proven is the relation (4.5), i.e. the existence of the foliation integrating the stable and unstable foliations of $\widetilde{\alpha}(a)$ for some $a \in S$ Anosov (the leaves of these foliations are given by the graphs of γ_x^a, respectively γ_x^{-a}, $x \in M$).

We will do this for Hölder cocycles over a TNS linear action on a torus in Lemma 5.4, thus proving Theorem 2.2. From this, the Franks–Manning classification and the results of §4 one can deduce Theorem 2.1. The reason we can prove Theorem 2.2 only for actions on a torus is that for (infra-)nilmanifolds the foliations obtained by intersecting stable and unstable foliations of commuting linear Anosov elements need not commute.

For smooth cocycles however, we can decide integrability on the level of distributions, via the theorem of Frobenius. This is done in Lemma 5.6. Note that this parallels the proof of Theorem 3.1: one can prove integrability of the corresponding distribution by showing that the $\mathfrak{g}$-valued 1-form ω on M satisfies the equation $d\omega + \frac{1}{2}\omega \wedge \omega = 0$.

Proof of Theorem 2.2. As mentioned in Remark (4) of §2, we may assume that the distributions E_i are constant. Call the foliations of M corresponding to the E_i's *minimal foliations*.

Since M is a torus, any subset of minimal foliations $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k$ generates an integrable foliation. If the integrable foliation is $\mathcal{F}$, we write $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k\}$.

The following lemma is immediate.

LEMMA 5.1. *Denote by $N(\alpha)$ the number of minimal foliations of the action α.*

There are constants $K_1 > K_0 > 1$, $\varepsilon_0 > 0$, $\delta_1 > 0$ and a size $\delta_0 > 0$ for the local foliations, $\varepsilon_0 < \delta_0 < \delta_1/N(\alpha)$, such that given two disjoint families $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k$ and $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_l$ of minimal foliations and $\mathcal{F} := \{\mathcal{F}_1, \dots, \mathcal{F}_k\}$, $\mathcal{G} := \{\mathcal{G}_1, \dots, \mathcal{G}_l\}$, $\mathcal{H} := \{\mathcal{F}, \mathcal{G}\}$, the following properties hold.

(P1) *For any $x \in M$, $y \in \mathcal{F}^{\rm loc}(x)$, $z \in \mathcal{G}^{\rm loc}(x)$ such that* $\mathrm{dist}_M(y, z) < \varepsilon_0$, *there is a unique $w := \mathcal{F}^{\rm loc}(z) \cap \mathcal{G}^{\rm loc}(y)$, and* $\max\{\mathrm{dist}_M(w, y), \mathrm{dist}_M(w, z)\} \leq K_1\, \mathrm{dist}_M(y, z)$.

(P2) *For any $x, y \in M$ such that* $\mathrm{dist}_M(x, y) < \varepsilon_0$ *and $y \in \mathcal{H}^{\rm loc}(x)$, there is a unique $w := \mathcal{F}^{\rm loc}(x) \cap \mathcal{G}^{\rm loc}(y)$, and* $\max\{\mathrm{dist}_M(w, x), \mathrm{dist}_M(w, y)\} \leq K_1\, \mathrm{dist}_M(x, y)$.

(P3) *If $x, y \in M$ are such that $y \in \mathcal{H}(x)$, x and y can be joined in $\mathcal{H}(x)$ by a path of length less than δ_1 and $\operatorname{dist}_M(x, y) < \delta_0$, then $y \in \mathcal{H}^{\mathrm{loc}}(x)$.*

In the rest of the proof the size of the local foliations will be the δ_0 given by the above lemma.

LEMMA 5.2. *There is a constant ε_1, $0 < \varepsilon_1 < \varepsilon_0$ such that the following holds: given any foliation $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k\}$, where the $\mathcal{F}_i$'s are minimal foliations, and $x \in M$, $z \in \mathcal{F}^{\mathrm{loc}}(x)$ with $\operatorname{dist}_M(x, z) < \varepsilon_1$, there exist $y_1 \in \mathcal{F}_1^{\mathrm{loc}}(x)$, $y_2 \in \mathcal{F}_2^{\mathrm{loc}}(y_1), \dots, y_{k-1} \in \mathcal{F}_{k-1}^{\mathrm{loc}}(y_{k-2})$ such that $z \in \mathcal{F}_k^{\mathrm{loc}}(y_{k-1})$. Moreover,*

$$\max_i\{\operatorname{dist}_M(x, y_i), \operatorname{dist}_M(z, y_i)\} \le K_1 \operatorname{dist}_M(x, z). \tag{5.1}$$

Proof. Let $\varepsilon_1 \le \varepsilon_0/(K_1^2 + K_1)$.

We construct the points y_i as follows. The families of foliations $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_i$ and $\mathcal{F}_{i+1}, \mathcal{F}_{i+2}, \dots, \mathcal{F}_k$ are both integrable; denote $\mathcal{G}_i := \{\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_i\}$ and $\widetilde{\mathcal{G}}_i := \{\mathcal{F}_{i+1}, \mathcal{F}_{i+2}, \dots, \mathcal{F}_k\}$. Then, by (P2) of Lemma 5.1, the local leaves $\mathcal{G}_i^{\mathrm{loc}}(x)$ and $\widetilde{\mathcal{G}}_i^{\mathrm{loc}}(z)$, which are both included in $\mathcal{F}(x)$, intersect in a unique point y_i, which satisfies (5.1) as well.

It remains to show that $y_i \in \mathcal{F}_i^{\mathrm{loc}}(y_{i-1})$. By our choice of ε_1 the distance between the points z and y_{i-1} is smaller than ε_0. Apply property (P2) for the foliations $\mathcal{F}_i$ and $\widetilde{\mathcal{G}}_i$, which span $\widetilde{\mathcal{G}}_{i-1}(z)$, and $y_{i-1} \in \widetilde{\mathcal{G}}_{i-1}^{\mathrm{loc}}(z)$. It follows that $\mathcal{F}_i^{\mathrm{loc}}(y_{i-1})$ and $\widetilde{\mathcal{G}}_i^{\mathrm{loc}}(z)$ intersect in a unique point ζ. Moreover,

$$\begin{aligned}
\operatorname{dist}_M(\zeta, x) &\le \operatorname{dist}_M(\zeta, y_{i-1}) + \operatorname{dist}_M(y_{i-1}, x)\\
&\le K_1 \operatorname{dist}_M(y_{i-1}, z) + K_1 \operatorname{dist}_M(x, z) \le (K_1^2 + K_1) \operatorname{dist}_M(x, z)\\
&\le (K_1^2 + K_1)\varepsilon_1 \le \varepsilon_0 < \delta_0.
\end{aligned}$$

Since $\zeta \in \mathcal{F}_i^{\mathrm{loc}}(y_{i-1})$ and $y_{i-1} \in \mathcal{G}_{i-1}^{\mathrm{loc}}(x)$, it follows that there is a path in $\mathcal{G}_i(x)$ between x and ζ of length at most $2\delta_0$, and then (P3) implies that $\zeta \in \mathcal{G}_i^{\mathrm{loc}}(x)$. However, $\mathcal{G}_i^{\mathrm{loc}}(x)$ intersects $\widetilde{\mathcal{G}}_i^{\mathrm{loc}}(z)$ in a unique point, y_i. Hence ζ has to coincide with y_i, and therefore $y_i \in \mathcal{F}_i^{\mathrm{loc}}(y_{i-1})$. □

Before we prove the main lemma, let us notice that a commutation similar to (4.5) automatically holds in some cases.

LEMMA 5.3. *Let $a, b, c \in S$ be partially hyperbolic diffeomorphisms. Assume $\mathcal{F}_1$ and $\mathcal{F}_2$ are minimal foliations such that $\mathcal{F}_1 \subset W^s(a) \cap W^s(c)$ and $\mathcal{F}_2 \subset W^s(b) \cap W^s(c)$. Then for any $x \in M$, $y \in \mathcal{F}_1^{\mathrm{loc}}(x)$, $z \in \mathcal{F}_2^{\mathrm{loc}}(x)$ and $w = \mathcal{F}_2^{\mathrm{loc}}(y) \cap \mathcal{F}_1^{\mathrm{loc}}(z)$ we have*

$$\gamma_x^c(w) = \gamma_y^b(w)\gamma_x^a(y) = \gamma_z^a(w)\gamma_x^b(z).$$

Proof. Apply first Lemma 4.2(iii) for γ^c and the families of points $\{x, y, w\}$ and $\{x, z, w\}$, and then use Lemma 4.2(i). □

LEMMA 5.4. *There is a constant ε_2, $0 < \varepsilon_2 < \varepsilon_1$ with the following property: let $a \in S$ be any Anosov diffeomorphism and $y, u \in M$ such that $\operatorname{dist}_M(y, u) < \varepsilon_2$. If $w_1 = W^u_{\mathrm{loc}}(y; a) \cap W^s_{\mathrm{loc}}(u; a)$ and $w_2 = W^s_{\mathrm{loc}}(y; a) \cap W^u_{\mathrm{loc}}(u; a)$, then*

$$\gamma_u^{-a}(w_2)\gamma_{w_1}^a(u) = \gamma_y^a(w_2)\gamma_{w_1}^{-a}(y).$$

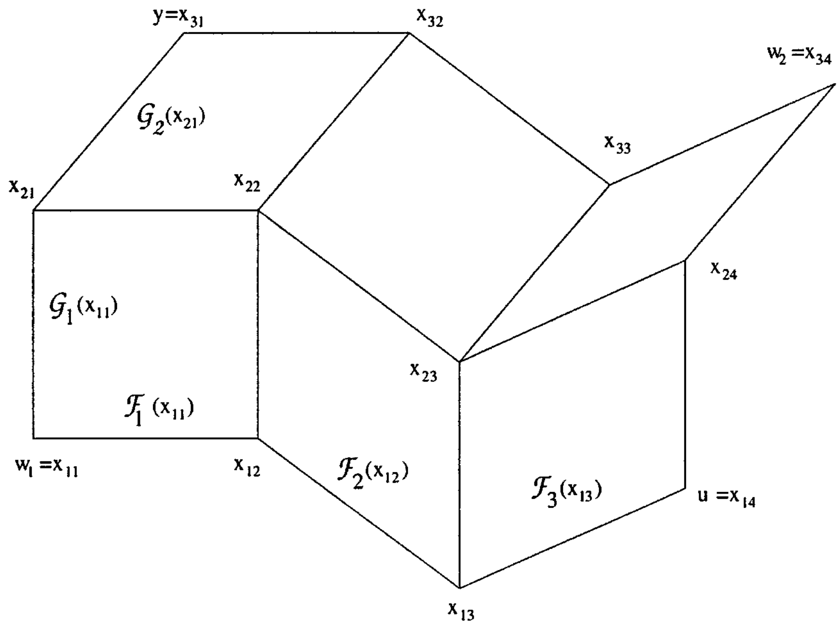

FIGURE 2.

Proof. Let $\{\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k\}$ and $\{\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_l\}$ be the two disjoint families of minimal foliations such that $W^s(a) = \{\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k\}$ and $W^u(a) = \{\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_l\}$.

Use Lemma 5.2 to find $x_{11} = w_1, x_{12}, \dots, x_{1,k+1} = u$ such that

$$x_{12} \in \mathcal{F}_1^{\text{loc}}(w_1), x_{13} \in \mathcal{F}_2^{\text{loc}}(x_{12}), \dots, x_{1,k+1} \in \mathcal{F}_k^{\text{loc}}(x_{1k}).$$

and $x_{21}, \dots, x_{l+1,1} = y$ such that

$$x_{21} \in \mathcal{G}_1^{\text{loc}}(x_{11}), x_{31} \in \mathcal{G}_2^{\text{loc}}(x_{21}), \dots, x_{l+1,1} \in \mathcal{G}_l^{\text{loc}}(x_{l1}).$$

We define recurrently the points x_{ij} for all $1 \le i \le l+1$ and $1 \le j \le k+1$ (Figure 2 illustrates the case $l = 2$ and $k = 3$): given the points x_{ij}, $x_{i+1,j} \in \mathcal{G}_i^{\text{loc}}(x_{ij})$ and $x_{i,j+1} \in \mathcal{F}_j^{\text{loc}}(x_{ij})$, we apply (P1) of Lemma 5.1 to define $x_{i+1,j+1} := \mathcal{F}_j^{\text{loc}}(x_{i+1,j}) \cap \mathcal{G}_i^{\text{loc}}(x_{i,j+1})$. Since there are only a finite number of minimal foliations, by taking ε_2 small enough we can assume that all the points $\{x_{ij}\}$ are in a neighborhood of diameter ε_0 of u and y.

We claim that $x_{l+1,k+1} = w_2$. Indeed, the family of local leaves

$$\mathcal{F}_1^{\text{loc}}(x_{l+1,1}), \mathcal{F}_2^{\text{loc}}(x_{l+1,2}), \dots, \mathcal{F}_k^{\text{loc}}(x_{l+1,k})$$

is contained in $W^s(y; a)$, hence there is a path in $W^s(y; a)$ of length less than $N(\alpha)\delta_0$ connecting $y = x_{l+1,1}$ and $x_{l+1,k+1}$. Since $\text{dist}_M(x_{l+1,k+1}, y) < \varepsilon_0 < \delta_0$, property (P3) implies that $x_{l+1,k+1} \in W^s_{\text{loc}}(y; a)$. Similarly, the family of local leaves

$$\mathcal{G}_1^{\text{loc}}(x_{1,k+1}), \mathcal{G}_2^{\text{loc}}(x_{2,k+1}), \dots, \mathcal{G}_l^{\text{loc}}(x_{l,k+1})$$

is contained in $W^u(u; a)$, hence (P3) and $\text{dist}_M(x_{l+1,k+1}, u) < \varepsilon_0 < \delta_0$ imply that $x_{l+1,k+1} \in W^u_{\text{loc}}(u; a)$. But $W^u_{\text{loc}}(u; a)$ and $W^s_{\text{loc}}(y; a)$ have w_2 as the unique point of intersection, which shows that $x_{l+1,k+1}$ coincides with w_2, as claimed.

Since the action is TNS, for each pair of minimal foliations $\mathcal{F}_j$, $\mathcal{G}_i$ there is a partially hyperbolic diffeomorphism whose stable manifold contains both of them. Therefore each quadruple $\{x_{ij}, x_{i+1,j}, x_{i,j+1}, x_{i+1,j+1}\}$ satisfies the hypothesis of Lemma 5.3 and we obtain that

$$\gamma^{-a}_{x_{i,j+1}}(x_{i+1,j+1})\gamma^{a}_{x_{ij}}(x_{i,j+1}) = \gamma^{a}_{x_{i+1,j}}(x_{i+1,j+1})\gamma^{-a}_{x_{ij}}(x_{i+1,j}). \tag{5.2}$$

But (5.2) implies

$$\begin{aligned}\gamma^{-a}_{x_{l,k+1}}(x_{l+1,k+1})\gamma^{-a}_{x_{l-1,k+1}}(x_{l,k+1})\dots\gamma^{-a}_{x_{1,k+1}}(x_{2,k+1})&\\ \gamma^{a}_{x_{1k}}(x_{1,k+1})\gamma^{a}_{x_{1,k-1}}(x_{1k})\dots\gamma^{a}_{x_{11}}(x_{12})&\\ = \gamma^{a}_{x_{l+1,k}}(x_{l+1,k+1})\gamma^{a}_{x_{l+1,k-1}}(x_{l+1,k})\dots\gamma^{a}_{x_{l+1,1}}(x_{l+1,2})&\\ \gamma^{-a}_{x_{l1}}(x_{l+1,1})\gamma^{-a}_{x_{l-1,1}}(x_{l1})\dots\gamma^{-a}_{x_{11}}(x_{21}).&\end{aligned} \tag{5.3}$$

To see this, define a total order on the set $\{(i, j) \mid 1 \le i \le l+1,\ 1 \le j \le k+1\}$ of indices by: $(i_1, j_1) \prec (i_2, j_2) \iff$ either $j_1 > j_2$ or $j_1 = j_2$ and $i_1 < i_2$. Now transform the left-hand side of (5.3) as follows: for indices ordered increasingly with respect to '$\prec$', at each step substitute the left-hand side of (5.2) by its right-hand side.

Finally, Lemma 4.2(iii) shows that (5.3) is equivalent to (4.5). □

Proof of Theorem 2.1. Let β be a δ-Hölder cocycle.

Consider an arbitrary TNS action α on a torus M. By Remark (5) of §2, there is a subgroup $\Gamma \subset \mathbb{Z}^k$ such that the linear action $\bar\alpha := (\alpha|_\Gamma)_*$ determined by the action induced on the first homotopy group is TNS. $\alpha|_\Gamma$ and $\bar\alpha$ are conjugated by a Hölder conjugacy h: $\alpha(a) = h \circ \bar\alpha(a) \circ h^{-1}$ for $a \in \Gamma$. Assume that both h and h^{-1} are ω-Hölder ($0 < \omega \le 1$).

Define $\bar\beta : \Gamma \times M \to \mathrm{GL}(d, \mathbb{R})$ by $\bar\beta(a, x) := \beta(a, h(x))$. Then $\bar\beta$ is a $\delta\omega$-Hölder cocycle over $\bar\alpha$. By Theorem 2.2, for β small enough there is a $(\omega\delta)$-Hölder function $\bar P : M \to \mathrm{GL}(d, \mathbb{R})$ and a representation $\pi : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ such that

$$\bar\beta(a, x) = \bar P(\bar\alpha(a)x)\pi(a)\bar P(x)^{-1}$$

for $a \in \Gamma$. Then $P : M \to \mathrm{GL}(d, \mathbb{R})$ given by $P(x) := \bar P(h^{-1}(x))$ is a $(\omega^2\delta)$-Hölder transfer map, and $\beta|_\Gamma$ is cohomologous to π via P. This yields a Γ-invariant foliation $\mathcal{F}$ of $M \times \mathrm{GL}(d, \mathbb{R})$ which has trivial holonomy and whose leaves are $(\omega^2\delta)$-Hölder.

We will show that $\mathcal{F}$ has actually δ-Hölder leaves and it is invariant for the full $\mathbb{Z}^k$ action, provided β is small enough. This implies the desired conclusion (see the proof of Lemma 4.4).

Indeed, choose $a \in \Gamma$ such that $\alpha(a)$ is an Anosov diffeomorphism. Assume that β is so small that

$$\begin{aligned}\lambda_-(a)^{\omega^2\delta} &< \mu_-(\beta, a)\mu_+(\beta, a)^{-1},\\ \lambda_+(a)^{\omega^2\delta} &> \mu_-(\beta, a)^{-1}\mu_+(\beta, a).\end{aligned} \tag{5.4}$$

Then $\{\gamma^a_x\}_{x\in M}$ and $\{\gamma^{-a}_x\}_{x\in M}$ are uniquely determined by conditions (1) and (2) of Lemma 4.1 in the $(\omega^2\delta)$-Hölder class, although these functions are actually δ-Hölder. But the foliation $\mathcal{F}$ gives a family of $(\omega^2\delta)$-Hölder functions by restriction to the stable, respectively unstable, foliations of a. Hence these functions coincide with $\{\gamma_x^{\pm a}\}_{x\in M}$, and then $\mathcal{F}$ has δ-Hölder leaves too: the fact that $F_{U,x}^a$ are δ-Hölder relies on the commutation relation (4.5), which holds due to the fact that the functions $F_{U,x}^a$ define a foliation. See the proof of Lemma 4.3. The same proof showed that the foliation defined by the functions $F_{U,x}^a$ is invariant under any element commuting with a, hence $\mathcal{F}$ is $\mathbb{Z}^k$ invariant.

The statement about the regularity of P follows from Theorem 2.4 in [**NT3**], which can be applied if (5.4) holds. We reproduce here the part that is relevant:

THEOREM 5.5. [**NT3**] *Let M be a compact manifold, $a \in \mathrm{Diff}^K(M)$ an Anosov diffeomorphism, and β, $\tilde{\beta}$ two C^K cocycles over the induced $\mathbb{Z}$ action taking values in* $\mathrm{GL}(d,\mathbb{R})$*, where $K = 1, 2, \dots, \omega$. Consider the expansion and contraction coefficients $\lambda_\pm = \lambda_\pm(a)$, $\mu_\pm = \mu_\pm(\beta, a)$ defined by (2.1) and (2.2). Assume that $\lambda_- < \mu_- \cdot \mu_+^{-1} \leq \mu_+ \cdot \mu_-^{-1} < \lambda_+$, and set*

$$\delta_0 = \max\left\{\frac{\ln(\mu_+/\mu_-)}{\ln\lambda_+}, \frac{\ln(\mu_-/\mu_+)}{\ln\lambda_-}\right\}.$$

If β and $\tilde{\beta}$ are cohomologous through a transfer map $P : M \to \mathrm{GL}(d,\mathbb{R})$ which is δ-Hölder for some $\delta > \delta_0$, then P is $C^{K-\varepsilon}$ for any small $\varepsilon > 0$. ($K - \varepsilon = K$ for $K \in \{1, \infty, \omega\}$.)

An important tool in the proof of the above result is the following theorem of Journé [**J**]: if a function is $C^{K+\alpha}$ along the leaves of two transverse foliations with uniformly smooth leaves, then the function is $C^{K+\alpha}$ ($K = 1, 2, \dots, \infty$, $0 < \alpha < 1$). For the analytic case one uses a theorem of de la Llave [**Ll1**] that relies on the cone method of [**HK**]. □

Proof of Theorem 2.3. By Remark (3) of §2, one can assume that the smooth distributions E_i are integrable. Denote by W_i the corresponding foliations of M. As mentioned at the beginning of §4, we may assume that $G = \mathrm{GL}(d,\mathbb{R})$. Let β be a smooth G-valued cocycle.

For an $\mathcal{A}$-invariant foliation W of M contained in the stable foliation of some element of S, denote by $\tilde{W}$ the foliation of $M \times G$ given by

$$\tilde{W}(x; g) = \{(t, \gamma_x^c(t)g) \mid t \in W(x)\}, \quad x \in M, g \in G,$$

where $c \in S$ is such that $E_i \subset W^s(c)$ (by Lemma 4.2(i), it does not matter which c we pick).

By Lemma 4.1(ii), $\widetilde{W_i}$ has smooth leaves which vary continuously in the C^∞ topology; let $\widetilde{E_i}$ be the distribution it determines in $T(M \times G)$. Similarly, for $c \in S$, let $\widetilde{E^s(c)}$ and $\widetilde{E^u(c)}$ be the distributions determined by $\widetilde{W^s(c)}$, respectively $\widetilde{W^u(c)}$, in $T(M \times G)$.

According to the discussion following Lemma 4.2, we attempt to construct the foliation $\mathcal{F}_\beta$ as the span of $\widetilde{W^s(a)}$ and $\widetilde{W^u(a)}$. Using distributions, this translates into showing that the distribution $\mathcal{D} := \widetilde{E^s(a)} + \widetilde{E^u(a)} = \sum_i \widetilde{E_i}$ is integrable. In order to prove this, by the theorem of Frobenius, we have to check that $\mathcal{D}$ is involutive: if X and Y are two (smooth enough) vector fields in $\mathcal{D}$, then $[X, Y] \in \mathcal{D}$.

LEMMA 5.6. *Under the hypotheses of Theorem 2.3, the distribution $\mathcal{D} \subset T(M \times G)$ defined above is smooth and involutive.*

Proof. Let $m = \dim M$ and $l = \dim G$. Since G is parallelizable, one can choose a smooth frame $\{Z_k\}_{k=1,l}$ of TG. Let $U \subset M$ be a small open set, and choose a smooth frame $\{X_j\}_{j=1,m}$ of TM over U such that each field X_j is contained in some E_i. In view of the construction of $\mathcal{D}$, one can find a unique frame $\{Y_j\}_{j=1,m}$ over $U \times G$ which spans $\mathcal{D}$ and has the form $Y_j = X_j + \sum_k \beta_{j,k} Z_k$, with $\beta_{j,k} : U \times G \to \mathbb{R}$.

Clearly the functions $\beta_{j,k}$ are smooth in the G variable. To show that $\beta_{j,k}$ is smooth along E_i, choose an element $c \in S$ for which $X_j, E_i \subset E^s(c)$. Since $Y_j \in \widetilde{E^s(c)} \subset \mathcal{D}$, the conclusion follows from the fact that $\widetilde{W^s(c)}$ has smooth leaves (by Lemma 4.2(ii)).

As in Lemma 3.3, this implies that $\mathcal{D}$ is smooth. To complete the proof, let $c \in S$ be such that $X_i, X_j \in E^s(c)|_U$. Then $Y_i, Y_j \in \widetilde{E^s(c)}$, which is involutive (being tangent to a foliation), hence $[Y_i, Y_j] \in \widetilde{E^s(c)} \subset \mathcal{D}$. □

This proves that the foliation $\mathcal{F}_\beta$ exists and has smooth leaves. The conclusion of the theorem now follows from Lemmas 4.4 and 4.5. The C^ω case follows from [**NT3**, Theorem 2.4], (see Theorem 5.5 above). □

6. *The derivative cocycle of a TNS action*

In this section we give an application of our results.

Let $\alpha, \widetilde{\alpha} : \mathbb{Z}^k \times \mathbb{T}^n \to \mathbb{T}^n$ be two smooth abelian actions. Given a finite set of generators $\{a_i\}$ of $\mathbb{Z}^k$, we say that $\widetilde{\alpha}$ is C^1 close to α if the diffeomorphisms $\alpha(a_i)$ and $\widetilde{\alpha}(a_i)$ are C^1 close for all i.

Let $P\mathbb{T}^n$ be the principal bundle of n-frames in the tangent bundle $T\mathbb{T}^n$. Let $\tau : \mathbb{T}^n \to P\mathbb{T}^n$ denote the standard framing of $\mathbb{T}^n$, corresponding to the natural identification $\Phi : T\mathbb{T}^n \overset{\sim}{\to}\!\!\to \mathbb{T}^n \times \mathbb{R}^n$. Let $\gamma_\alpha : \mathbb{Z}^k \times \mathbb{T}^n \to \mathrm{GL}(n, \mathbb{R})$ denote the derivative cocycle of the action α with respect to the section τ, i.e.

$$D\alpha(a)\tau(x) = \tau(\alpha(a)x)\gamma_\alpha(a, x),$$

for all $a \in \mathbb{Z}^k$, $x \in \mathbb{T}^n$.

THEOREM 6.1. *Let $\alpha : \mathbb{Z}^k \times \mathbb{T}^n \to \mathbb{T}^n$ be a faithful linear TNS action and $\widetilde{\alpha} : \mathbb{Z}^k \times \mathbb{T}^n \to \mathbb{T}^n$ an action C^1 close to α. Then the derivative cocycle $\gamma_{\widetilde{\alpha}}$ is cohomologous to a constant cocycle. The transfer map is Hölder.*

Before starting the proof, we recall some facts from the theory of partially hyperbolic diffeomorphisms, as presented in [**BP**].

Let M be a compact Riemannian manifold and f a smooth diffeomorphism of M. We consider the Banach space $\Gamma^0(M)$ of continuous vector fields on M, on which f acts as an invertible bounded linear operator f_*. We complexify $\Gamma^0(M)$. The Mather spectrum $\sigma(f)$ of f is the spectrum of f_* on this complex Banach space. If the non-periodic points of f are dense in M then $\sigma(f)$ is a union of circles $\{|z| = a\}$. See [**Mat**].

Assume now that the spectrum of f consists of p components $\{S_i\}_{i=1}^p$, where S_i is contained in an annulus with radii λ_i and μ_i,

$$0 < \lambda_1 \le \mu_1 < \cdots < \lambda_p \le \mu_p.$$

Then there is a decomposition

$$\Gamma^0(M) = \Gamma_1 \oplus \cdots \oplus \Gamma_p,$$

where Γ_i are f_* invariant submodules of $\Gamma_0(M)$ for which $\sigma(f_*|_{\Gamma_i}) = S_i$. To each submodule Γ_i there corresponds a distribution E_i. The distributions E_i are η-Hölder for some η that can be bounded from below by a quantity depending continuously on $\{\lambda_i, \mu_i\}_{i=1}^p$ (see [**BP**, Theorem 2.1 and relation (2.12)]). Denote by d_{C^1} the metric in the space of C^1 diffeomorphisms of M induced by the Riemannian metric, and by d the metric in the space of continuous distributions of M:

$$d(E_1, E_2) = \max_{x\in M} \max_{v_1\in E_1} \max_{v_2\in E_2} \left\| \frac{v_1}{\|v_1\|} - \frac{v_2}{\|v_2\|} \right\|.$$

Then for any $\varepsilon > 0$ sufficiently small, there exists a $\delta > 0$ such that for any g diffeomorphism with $d_{C^1}(f, g) < \delta$, the spectrum of g_* is contained in a union of annuli with radii $\lambda_i - \varepsilon$ and $\mu_i + \varepsilon$, and $d(E_i^f, E_i^g) < \varepsilon$.

Proof of Theorem 6.1. Let $a \in \mathcal{A} := \alpha(\mathbb{Z}^k)$ be a linear Anosov diffeomorphism. Then the spectrum of a_* consists of a finite number of circles, and the distribution corresponding to a given circle is invariant under the full $\mathbb{Z}^k$ action. Moreover, considering intersections of such invariant distributions for a finite number of elements in $\mathcal{A}$, one can find an $\mathcal{A}$-invariant splitting of the tangent bundle $T\mathbb{T}^n \cong E_1 \oplus E_2 \oplus \cdots \oplus E_p$ with the property that for each $a \in \mathcal{A} - \{I\}$, the spectrum of $a_*|_{E_i}$ is contained in a circle. We will denote by $\lambda_i(a)$ the radius of the circle. Note that $\lambda_i : \mathcal{A} \to \mathbb{R}$ is a homomorphism. Let $Z \subset \mathbb{Z}^k$ be the pre-image under α of the set of elements introduced by the TNS condition.

Let $\varepsilon > 0$ be given. Taking $\widetilde{\alpha}$ sufficiently C^1 close to α, one can find an $\widetilde{\alpha}(\mathbb{Z}^k)$-invariant splitting of the tangent bundle $T\mathbb{T}^n = \widetilde{E}_1 \oplus \widetilde{E}_2 \oplus \cdots \oplus \widetilde{E}_p$ such that the spectrum of $\widetilde{\alpha}(a)_*|_{\widetilde{E}_i}$ is pinched between $\lambda_i(\alpha(a)) - \varepsilon$ and $\lambda_i(\alpha(a)) + \varepsilon$ for any $a \in Z$, $i = 1, 2, \dots, p$. Moreover, since $\widetilde{E}_i$ is close to E_i and Hölder, one can choose a Hölder identification between $\widetilde{E}_i$ and E_i (e.g. take the orthogonal projection in each fiber). Since the sub-bundles $E_i \subset T\mathbb{T}^n$ are smoothly trivial, we conclude that there are Hölder bundle maps $\Psi_i : \Phi(\widetilde{E}_i) \to \mathbb{T}^n \times V_i$, where $V_i \subset \mathbb{R}^n$ are vector subspaces.

Consider the Hölder cocycles $\gamma_i : \mathbb{Z}^k \times \mathbb{T}^n \to \mathrm{GL}(V_i)$ obtained by restricting the derivative cocycle $\gamma_{\widetilde{\alpha}}$ to $\Phi(\widetilde{E}_i)$ and conjugating by Ψ_i. Note that the contraction and expansion coefficients (2.2) of $\gamma_i|_{\langle a\rangle}$ are exactly the radii of the annulus bounding the spectrum of $\widetilde{\alpha}(a)_*$ on $\widetilde{E}_i$. (Indeed, conjugation by a continuous bundle map does not affect the spectrum and the equality follows from the spectral mapping theorem.) Hence, by taking $\widetilde{\alpha}$ closer to α, the cocycle $\bar{\gamma}_i(a, x) := \lambda_i(\alpha(a))^{-1}\gamma_i(a, x)$ can be made as small as desired while keeping its Hölder class away from zero. Then, by Theorem 2.1, there are homomorphisms $\pi_i : \mathbb{Z}^k \to \mathrm{GL}(V_i)$ and Hölder transfer maps $P_i : \mathbb{T}^n \to \mathrm{GL}(V_i)$ such that

$$\gamma_i(x, a) = P_i(\alpha(a)x)(\lambda_i(\alpha(a))\pi_i(a))P_i(x)^{-1}.$$

Therefore $\gamma_{\widetilde{\alpha}}$ is cohomologous to $\oplus\lambda_i \circ \alpha \cdot \pi_i$ via the Hölder transfer map $\oplus\Psi_i^{-1}P_i$. $\square$

7. *Examples of TNS actions and related questions*

Example 1. Let $\mathrm{SL}(n,\mathbb{R})$ be the group of invertible matrices of determinant one. Let $\mathbf{T} \subset \mathrm{SL}(n,\mathbb{R})$ be a maximal torus such that $\mathbf{T} \cong \mathbb{R}^{n-1}$. It follows from a theorem of Prasad–Raghunathan (see [**PR**, Theorem 7.1]), that there is $g \in \mathrm{SL}(n,\mathbb{R})$ such that $\mathcal{A} := g\mathbf{T}g^{-1} \cap \mathrm{SL}(n,\mathbb{Z})$ is a cocompact lattice in $\mathbf{T}$. In particular, it follows that:

(1) $\mathcal{A} - \{I\}$ consists of hyperbolic matrices;

(2) the elements of $\mathcal{A}$ are simultaneously diagonalizable over $\mathbb{R}$;

(3) $\mathcal{A}$ is isomorphic to a free abelian group of rank $n-1$;

(4) if $v_1,\dots,v_n \in \mathbb{R}^n$ is a basis of simultaneous eigenvectors for the group $\mathcal{A}$, and $\lambda_i : \mathcal{A} \to \mathbb{R}^n$ is the character of $\mathcal{A}$ defined via $Av := \lambda(A)v_i$, $A \in \mathcal{A}$, then for any strictly non-empty subset J of $\{1,\dots,n\}$, there exists $A \in \mathcal{A}$, such that $\lambda_j(A) < 1$, for $j \in J$, and $\lambda_i(A) > 1$, for $i \notin J$.

Using property (4), it follows that the natural action of $\mathcal{A}$ on $\mathbb{T}^n$ is a TNS $\mathbb{Z}^{n-1}$ action.

This example was investigated by Katok and Lewis in [**KL**]. They proved that the natural action of $\mathcal{A}$ on $\mathbb{T}^n$ is C^∞ rigid.

Example 2. Consider the following two matrices in $\mathrm{SL}(4,\mathbb{Z})$:

$$A = \begin{pmatrix} 6 & 13 & 1 & -4 \\ 4 & 10 & 1 & -3 \\ 3 & 7 & 1 & -2 \\ 2 & 5 & 1 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & -1 & 2 & 1 \\ -1 & -2 & 2 & 3 \end{pmatrix}.$$

One can check that A and B are hyperbolic and $AB = BA$. Therefore A and B generate an Anosov $\mathbb{Z}^2$ action α on $\mathbb{T}^4$.

There is an ordered base $\{e_1, e_2, e_3, e_4\}$ in $\mathbb{R}^4$ in which both A and B are diagonalizable. The signs of the Lyapunov exponents are $(+,-,-,-)$ for A, $(-,-,-,+)$ for B and $(+,-,-,+)$ for AB. Denote $V_1 = \mathrm{span}\{e_1\}$, $V_2 = \mathrm{span}\{e_2, e_3\}$, $V_3 = \mathrm{span}\{e_4\}$. Then V_1, V_2, V_3 induce a splitting of the tangent bundle $T\mathbb{T}^4$ which satisfies the definition of a TNS action. The set S is $\{A, B, B^{-1}A^{-1}\}$.

Example 3. We describe now an example of a TNS $\mathbb{Z}^3$ action on a nilmanifold. In our search for this example [**Q**] was useful. Let $\mathfrak{n}$ be the 2-step nilpotent Lie algebra generated by $\{e_i; 1 \le i \le 10\}$, with the relations $[e_1,e_2] = e_5$, $[e_1,e_3] = e_6$, $[e_1,e_4] = e_7$, $[e_2,e_3] = e_8$, $[e_2,e_4] = e_9$, $[e_3,e_4] = e_{10}$, and all the other brackets between the generators are zero. Let $C = \mathrm{span}_{\mathbb{Z}}\{e_i\}$. Denote $N = \exp(\mathfrak{n})$ and $\Gamma = \exp(2C)$. Then N is a connected, simply connected nilpotent Lie group, and Γ is a cocompact lattice in N.

Consider the standard representation of $\mathrm{SL}(4,\mathbb{Z})$ on $\mathrm{span}\{e_i; 1 \le i \le 4\}$. Then, using the relations between e_i's, we find a representation of $\mathrm{SL}(4,\mathbb{Z})$ on $\mathrm{span}\{e_i; 5 \le i \le 10\}$. So we have a representation of $\mathrm{SL}(4,\mathbb{Z})$ on $\mathfrak{n}$, and therefore an action on N, which invariates Γ. An abelian subgroup generated by three hyperbolic matrices can be found in $\mathrm{SL}(4,\mathbb{Z})$, using the theorem of Prasad–Raghunathan. Using the property (4) exhibited in Example 1, it is easy to verify that the $\mathbb{Z}^3$ action on the nilmanifold N/Γ is a TNS action.

Finally, we would like to mention that so far we have not found an example of a linear TNS action on an infranilmanifold that is not a nilmanifold. It would also be interesting to find examples of linear TNS actions that are non-diagonalizable.

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