MEMORY LOSS FOR TIME-DEPENDENT PIECEWISE EXPANDING SYSTEMS IN HIGHER DIMENSION

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Abstract. We prove a counterpart of exponential decay of correlations for non-stationary systems. Namely, given two probability measures absolutely continuous with respect to a reference measure, their quasi-Hölder distance (and in particular their L^1 distance) decreases exponentially under action by compositions of arbitrarily chosen maps close to those that are both piecewise expanding and mixing in a certain sense.

1. Introduction

This paper studies statistical properties of time-dependent dynamical systems. In such systems the dynamical model itself is allowed to vary with time. An important example is the flow generated by a nonautonomous vector field. Perhaps the vector field depends on physical parameters that vary with time. We address memory loss for time-dependent dynamical systems, an analog of decay of correlations.

The memory loss problem has been studied extensively in the contexts of stochastic differential equations (SDEs), random dynamical systems[1], and autonomous (time-independent) deterministic dynamical systems. An SDE of the form

$$\mathrm{d}x_t = a(x_t)\,\mathrm{d}t + \sum_{i=1}^{n} b_i(x_t) \circ \mathrm{d}W_t^i$$

gives rise to a stochastic flow of diffeomorphisms in which almost every Brownian path defines a time-dependent flow (see *e.g.* [21]). Lyapunov exponents for such flows are known to be well-defined, nonrandom (they do not depend on the realization of the noise), and constant almost everywhere in phase space if the system is ergodic. Ergodic systems for which the greatest Lyapunov exponent $\lambda_{\max}$ is negative exhibit a phenomenon known as *random sinks*. Under suitable conditions, any ensemble of initial conditions will coalesce near a unique equilibrium point that evolves in time [23]. This phenomenon occurs in dissipative systems such as the Navier-Stokes system (see *e.g.* [29, 30]) and in certain coupled oscillator networks modeling neuronal activity [25]. Memory loss also occurs if $\lambda_{\max} > 0$, for in this case initial distributions will track random SRB measures (see [24]) rather than random sinks. For further information about random dynamical systems, see *e.g.* [2, 3].

We say that an autonomous deterministic system exhibits *memory loss in the statistical sense* if there exists a unique invariant measure ν that attracts absolutely continuous distributions $\rho_0 \ll \nu$, that is $\rho_t \to \nu$ as $t \to \infty$ where ρ_t denotes the dynamical evolution of ρ_0. Both the nature and speed of the convergence are of interest. Statistical memory loss and the closely related notion of decay of correlations have received a great deal of attention in this context (see *e.g.* [11, 15, 16, 17, 26, 27, 31, 32, 39, 45, 46]). Since time-dependent deterministic systems are out of equilibrium, we

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[1] In the context of random dynamical systems, the maps are chosen according to a known distribution.

associate statistical memory loss not with initial distributions converging to invariant measures but rather with distances between pairs of initial distributions decreasing as the distributions evolve.

Important classes of time-dependent systems include dynamical systems with time-varying parameters and physical processes that take place in evolving environments. For example, consider a Lorentz gas (Sinai billiard) in which some of the scatterers move, perhaps due to bombardment by light particles. See [12] for an effort to model the movement of a heavy particle in this context. Stenlund, Young, and Zhang introduce a model of Sinai billiards with moving scatterers and prove an exponential memory loss result for this model [41]. Open systems (systems with holes) provide another important example, for perhaps the holes move over time. Mohapatra and Ott formulate a notion of conditional memory loss for time-dependent open systems and prove that this type of memory loss occurs at an exponential rate for a class of one-dimensional piecewise-smooth expanding maps with holes [33].

We do not assume any *a priori* knowledge of any statistical properties of the evolution of the dynamical model. Indeed, the stationarity of the process is irrelevant from our point of view. By contrast, knowledge of the statistical properties of the process typically plays a central role in random dynamical systems.

In this paper we focus on time-dependent discrete-time systems: compositions of the form $f_n \circ f_{n-1} \circ \cdots \circ f_1$, where $(f_i)_i$ is a finite or infinite sequence of maps from a space X into itself. Identify probability measures absolutely continuous with respect to the reference measure μ on X with their their density functions (Radon-Nikodym derivatives with respect to μ). We say that the system exhibits exponential memory loss in the statistical sense if given any initial densities φ_0 and ψ_0, the evolved densities φ_t and ψ_t satisfy $\|\varphi_t - \psi_t\|_{L^1(\mu)} \leqslant C_{\varphi_0,\psi_0} e^{-\Lambda t}$ for some $\Lambda > 0$ independent of the initial measures.

We focus specifically on time-dependent piecewise $C^{1+\alpha}$ expanding systems in dimension at least two. Here the phase space X is a Riemannian manifold and for each map f_i, there exists a finite or countably infinite collection $\{U_{ij}\}$ of pairwise disjoint open subsets of X such that $\bigcup_j U_{ij}$ has full Riemannian volume for each i and $f_i|U_{ij}$ is smooth for all relevant i and j. The time-independent case (iterates of a single piecewise smooth expanding map) has received substantial attention, particularly with respect to the existence of absolutely continuous invariant probability measures and exponential decay of correlations [1, 5, 7, 8, 9, 10, 13, 18, 20, 34, 38, 40, 42, 43, 44].

The ergodic theory of a time-independent piecewise smooth expanding system can be subtle in higher dimension because the domains on which the map is smooth can have complicated boundaries. This subtlety is magnified in the time-dependent case, wherein both the maps and the domains on which the maps are smooth can vary with time.

In this paper we prove that certain time-dependent piecewise $C^{1+\alpha}$ expanding systems in higher dimension exhibit exponential memory loss in the statistical sense. The current work builds on previous work covering time-dependent expanding systems and time-dependent piecewise expanding systems in one dimension [22, 37]. This previous work uses the method of coupling, introduced in [46] and developed later in *e.g.* [6, 11, 12]. Here we use cones and the Hilbert projective metric (see *e.g.* [4, 26, 35, 36]. One could argue that coupling and the cone method are preferable to the spectral approach when dealing with time-dependent deterministic systems. For many random dynamical systems, one can study an averaged Perron-Frobenius operator (see *e.g.* [14]); one cannot average in the time-dependent deterministic context.

2. Statement of results

We begin by defining a class of piecewise $C^{1+\alpha}$ expanding maps with good ergodic properties. The setup is based on that introduced by Saussol [40]. We describe perturbations of these maps and define the space of quasi-Hölder densities. Saussol proved that maps in this class admit finitely many ergodic absolutely continuous invariant probability measures (ACIPs) with quasi-Hölder densities.

We state the local and global versions of our results at the end of this section; the global statement is a straightforward consequence of the local result.

Remark 2.1. Throughout this paper we use a fixed Hölder exponent $0 < \alpha \leqslant \mathrm{Lip}$; unless otherwise stated, the maps are (piecewise) $C^{1+\alpha}$. For maps in $C^{1+\mathrm{Lip}}$ we use $\alpha = 1$ in the computations.

For $f : U \subset \mathbb{R}^k \to \mathbb{R}^\ell$ use as $C^{1+\alpha}$ and C^2 norms

$$\|f\|_{C^{1+\alpha}} := \|f\|_{C^0} + \|Df\|_{C^\alpha}, \qquad \|f\|_{C^2} := \|f\|_{C^0} + \|Df\|_{C^0} + \|D^2 f\|_{C^0}. \tag{2.1}$$

These extend in a straightforward way to the case when the domain or range is a subset of a compact manifold.[2]

Remark 2.2. For simplicity of exposition we consider only maps on the N-dimensional torus. However, one can extend the proofs to arbitrary compact manifolds.

Let $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$ denote the N-dimensional torus and μ the normalized Lebesgue measure on $\mathbb{T}^N$. For a set $S \subset \mathbb{T}^N$ and $\varepsilon > 0$, define

$$B_\varepsilon(S) := \bigcup_{x \in S} B_\varepsilon(x),$$

where $B_\varepsilon(x)$ or $B(x, \varepsilon)$ stand for the open ball of radius ε centered at x.

2.1. **Domains for piecewise continuous maps; the classes $\mathcal{R}(K)$, $C(\mathcal{A})$ and $C(\mathcal{R})$.** Abusing notation, we refer to a family of open pairwise disjoint sets that covers $\mathbb{T}^N$ up to measure zero (e.g., $\mathcal{A} \in \mathcal{R}$ below) as an *open partition*. The maps we consider are piecewise continuous on such open partitions.

Definition 2.3 (partitions)**.** Consider in $\mathbb{T}^N$ a finite family of pairwise disjoint open sets $\mathcal{A} := \{U_i : 1 \leqslant i \leqslant M\}$.

We say that $\mathcal{A} \in \mathcal{R}(K)$ if it covers $\mathbb{T}^N$ up to measure zero and each U_i has boundaries bounded piecewise in C^2 by K. More precisely:

(1) $\mu(\mathbb{T}^N \setminus \bigcup_i U_i) = 0$;
(2) for each i there are finitely many compact C^2 embedded codimension-one submanifolds $\{\Gamma_{ij}\}_j$ of $\mathbb{T}^N$ such that ∂U_i is contained in the union $\bigcup_j \Gamma_{ij}$;
(3) the C^2 norm of each Γ_{ij} is strictly less than K; that is, for each Γ_{ij} there are finitely many C^2 charts $\Phi_{\ell;ij} : B_N \subset \mathbb{R}^N \to W_{\ell;ij} \subset \mathbb{T}^N$, where B_N is the unit ball of $\mathbb{R}^N$, such that each $\Phi_{\ell;ij}$ and $\Phi_{\ell;ij}^{-1}$ have C^2-norm less than K and $\Gamma_{ij} \subset \bigcup_\ell \Phi_{\ell;ij}(B_N \cap (\mathbb{R}^{N-1} \oplus \{0\}))$.

Denote by $\mathcal{R}$ the union $\bigcup_{K>0} \mathcal{R}(K)$.

For $\mathcal{A} \in \mathcal{R}$ we define, using the above notation

$$\kappa(\mathcal{A}) := \sup_{x \in \mathbb{T}^N} \#\{\Gamma_{ij} : x \in \Gamma_{ij}\},$$

which gives the maximum number of boundary components to which a point belongs.

Definition 2.4 (nearby partitions)**.** We say that two open partitions $\mathcal{A} = \{U_i\}_i$ and $\widetilde{\mathcal{A}} = \{\widetilde{U}_i\}_i$ in $\mathcal{R}(K)$ are δ-close if:

(1) the families $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ have the same number of elements, and there is also a correspondence between the bounding submanifolds Γ_{ij} and $\widetilde{\Gamma}_{ij}$;
(2) for each i, $\mathrm{d}_{\mathrm{Hausd}}(U_i, \widetilde{U}_i) < \delta$, where $\mathrm{d}_{\mathrm{Hausd}}$ denotes Hausdorff distance;[3]

[2] The particular formulas used for the norms in (2.1) are not important.
[3] Recall that $\mathrm{d}_{\mathrm{Hausd}}(A, B) := \max\{\sup\{\mathrm{dist}(x, B) \mid x \in A\}, \sup\{\mathrm{dist}(y, A) \mid y \in B\}\}$.

(3) for each i, j, the bounding submanifolds Γ_{ij} and $\widetilde{\Gamma}_{ij}$ are less than δ apart in Hausdorff distance.

This defines a topology on $\mathcal{R}(K)$.[4] It is not hard to see that

Lemma 2.5. *Given $\mathcal{A} \in \mathcal{R}(K)$, there is $\delta > 0$ such that $\kappa(\widetilde{\mathcal{A}}) \leqslant \kappa(\mathcal{A})$ for each $\widetilde{\mathcal{A}} \in \mathcal{R}(K)$ that is δ-close to $\mathcal{A}$.*

Proof. Indeed, if $x_n \in \mathbb{T}^N$ is a point in k boundary components of $\mathcal{A}_n$ with $\mathcal{A}_n$ being δ_n-close to $\mathcal{A}$, $\delta_n \to 0$, then we can select a subsequence so that $x_n \to x$ and $x_n \in \cap_{(i,j)\in J}\Gamma^n_{ij}$ for a set J with k elements, with each boundary component Γ^n_{ij} of $\mathcal{A}_n$ being δ_n-close to the corresponding boundary component Γ_{ij} of $\mathcal{A}$. As $n \to \infty$, this implies that $x \in \cap_J \Gamma_{ij}$. ■

Definition 2.6 (piecewise continuous maps)**.** For $\mathcal{A} \in \mathcal{R}$, we write $f \in C(\mathcal{A})$ for functions $f : \mathbb{T}^N \to \mathbb{T}^N$ that are continuous on each $U \in \mathcal{A}$. Denote by $C(\mathcal{R})$ the union of all $C(\mathcal{A})$ with $\mathcal{A} \in \mathcal{R}$.

2.2. Piecewise expanding maps; the classes $\mathcal{M}$ and $\mathcal{M}^*$.

We describe next the piecewise $C^{1+\alpha}$ expanding maps that we consider.

The main estimate we need, a Lasota-Yorke inequality derived by Saussol [40, Lemma 4.1], requires the properties described later in Definition 4.1; we consider a class $\mathcal{M}^*$ of maps satisfying those. We first introduce a class of piecewise expanding maps $\mathcal{M}$ that is easier to describe.

Notation 2.7. Denote by $\xi_N = \pi^{N/2}/(N/2)!$ the volume of the unit ball in $\mathbb{R}^N$.[5]

Definition 2.8 (the class $\mathcal{M}$)**.** For $0 < s < 1$, $K > 0$, $\kappa > 0$ such that (2.2) holds,

$$s^\alpha + \left(\frac{4s\kappa}{1-s}\right)\left(\frac{\xi_{N-1}}{\xi_N}\right) < 1, \tag{2.2}$$

denote by $\mathcal{M}(s, K, \kappa)$ the piecewise $C^{1+\alpha}$ maps $f : \mathbb{T}^N \to \mathbb{T}^N$ that satisfy the following properties:

(1) $f \in C(\mathcal{A}_1)$ with $\mathcal{A}_1 = \{U_i : 1 \leqslant i \leqslant M\} \in \mathcal{R}(K)$; we will refer to $\mathcal{A}_1$ by $\mathcal{A}_1(f)$
(2) (backward contraction) for each i, $f|_{U_i}$ is injective with a differentiable inverse and $\|D[(f|_{U_i})^{-1}]\| < s$
(3) for each i, both $f|_{U_i}$ and all its partial derivatives extend to continuous functions on the closure of U_i
(4) for each i, $\|f|_{U_i}\|_{1+\alpha} < K$
(5) $\kappa(\mathcal{A}_1) \leqslant \kappa$.

Definition 2.9 (the class $\mathcal{M}^*$)**.** Denote by $\mathcal{M}^*(s, K, \kappa, \varepsilon_0)$ the set of maps $f \in \mathcal{M}(s, K, \kappa)$ that can be extended to a neighborhood of the original sets U_i as follows (we use the notations of Definition 2.8): for each i, there is an open set $V_i \supset \overline{U_i}$ and an extension $f_{(i)} : V_i \to \mathbb{T}^N$ of $f|_{U_i}$ such that

(1) $f_{(i)}(V_i) \supset \overline{B_{\varepsilon_0}(f(U_i))}$
(2) $f_{(i)}$ is a C^1 diffeomorphism from V_i to its image
(3) (backward contraction of extensions) $\|D[f^{-1}_{(i)}]\| < s$ on $f_{(i)}(V_i)$
(4) $\|f_{(i)}\|_{1+\alpha} < K$ on V_i.

Remark 2.10. For iterates of a single piecewise expanding map, not requiring a balance between complexity and expansion (as is condition (2.2)) can lead to maps with no ACIPs (see [9]).

[4] One can define a fundamental system of neighborhoods, as we do later in Section 2.3 for maps.
[5] $(N/2)!$ stands for $\Gamma(N/2+1)$.

2.3. **Perturbations of maps in $\mathcal{M}$; the neighborhoods $\mathcal{N}(f,\delta;s,K,\kappa)$.** We describe next perturbations of $f \in \mathcal{M}$. This is similar to the topology used in [14, §2.4] and [19].

Definition 2.11 (nearby maps; $\mathcal{N}(f,\delta;s,K,\kappa)$)**.** Let $f, \widetilde{f} \in \mathcal{M}(s,K,\kappa)$. We say that $\widetilde{f}$ is a δ-*perturbation of* f, denoted $\widetilde{f} \in \mathcal{N}(f,\delta) = \mathcal{N}(f,\delta;s,K,\kappa)$, if the following hold.

(1) $\widetilde{f} \in C(\widetilde{\mathcal{A}}_1)$ with $\widetilde{\mathcal{A}}_1 \in \mathcal{R}(K)$ δ-close to $\mathcal{A}_1(f)$, as described in Definition 2.4; [6]
(2) outside a δ-neighborhood of the boundaries,[7] the maps are δ-close in $C^{1+\alpha}$:

$$\|f|_{W_i} - \widetilde{f}|_{W_i}\|_{C^{1+\alpha}} < \delta \qquad \text{for each } i,$$

where

$$W_i := \{x \in U_i \cap \widetilde{U}_i \mid \mathrm{dist}(x, U_i^c) > \delta, \mathrm{dist}(x, \widetilde{U}_i^c) > \delta\}.$$

It is not difficult to check that as f and δ vary, the sets $\mathcal{N}(f,\delta;s,K,\kappa)$ form a fundamental system of neighborhoods,[8] so they define a topology on $\mathcal{M}(s,K,\kappa)$.

2.4. **Mixing maps; the class $\mathcal{E}$.** Arbitrary compositions of piecewise $C^{1+\alpha}$ expanding maps in $\mathcal{M}^*$ do not necessarily exhibit exponential loss of memory. Indeed, a system defined by iterating a single piecewise $C^{1+\alpha}$ expanding map may not even be ergodic, and decay of correlations (memory loss) in this context requires mixing. We therefore formulate a type of mixing condition.

Definition 2.12 (the class $\mathcal{E}$)**.** Let $\zeta_1 \in (0,1)$ and $\zeta_2 \in (1,\infty)$. We say a map $f : \mathbb{T}^N \to \mathbb{T}^N$ belongs to $\mathcal{E}(\zeta_1,\zeta_2)$ if for every finite partition $\mathcal{H}$ of $\mathbb{T}^N$ into hypercubes, there exists $J(\mathcal{H},\zeta_1,\zeta_2)$ such that for every $H_1, H_2 \in \mathcal{H}$, we have

$$\zeta_1 < \frac{\mu(H_1 \cap f^{-i}(H_2))}{\mu(H_1)\mu(H_2)} < \zeta_2 \tag{2.3}$$

for every $i \geqslant J(\mathcal{H},\zeta_1,\zeta_2)$.

Remark 2.13. For fixed ζ_1, ζ_2, $\mathcal{H}$, and i, (2.3) is an open condition with respect to the topology we have defined on $\mathcal{M}^*$. This is precisely how we use (2.3) in the proof of Theorem 2.16.

2.5. **Densities; the set $\mathcal{D}$.** We consider densities that are quasi-Hölder. These quasi-Hölder spaces were considered by Saussol [40], where more details can be found.

For $\varphi \in L^1(\mu)$ and a Borel set $S \subset \mathbb{T}^N$, define the oscillation of φ on S by

$$\mathrm{osc}(\varphi, S) := \mathrm{Esup}(\varphi, S) - \mathrm{Einf}(\varphi, S).$$

Given $\varepsilon_0 > 0$, define the seminorm

$$|\varphi|_{\alpha,\varepsilon_0} := \sup_{0<\varepsilon\leqslant\varepsilon_0} \varepsilon^{-\alpha} \int_{\mathbb{T}^N} \mathrm{osc}(\varphi, B_\varepsilon(x))\,\mathrm{d}\mu(x).$$

The seminorms $|\varphi|_{\alpha,\varepsilon_0}$ are equivalent for different ε_0's.[9] Define

$$\mathrm{OSC}_\alpha := \{\varphi \in L^1(\mu) : |\varphi|_{\alpha,\varepsilon_0} < \infty\}.$$

This space does not depend on ε_0, and contains the α-Hölder functions. Define the norm $\|\cdot\|_{\alpha,\varepsilon_0}$ on OSC_α by

$$\|\varphi\|_{\alpha,\varepsilon_0} := \|\varphi\|_{L^1(\mu)} + |\varphi|_{\alpha,\varepsilon_0}. \tag{2.4}$$

[6]Note that one can represent an $f \in \mathcal{M}$ with more than one choice of $\mathcal{A}_1(f)$.

[7]By Definition 2.4, the boundaries of $\mathcal{A}_1$ and $\widetilde{\mathcal{A}}_1$ have Hausdorff distance at most δ.

[8]If $g \in \mathcal{N}(f_1,\delta_1;s,K,\kappa) \cap \mathcal{N}(f_2,\delta_2;s,K,\kappa)$ then $\mathcal{N}(g,\delta;s,K,\kappa) \subset \mathcal{N}(f_1,\delta_1;s,K,\kappa) \cap \mathcal{N}(f_2,\delta_2;s,K,\kappa)$ for some $\delta > 0$.

[9]For $0 < \varepsilon_1 < \varepsilon_2$ there are finitely many vectors v_i such that $B_{\varepsilon_2}(x) \subset \cup_i B_{\varepsilon_1}(x + v_i)$ for all x.

Equipped with this norm, OSC_α is a Banach space and the unit ball of OSC_α is precompact in $L^1(\mu)$.

Our memory loss results hold for densities in the set

$$\mathcal{D} := \{\varphi \in \mathrm{OSC}_\alpha : \varphi \geqslant 0,\ \|\varphi\|_{L^1(\mu)} = 1\}.$$

Remark 2.14. Saussol [40] proves that maps somewhat more general[10] than those in $\mathcal{M}^*$ admit an ACIP, whose density is in $\mathcal{D}$. This is obtained by proving that the Perron-Frobenius operator satisfies a Lasota-Yorke inequality on OSC_α, and therefore is quasi-compact.

2.6. **Results.** As mentioned earlier, we need both sufficient expansion (for condition (2.2) to hold), and some form of mixing (of the type exhibited by maps in $\mathcal{E}$).

We formulate two types of results: *local*, governing arbitrary compositions of maps chosen from a small neighborhood of a fixed map in $\mathcal{E} \cap \mathcal{M}^*$, and *global*, wherein we move through a union of such neighborhoods; the latter is a simple consequence of the former.

We begin by setting some notation.

Notation 2.15. Given maps $f_i : \mathbb{T}^N \to \mathbb{T}^N$ for $i \in \mathbb{N}$, denote $F_{m,k} := f_m \circ \cdots \circ f_k$ if $m \geqslant k$. Write F_m for $F_{m,1}$.

For a (non-singular) map $f : \mathbb{T}^N \to \mathbb{T}^N$, denote by $\mathcal{P}_f : L^1(\mu) \to L^1(\mu)$ the associated Perron-Frobenius operator:

$$\int_{\mathbb{T}^N} (\varphi \circ f) \cdot \psi \,\mathrm{d}\mu = \int_{\mathbb{T}^N} \varphi \cdot \mathcal{P}_f(\psi) \,\mathrm{d}\mu \qquad \varphi \in L^\infty(\mu),\ \ \psi \in L^1(\mu).$$

In other words, the Perron-Frobenius map $\mathcal{P}_f$ describes the action of f on ACIMs:

$$f^*(\psi \,\mathrm{d}\mu) = \mathcal{P}_f(\psi) \,\mathrm{d}\mu.$$

Recall that $\mathcal{P}_f$ does not increase the L^1-norm,

$$\|\mathcal{P}_f(\psi)\|_{L^1} \leqslant \|\psi\|_{L^1}. \tag{2.5}$$

2.6.1. *Local result.* The main local theorem states that given two densities in $\mathcal{D}$, their distance with respect to $\|\cdot\|_{\alpha,\varepsilon_0}$ decreases at an exponential rate under action by arbitrary composition of maps that are close to a single map in $\mathcal{E} \cap \mathcal{M}^*$.

Theorem 2.16. *Let $g \in \mathcal{E}(\zeta_1, \zeta_2) \cap \mathcal{M}^*(s, K, \kappa, \varepsilon_0)$. There exist $\Lambda < 1$ and $\delta > 0$ such that given $\varphi, \psi \in \mathcal{D}$, there exists $C_{\varphi,\psi} > 0$ with the following property: for any sequence $(f_i)_{i=1}^\infty$ in $\mathcal{N}(g, \delta; s, K, \kappa) \cap \mathcal{M}^*(s, K, \kappa, \varepsilon_0)$, we have*

$$\int_{\mathbb{T}^N} |\mathcal{P}_{F_m}(\varphi) - \mathcal{P}_{F_m}(\psi)| \,\mathrm{d}\mu \leqslant \|\mathcal{P}_{F_m}(\varphi) - \mathcal{P}_{F_m}(\psi)\|_{\alpha,\varepsilon_0} \leqslant C_{\varphi,\psi}\Lambda^m \tag{2.6}$$

for all $m \in \mathbb{N}$.

One can relax the hypotheses of the above theorem because, by the Whitney extension theorem, maps in $\mathcal{M}$ close to $f \in \mathcal{M}^*$ are in $\mathcal{M}^*$ provided the boundaries are suitable.

[10]The main difference is that there can be countable many U_i's, and their boundaries need not be piecewise smooth. In that case condition (2.2) is replaced by (PE5), described in Section 4.

2.6.2. *Global result.* We note that many global formulations are possible. We give below such a result, but one can replace continuity of ω with weaker assumptions.

Let $a < b$ and let $\omega : [a, b] \to \mathcal{M}^*$ be a map. We discretize ω by considering sequences of the form $(\omega(t_i))_i$ where $a \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant b$. Let $\Omega_m := \omega(t_m) \circ \cdots \circ \omega(t_1)$.

Theorem 2.17. *Let $\omega : [a, b] \to \mathcal{M}^*(s, K, \kappa, \varepsilon_0)$ be a continuous map. Assume that*

$$\omega([a, b]) \subset \bigcup_{i=1}^{M} \mathcal{N}(g_i, \delta_i; s, K, \kappa)$$

where $g_i \in \mathcal{M}^*(s, K, \kappa, \varepsilon_0)$ are maps for which Theorem 2.16 holds with a corresponding $\delta_i > 0$.

Then there exists $\Lambda < 1$ such that the following holds for any discretized sequence $(\omega(t_i))_i$: for every $\varphi, \psi \in \mathcal{D}$, there exists $C'_{\varphi,\psi} > 0$ such that

$$\int_{\mathbb{T}^N} |\mathcal{P}_{\Omega_m}(\varphi) - \mathcal{P}_{\Omega_m}(\psi)| \, \mathrm{d}\mu \leqslant \|\mathcal{P}_{\Omega_m}(\varphi) - \mathcal{P}_{\Omega_m}(\psi)\|_{\alpha,\varepsilon_0} \leqslant C'_{\varphi,\psi}\Lambda^m \tag{2.7}$$

for all relevant $m \in \mathbb{N}$.

3. Proof of Theorem 2.16

We use the theory of cones and a projective metric known as the Hilbert metric (see *e.g.* [26, 28]). Saussol [40] uses this theory to obtain precise estimates on rates of correlation decay for maps in $\mathcal{M}^*$.

The proof proceeds as follows. We define a suitable cone $\mathcal{C}_a \subset \mathrm{OSC}_\alpha$. We then find a time $T \in \mathbb{N}$ such that $\mathcal{P}_{F_{i+T-1,i}}$ maps $\mathcal{C}_a$ strictly inside itself for all $i \in \mathbb{N}$. The diameter of $\mathcal{P}_{F_{i+T-1,i}}(\mathcal{C}_a)$ with respect to the Hilbert metric is bounded uniformly in i. The general theory of cones then implies that $\mathcal{P}_{F_{i+T-1,i}}$ is a contraction on $\mathcal{C}_a$ with a contraction factor that is uniform in i. Theorem 2.16 follows.

3.1. **Invariance of a suitable convex cone.** The following Lasota-Yorke inequality provides control of the oscillation seminorm of functions in OSC_α under the action of the Perron-Frobenius operator. It is the crucial estimate that allows us to analyze the action of the Perron-Frobenius operator on $\mathcal{C}_a$. For completeness we provide the proof in Section 4.

Proposition 3.1 (Lasota-Yorke inequality [40])**.** *Let s and κ be such that (2.2) is satisfied, and $K, \varepsilon_0 > 0$. There are positive constants $\varepsilon_{LY} = \varepsilon_{LY}(s, K, \kappa, \varepsilon_0, N) \leqslant \varepsilon_0$, $\gamma_{LY} = \gamma_{LY}(s, K, \kappa, \varepsilon_{LY}) < 1$ and $K_{LY} = K_{LY}(s, K, \kappa, \varepsilon_{LY})$, such that:*

if $f \in \mathcal{M}^*(s, K, \kappa, \varepsilon_0)$ then $\mathcal{P}_f$ maps OSC_α to itself and

$$|\mathcal{P}_f(\varphi)|_{\alpha,\varepsilon_{LY}} \leqslant \gamma_{LY} \, |\varphi|_{\alpha,\varepsilon_{LY}} + K_{LY}\|\varphi\|_{L^1(\mu)} \qquad \varphi \in \mathrm{OSC}_\alpha \,. \tag{3.1}$$

Proof. See Section 4, where we recall the proof of Saussol [40]. The constants ε_{LY}, γ_{LY} and K_{LY} are described there. ∎

Notation 3.2. We fix the above parameters s, K, κ, ε_0; the constants ε_{LY}, γ_{LY} and K_{LY} are those given in Proposition 3.1.

We now define $\mathcal{C}_a$ and study the action of the Perron-Frobenius operator on it. The following parameters are used throughout the proof of Theorem 2.16.

(P1) $0 < \sigma < 1$

(P2) $\varepsilon_{\mathcal{H}}$: choose such that $\varepsilon_{\mathcal{H}} \leqslant \varepsilon_{LY}$. Let $\mathcal{H}$ be a partition of $\mathbb{T}^N$ into hypercubes such that

$$\sup_{H\in\mathcal{H}} \mathrm{diam}(H) \leqslant \varepsilon_{\mathcal{H}}.$$

(P3) $T \in \mathbb{N}$: choose such that $T \geqslant J(\mathcal{H}, \zeta_1, \zeta_2)$.

(P4) $a > 0$: the aperture of the cone $\mathcal{C}_a$. We choose the parameters $\varepsilon_{\mathcal{H}}$, T, and a such that

$$\zeta_1 - \zeta_2 \varepsilon_{\mathcal{H}}^{\alpha} a > 0,$$

$$(\zeta_1 - \zeta_2 \varepsilon_{\mathcal{H}}^{\alpha} a)^{-1} \left(\gamma_{LY}^T a + \frac{K_{LY}}{1 - \gamma_{LY}} \right) \leqslant \sigma a.$$

The inequalities in (P4) may be simultaneously satisfied by first choosing T sufficiently large, then a sufficiently large, then $\varepsilon_{\mathcal{H}}$ sufficiently small, and finally increasing T if necessary so that (P3) holds.

Once parameter selection is complete, let δ be sufficiently small so that (2.3) holds with ζ_1, ζ_2, and $\mathcal{H}$ fixed for every composition of T maps chosen from $\mathcal{N}(g, \delta; s, K, \kappa) \cap \mathcal{M}^*(s, K, \kappa, \varepsilon_0)$.

Define

$$\mathcal{C}_a = \left\{ \varphi \in L^1(\mu) : \varphi \neq 0, \ \ \varphi \geqslant 0, \ \ |\varphi|_{\alpha, \varepsilon_{LY}} \leqslant a E[\varphi | \mathcal{H}] \right\}.$$

We now show that for every $i \in \mathbb{N}$, $\mathcal{P}_{F_{i+T-1,i}}$ maps $\mathcal{C}_a$ into $\mathcal{C}_{\sigma a}$. The following two lemmas accomplish this.

Lemma 3.3. *For every $\varphi \in \mathcal{C}_a$ and $i \in \mathbb{N}$ we have*

$$(\zeta_1 - \zeta_2 \varepsilon_{\mathcal{H}}^{\alpha} a) \int_{\mathbb{T}^N} \varphi \, \mathrm{d}\mu \leqslant E[\mathcal{P}_{F_{i+T-1,i}}(\varphi) | \mathcal{H}] \leqslant \zeta_2 (1 + a \varepsilon_{\mathcal{H}}^{\alpha}) \int_{\mathbb{T}^N} \varphi \, \mathrm{d}\mu. \tag{3.2}$$

Proof of Lemma 3.3. Write $F = F_{i+T-1,i}$. For $x \in \mathbb{T}^N$, let $H(x)$ denote the element of $\mathcal{H}$ that contains x. We have

$$\begin{aligned} E[\mathcal{P}_F(\varphi) | \mathcal{H}](x) &= \frac{1}{\mu(H(x))} \int_{H(x)} \mathcal{P}_F(\varphi) \, \mathrm{d}\mu \\ &= \frac{1}{\mu(H(x))} \int_{F^{-1}(H(x))} \varphi \, \mathrm{d}\mu \\ &= \frac{1}{\mu(H(x))} \sum_{H' \in \mathcal{H}} \int_{H' \cap F^{-1}(H(x))} \varphi(z) \, \mathrm{d}\mu(z). \end{aligned}$$

Bounding φ from below, for μ almost every z in $H' \cap F^{-1}(H(x))$ we have

$$\begin{aligned} \varphi(z) &\geqslant \left(\frac{1}{\mu(H')} \int_{H'} \varphi \, \mathrm{d}\mu \right) - \operatorname{osc}(\varphi, H') \\ &\geqslant \frac{1}{\mu(H')} \left(\int_{H'} \varphi \, \mathrm{d}\mu - \int_{H'} \operatorname{osc}(\varphi, B(y, \varepsilon_{\mathcal{H}})) \, \mathrm{d}y \right). \end{aligned}$$

Integrating gives

$$\begin{aligned} E[\mathcal{P}_F(\varphi) | \mathcal{H}](x) &\geqslant \sum_{H' \in \mathcal{H}} \frac{1}{\mu(H(x))} \int_{H' \cap F^{-1}(H(x))} \frac{1}{\mu(H')} \left(\int_{H'} \varphi \, \mathrm{d}\mu - \int_{H'} \operatorname{osc}(\varphi, B(y, \varepsilon_{\mathcal{H}})) \, \mathrm{d}y \right) \mathrm{d}\mu(z) \\ &= \sum_{H' \in \mathcal{H}} \frac{\mu(H' \cap F^{-1}(H(x)))}{\mu(H(x)) \mu(H')} \left(\int_{H'} \varphi \, \mathrm{d}\mu - \int_{H'} \operatorname{osc}(\varphi, B(y, \varepsilon_{\mathcal{H}})) \, \mathrm{d}y \right) \\ &\geqslant \zeta_1 \int_{\mathbb{T}^N} \varphi \, \mathrm{d}\mu - \zeta_2 \, |\varphi|_{\alpha, \varepsilon_{LY}} \, \varepsilon_{\mathcal{H}}^{\alpha} \\ &\geqslant (\zeta_1 - \zeta_2 \varepsilon_{\mathcal{H}}^{\alpha} a) \int_{\mathbb{T}^N} \varphi \, \mathrm{d}\mu. \end{aligned}$$

The upper bound

$$E[\mathcal{P}_F(\varphi)|\mathcal{H}](x) \leqslant \zeta_2(1 + a\varepsilon_{\mathcal{H}}^{\alpha}) \int_{\mathbb{T}^N} \varphi \, \mathrm{d}\mu$$

is established in a similar fashion.

■

Lemma 3.4. *For every $i \in \mathbb{N}$ we have*

$$\mathcal{P}_{F_{i+T-1,i}}(\mathcal{C}_a(\mathcal{H})) \subset \mathcal{C}_{\sigma a}(\mathcal{H}).$$

Proof of Lemma 3.4. Write $F = F_{i+T-1,i}$. Iterating (3.1) and using (3.2), we have

$$\begin{aligned}
|\mathcal{P}_F(\varphi)|_{\alpha,\varepsilon_{LY}} &\leqslant \gamma_{LY}^T |\varphi|_{\alpha,\varepsilon_{LY}} + \frac{K_{LY}}{1-\gamma_{LY}} \|\varphi\|_{L^1(\mu)} \\
&\leqslant \|\varphi\|_{L^1(\mu)} \left(\gamma_{LY}^T a + \frac{K_{LY}}{1-\gamma_{LY}} \right) \\
&\leqslant (\zeta_1 - \zeta_2 \varepsilon_{\mathcal{H}}^{\alpha} a)^{-1} \left(\gamma_{LY}^T a + \frac{K_{LY}}{1-\gamma_{LY}} \right) E[\mathcal{P}_F(\varphi)|\mathcal{H}] \\
&\leqslant \sigma a E[\mathcal{P}_F(\varphi)|\mathcal{H}].
\end{aligned}$$

■

3.2. $\mathcal{P}_{F_{i+T-1,i}}$ **is a contraction on** $\mathcal{C}_a$**.** See *e.g.* [4, 26, 35, 36] for information on cones and the Hilbert projective metric. Here we briefly introduce what we need in our context.

Define the partial order $\prec$ on $\mathcal{C}_a$ by declaring that $\varphi \prec \psi$ if $\psi - \varphi \in \mathcal{C}_a$. The Hilbert metric Θ is defined on $\mathcal{C}_a$ by

$$\Theta(\varphi, \psi) = \log \left(\frac{\inf \{ s > 0 : \psi \prec s\varphi \}}{\sup \{ r > 0 : r\varphi \prec \psi \}} \right).$$

Theorem 3.5 ([4])**.** *Let $i \in \mathbb{N}$. Define*

$$\Delta_i = \sup_{\varphi^*, \psi^* \in \mathcal{P}_{F_{i+T-1,i}}(\mathcal{C}_a)} \Theta(\varphi^*, \psi^*).$$

For every $\varphi, \psi \in \mathcal{C}_a$, we have

$$\Theta(\mathcal{P}_{F_{i+T-1,i}}(\varphi), \mathcal{P}_{F_{i+T-1,i}}(\psi)) \leqslant \tanh\left(\frac{\Delta_i}{4}\right) \Theta(\varphi, \psi).$$

Here $\tanh(\infty) = 1$.

The following lemma provides an upper bound on the diameter of $\mathcal{P}_{F_{i+T-1,i}}(\mathcal{C}_a)$ in $\mathcal{C}_a$ that is uniform in i.

Lemma 3.6. *For every $i \in \mathbb{N}$ and all $\varphi, \psi \in \mathcal{C}_a$, we have*

$$\Theta(\mathcal{P}_{F_{i+T-1,i}}(\varphi), \mathcal{P}_{F_{i+T-1,i}}(\psi)) \leqslant \Delta := 2 \log \left(\frac{1+\sigma}{1-\sigma} \right) + 2 \log \left(\frac{\zeta_2 (1 + a\varepsilon_{\mathcal{H}}^{\alpha})}{\zeta_1 - \zeta_2 \varepsilon_{\mathcal{H}}^{\alpha} a} \right) \tag{3.3}$$

Proof of Lemma 3.6. Let $\varphi^*, \psi^* \in \mathcal{C}_{\sigma a}$. Let r and s be such that

$$r\varphi^* \prec \psi^* \prec s\varphi^*.$$

Looking at $\psi^* - r\varphi^*$, we have

$$\begin{aligned}
|\psi^* - r\varphi^*|_{\alpha,\varepsilon_{LY}} &\leqslant |\psi^*|_{\alpha,\varepsilon_{LY}} + r |\varphi^*|_{\alpha,\varepsilon_{LY}} \\
&\leqslant \sigma a E[\psi^*|\mathcal{H}] + r\sigma a E[\varphi^*|\mathcal{H}].
\end{aligned}$$

Therefore $\psi^* - r\varphi^* \in \mathcal{C}_a$ if

$$\sigma a E[\psi^*|\mathcal{H}] + r\sigma a E[\varphi^*|\mathcal{H}] \leqslant aE[\psi^* - r\varphi^*|\mathcal{H}],$$

or equivalently,

$$r \leqslant \left(\frac{1-\sigma}{1+\sigma}\right)\frac{E[\psi^*|\mathcal{H}]}{E[\varphi^*|\mathcal{H}]}. \tag{3.4}$$

Arguing analogously, $s\varphi^* - \psi^* \in \mathcal{C}_a$ if

$$\left(\frac{1+\sigma}{1-\sigma}\right)\frac{E[\psi^*|\mathcal{H}]}{E[\varphi^*|\mathcal{H}]} \leqslant s. \tag{3.5}$$

Bounds (3.4) and (3.5) imply

$$\begin{aligned}\Theta(\varphi^*,\psi^*) &\leqslant \log\left(\left(\frac{1+\sigma}{1-\sigma}\right)\operatorname{Esup}\left(\frac{E[\psi^*|\mathcal{H}]}{E[\varphi^*|\mathcal{H}]}\right)\right) - \log\left(\left(\frac{1-\sigma}{1+\sigma}\right)\operatorname{Einf}\left(\frac{E[\psi^*|\mathcal{H}]}{E[\varphi^*|\mathcal{H}]}\right)\right)\\ &= 2\log\left(\frac{1+\sigma}{1-\sigma}\right) + \log\left(\left\|\frac{E[\psi^*|\mathcal{H}]}{E[\varphi^*|\mathcal{H}]}\right\|_{L^\infty}\left\|\frac{E[\varphi^*|\mathcal{H}]}{E[\psi^*|\mathcal{H}]}\right\|_{L^\infty}\right). \end{aligned}\tag{3.6}$$

Now let $\varphi,\psi \in \mathcal{C}_a$. Write $F_{i+T-1,i} = F$. Using Lemma 3.4, estimate (3.6), and Lemma 3.3, we have

$$\Theta(\mathcal{P}_F(\varphi),\mathcal{P}_F(\psi)) \leqslant \Delta := 2\log\left(\frac{1+\sigma}{1-\sigma}\right) + 2\log\left(\frac{\zeta_2(1+a\varepsilon_{\mathcal{H}}^\alpha)}{\zeta_1 - \zeta_2\varepsilon_{\mathcal{H}}^\alpha a}\right).$$

∎

Since the diameter of $\mathcal{P}_{F_{i+T-1,i}}(\mathcal{C}_a)$ is finite, Theorem 3.5 implies that $\mathcal{P}_{F_{i+T-1,i}}$ is a contraction on $\mathcal{C}_a$.

Proposition 3.7. *For every $i \in \mathbb{N}$ and all $\varphi,\psi \in \mathcal{C}_a$, we have*

$$\Theta(\mathcal{P}_{F_{i+T-1,i}}(\varphi),\mathcal{P}_{F_{i+T-1,i}}(\psi)) \leqslant \tanh\left(\frac{\Delta}{4}\right)\Theta(\varphi,\psi). \tag{3.7}$$

3.3. Comparing the OSC_α distance to the Θ distance.

Lemma 3.8. *For every $m \in \mathbb{N}$ and all $\varphi,\psi \in \mathcal{D}\cap\mathcal{C}_a$, we have*

$$\|\mathcal{P}_{F_{mT}}(\psi) - \mathcal{P}_{F_{mT}}(\varphi)\|_{\alpha,\varepsilon_{LY}} \leqslant (a(1+\varepsilon_{\mathcal{H}}^\alpha) + (2+a))\,\Theta(\mathcal{P}_{F_{mT}}(\psi),\mathcal{P}_{F_{mT}}(\varphi)). \tag{3.8}$$

Proof of Lemma 3.8. Write $F = F_{mT}$. Suppose $r,s \geqslant 0$ are such that $r \leqslant 1 \leqslant s$ and

$$r\mathcal{P}_F(\varphi) \prec \mathcal{P}_F(\psi) \prec s\mathcal{P}_F(\varphi).$$

Estimating the L^1 norm of $\mathcal{P}_F(\psi) - \mathcal{P}_F(\varphi)$, we have

$$\begin{aligned}\|\mathcal{P}_F(\psi) - \mathcal{P}_F(\varphi)\|_{L^1(\mu)} &= \int_{\mathbb{T}^N} |\mathcal{P}_F(\psi) - (r + (1-r))\mathcal{P}_F(\varphi)|\,\mathrm{d}\mu\\ &\leqslant \int_{\mathbb{T}^N} |\mathcal{P}_F(\psi) - r\mathcal{P}_F(\varphi)|\,\mathrm{d}\mu + (1-r)\\ &\leqslant |\mathcal{P}_F(\psi) - r\mathcal{P}_F(\varphi)|_{\alpha,\varepsilon_{LY}}\,\varepsilon_{\mathcal{H}}^\alpha + \int_{\mathbb{T}^N} \mathcal{P}_F(\psi) - r\mathcal{P}_F(\varphi)\,\mathrm{d}\mu + (1-r)\\ &= |\mathcal{P}_F(\psi) - r\mathcal{P}_F(\varphi)|_{\alpha,\varepsilon_{LY}}\,\varepsilon_{\mathcal{H}}^\alpha + 2(1-r). \end{aligned}\tag{3.9}$$

Estimating the oscillation seminorm of $\mathcal{P}_F(\psi) - \mathcal{P}_F(\varphi)$, we have

$$|\mathcal{P}_F(\psi) - \mathcal{P}_F(\varphi)|_{\alpha,\varepsilon_{LY}} \leqslant |\mathcal{P}_F(\psi) - r\mathcal{P}_F(\varphi)|_{\alpha,\varepsilon_{LY}} + (1-r)\,|\mathcal{P}_F(\varphi)|_{\alpha,\varepsilon_{LY}}. \tag{3.10}$$

Estimates (3.9) and (3.10) imply

$$\|\mathcal{P}_F(\psi) - \mathcal{P}_F(\varphi)\|_{\alpha,\varepsilon_{LY}} \leqslant a(1+\varepsilon_{\mathcal{H}}^\alpha)E[\mathcal{P}_F(\psi) - r\mathcal{P}_F(\varphi)|\mathcal{H}] + (1-r)(2 + aE[\mathcal{P}_F(\varphi)|\mathcal{H}]).$$

Integrating gives

$$\|\mathcal{P}_F(\psi) - \mathcal{P}_F(\varphi)\|_{\alpha,\varepsilon_{LY}} \leqslant (1-r)(a(1+\varepsilon_{\mathcal{H}}^\alpha) + (2+a)).$$

Finally, since $1 - r \leqslant -\log(r) \leqslant \log(s/r)$ for $s \geqslant 1$, we conclude that

$$\|\mathcal{P}_F(\psi) - \mathcal{P}_F(\varphi)\|_{\alpha,\varepsilon_{LY}} \leqslant (a(1+\varepsilon_{\mathcal{H}}^\alpha) + (2+a))\,\Theta(\mathcal{P}_F(\psi), \mathcal{P}_F(\varphi)).$$

■

3.4. **Completion of the proof of Theorem 2.16.** Let $\varphi, \psi \in \mathcal{D} \cap \mathcal{C}_a$. Let $m \in \mathbb{N}$. Write $m = kT + i$, where $k \in \mathbb{Z}^+$ and $0 \leqslant i < T$. Using (3.1), Lemma 3.8, and Proposition 3.7, we have

$$\begin{aligned}
&\|\mathcal{P}_{F_m}(\varphi) - \mathcal{P}_{F_m}(\psi)\|_{\alpha,\varepsilon_{LY}} \\
&\leqslant \left(1 + \frac{K_{LY}}{1-\gamma_{LY}}\right) \|\mathcal{P}_{F_{kT}}(\varphi) - \mathcal{P}_{F_{kT}}(\psi)\|_{\alpha,\varepsilon_{LY}} \\
&\leqslant \left(1 + \frac{K_{LY}}{1-\gamma_{LY}}\right) (a(1+\varepsilon_{\mathcal{H}}^\alpha) + (2+a))\,\Theta(\mathcal{P}_{F_{kT}}(\varphi), \mathcal{P}_{F_{kT}}(\psi)) \\
&\leqslant \left(1 + \frac{K_{LY}}{1-\gamma_{LY}}\right) (a(1+\varepsilon_{\mathcal{H}}^\alpha) + (2+a)) \cdot \max\{\Delta, 1\} \cdot \tanh^{-2}\left(\frac{\Delta}{4}\right) \left(\tanh\left(\frac{\Delta}{4}\right)^{1/T}\right)^m.
\end{aligned}$$

For general $\varphi, \psi \in \mathcal{D}$, choose $\eta_{\varphi,\psi} > 0$ sufficiently large so that

$$\frac{\varphi + \eta_{\varphi,\psi}}{1 + \eta_{\varphi,\psi}} \in \mathcal{D} \cap \mathcal{C}_a, \qquad \frac{\psi + \eta_{\varphi,\psi}}{1 + \eta_{\varphi,\psi}} \in \mathcal{D} \cap \mathcal{C}_a.$$

We have established (2.6) with

$$\begin{aligned}
C_{\varphi,\psi} &= K_{\varepsilon_{LY},\varepsilon_0}(1 + \eta_{\varphi,\psi}) \left(1 + \frac{K_{LY}}{1-\gamma_{LY}}\right) (a(1+\varepsilon_{\mathcal{H}}^\alpha) + (2+a)) \cdot \max\{\Delta, 1\} \cdot \tanh^{-2}\left(\frac{\Delta}{4}\right) \\
\Lambda &= \tanh\left(\frac{\Delta}{4}\right)^{1/T}.
\end{aligned}$$

Here the constant $K_{\varepsilon_{LY},\varepsilon_0}$ accounts for the equivalence of $\|\cdot\|_{\alpha,\varepsilon_{LY}}$ and $\|\cdot\|_{\alpha,\varepsilon_0}$.

4. Proof of Proposition 3.1

This result is in Saussol [40], for a class of piecewise expanding maps in which (PE5) below replaces (2.2). We are repeating the proof here to clarify how the constants are determined.

For reference, in Definition 4.1 we describe (using partially his notation) maps considered by Saussol [40, §2]. In Saussol's case the family $\{U_i\}_i$ can be countable and the domain of f is a compact set $\Omega \subset \mathbb{R}^N$ that is equal to the closure of its interior. We rewrote the properties for maps on $\mathbb{T}^N$.

Definition 4.1 (Saussol [40])**.** We say that $f : \mathbb{T}^N \to \mathbb{T}^N$ is an *admissible piecewise expanding* map if there exist a finite collection $\mathcal{A}_1 = \mathcal{A}_1(f) = \{U_i : 1 \leqslant i \leqslant M\}$ of pairwise disjoint open sets, $0 < \varepsilon_* < 1/2$, $s < 1$, $c_{\mathrm{detH}} > 0$ such that the following hold.

(PE1) (extension) For every $1 \leqslant i \leqslant M$ there exists an open set V_i satisfying $V_i \supset \overline{U}_i$ such that

- **(a)** $f|U_i$ extends to a map $f_{(i)} : V_i \to \mathbb{T}^N$
- **(b)** $f_{(i)}(V_i) \supset \overline{B_{\varepsilon_*}(f(U_i))}$.

(PE2) (regularity)

- **(a)** the map $f_{(i)}$ is a C^1 diffeomorphism from V_i to $f_{(i)}(V_i)$ and

(b) the determinant of $D[f_{(i)}^{-1}]$ is uniformly α-Hölder in the following sense:[11] for $0 < \varepsilon \leqslant s\varepsilon_*$, $z \in V_i$ and $x \in B_\varepsilon(z) \cap V_i$,

$$\left|\frac{1}{\det D_x f_{(i)}} - \frac{1}{\det D_z f_{(i)}}\right| \leqslant c_{\mathrm{detH}} \frac{\varepsilon^\alpha}{|\det D_z f_{(i)}|}$$

(PE3) $\mu\left(\mathbb{T}^N \setminus \bigcup_{i=1}^M U_i\right) = 0$.

(PE4) (backward contraction) For all $1 \leqslant i \leqslant M$ and $x, y \in f_{(i)}(V_i)$, we have

$$\mathrm{dist}_{\mathbb{T}^N}(x,y) \leqslant \varepsilon_* \implies \mathrm{dist}_{\mathbb{T}^N}\left((f_{(i)})^{-1}(x), (f_{(i)})^{-1}(y)\right) < s \cdot \mathrm{dist}_{\mathbb{T}^N}(x,y),$$

where $\mathrm{dist}_{\mathbb{T}^N}$ is the standard metric on $\mathbb{T}^N$.

(PE5) (cutting versus expansion) For $0 < \varepsilon \leqslant \varepsilon_*$ introduce the following constants and assume $\eta(f, \varepsilon_*)$ is finite[12]

$$\rho(f, \varepsilon, \varepsilon_*) := \sup_{x \in \mathbb{T}^N} \sum_{i=1}^M \frac{\mu\left(\left(f_{(i)}\right)^{-1}\left(B_\varepsilon(\partial f(U_i))\right) \cap B_{(1-s)\varepsilon_*}(x)\right)}{\mu(B_{(1-s)\varepsilon_*}(x))},$$

$$\eta(f, \varepsilon_*) := s^\alpha + 2 \sup_{\varepsilon \leqslant \varepsilon_*} \frac{\rho(f, \varepsilon, \varepsilon_*)}{\varepsilon^\alpha} {\varepsilon_*}^\alpha.$$

Following Saussol, we derive in Section 4.3 the following Lasota-Yorke inequality:

Theorem 4.2 (Saussol [40], Lemma 4.1)**.** *Assume f satisfies* (PE1)-(PE5)*. Then*

$$|\mathcal{P}_f(\varphi)|_{\alpha,\varepsilon_*} \leqslant \gamma_S |\varphi|_{\alpha,\varepsilon_*} + K_S \|\varphi\|_{L^1(\mu)}, \tag{4.1}$$

where

$$\gamma_S = (1 + c_{\mathrm{detH}} s^\alpha {\varepsilon_*}^\alpha)\, \eta(f, \varepsilon_*), \tag{4.2}$$

$$K_S = 2c_{\mathrm{detH}} s^\alpha + 2\,(1 + c_{\mathrm{detH}} s^\alpha {\varepsilon_*}^\alpha) \left(\sup_{\varepsilon \leqslant \varepsilon_*} \frac{\rho(f, \varepsilon, \varepsilon_*)}{\varepsilon^\alpha}\right). \tag{4.3}$$

Proof. See Section 4.3. ∎

4.1. **Proof of Proposition 3.1.** We verify that maps in $\mathcal{M}^*(s, K, \kappa, \varepsilon_0)$ satisfy the properties (PE1)-(PE5) and estimate the constants. Let $f \in \mathcal{M}^*(s, K, \kappa, \varepsilon_0)$. For $\varepsilon_{LY} \leqslant \varepsilon_0$, the properties that are not clear are (b) in (PE2) and (PE5).

Condition (PE2)(b)**:** Since $\|D_y f_{(i)}\| \leqslant K$ for $y \in V_i$ and $\|Df_{(i)}\|_{C^\alpha(V_i)} \leqslant K$, it follows that for $x, z \in V_i$,

$$\begin{aligned} |(\det D_x f_{(i)})^{-1} - (\det D_z f_{(i)})^{-1}| &= \left|\det D_x f_{(i)} - \det D_z f_{(i)}\right| |\det D_x f_{(i)}|^{-1} |\det D_z f_{(i)}|^{-1} \\ &\leqslant C_{\det}(N, K, s) \|Df_{(i)}\|_{C^\alpha} |\det D_z f_{(i)}|^{-1} \,\mathrm{dist}_{\mathbb{T}^N}(x,z)^\alpha \end{aligned}$$

where

$$\begin{aligned} C_{\det}(N, K, s) = \sup &\left\{\frac{|\det A - \det B|}{\|A - B\|} \mid A, B \in \mathrm{Mat}_{N\times N}(\mathbb{R}), A \neq B, \|A\|, \|B\| \leqslant K\right\} \\ &\cdot \sup\left\{|\det\left(A^{-1}\right)| \mid A \in \mathrm{GL}(N, \mathbb{R}), \|A^{-1}\| \leqslant s < 1\right\} < \infty. \end{aligned}$$

In conclusion,

$$c_{\mathrm{detH}} \leqslant C_{\det}(N, K, s) K. \tag{4.4}$$

[11]Saussol has this stated for $|\det D_u[f_{(i)}^{-1}] - \det D_v[f_{(i)}^{-1}]|$ with $w \in f_{(i)}(V_i)$, $u, v \in B_\varepsilon(w) \cap f_{(i)}(V_i)$.

[12]Saussol requires in addition that $\eta_{\varepsilon_*}(f) := \sup_{\gamma \leqslant \varepsilon_*} \eta(f, \gamma) < 1$.

Condition (PE5)**:** Saussol [40, Lemma 2.1] shows that (2.2) implies $\eta_{\varepsilon_*}(f) < 1$ if ε_* is chosen sufficiently small. The key estimate are equations (3) and (4) in the proof of Saussol's Lemma 2.1. In Lemma 4.3, proven below, we repeat this argument with a more precise statement.

Lemma 4.3. *Let $\Gamma \subset \mathbb{T}^N$ be a compact codimension-one embedded submanifold with C^2-norm bounded by K. Then*

$$\sup_x \mu(B_\nu(\Gamma) \cap B_\omega(x)) = 2\nu\omega^{N-1}\xi_{N-1}(1 + o_K(1)) \text{ as } \max\{\nu, \omega\} \to 0^+$$

where $o_K(1)$ means that this asymptotic is determined only by the value K.

To estimate $\rho(f, \varepsilon, \gamma)$, because of the backward contraction, we need Lemma 4.3 with $\nu = s\varepsilon < \gamma$ and $\omega = (1-s)\gamma$, where $0 < \varepsilon \leqslant \gamma \leqslant \varepsilon_0$:

$$\rho(f, \varepsilon, \gamma) \leqslant \kappa \frac{2(s\varepsilon)[(1-s)\gamma]^{N-1}\xi_{N-1}(1 + o_K(1))}{[(1-s)\gamma]^N \xi_N} = \kappa \frac{2s\varepsilon\xi_{N-1}}{(1-s)\gamma\xi_N}(1 + o_K(1)) \tag{4.5}$$

hence

$$\sup_{\varepsilon \leqslant \gamma} \frac{\rho(f, \varepsilon, \gamma)}{\varepsilon^\alpha} \leqslant \kappa \frac{2s}{1-s} \cdot \frac{\xi_{N-1}}{\xi_N} \cdot \gamma^{-\alpha}(1 + o_K(1)) \tag{4.6}$$

with

$$o_K(1) \to 0 \text{ as } \gamma \to 0^+.$$

so (PE5) holds.

Proof of Proposition 3.1. We conclude that one can apply Theorem 4.2 for maps in $f \in \mathcal{M}^*(s, K, \kappa, \varepsilon_0)$ and $\varepsilon_{LY} = \varepsilon_* \leqslant \varepsilon_0$. Bounds for c_{detH}, $\rho(f, \cdot, \varepsilon_*)$ and $\eta(f, \varepsilon_*)$ are determined by the dimension N and s, K, κ, using (4.4), (4.5) and (4.6). We obtain

$$\gamma_{LY} \leqslant (1 + c_{\mathrm{detH}} s^\alpha \varepsilon_{LY}^\alpha) \left(s^\alpha + 2\kappa \frac{2s\xi_{N-1}}{(1-s)\xi_N}(1 + o_K(1)) \right), \tag{4.7}$$

$$K_{LY} \leqslant 2c_{\mathrm{detH}} s^\alpha + 4\kappa \left(1 + c_{\mathrm{detH}} s^\alpha \varepsilon_{LY}^\alpha\right) \frac{s\xi_{N-1}}{(1-s)\xi_N} \varepsilon_{LY}^{-\alpha}(1 + o_K(1)) \tag{4.8}$$

with

$$o_K(1) \to 0 \text{ as } \varepsilon_{LY} \to 0^+.$$

Then, given (2.2) and taking into account the bound (4.4) for c_{detH}, there exists ε_{LY} sufficiently small, determined only by the dimension N and $s, K, \kappa, \varepsilon_0$, such that $\gamma_{LY} < 1$. ■

Proof of Lemma 4.3. We only sketch the idea, for more details see [40, Lemma 2.1]. Through a chart, one can map locally Γ into a hyperplane of $\mathbb{R}^N$. Up to a small distortion, we have now to intersect a ν-neighborhood of the hyperplane by a sphere of radius ω. The largest volume occurs when the center of the sphere is on the hyperplane, and then the intersection is close to a cylinder of height 2ν and radius ω. ■

4.2. **Preliminaries to the Proof of Theorem 4.2.** We begin by showing that OSC_α continuously injects into $L^\infty(\mu)$.

Lemma 4.4. *Let $\varphi \in L^1(\mu)$. If $a, b, c > 0$ satisfy $a + b \leqslant c \leqslant 1/2$, then for all $x \in \mathbb{T}^N$, we have*

$$\mathrm{Esup}(\varphi, B(x, a)) \leqslant \frac{1}{\mu(B(x, b))} \int_{B(x,b)} \left[\varphi(y) + \mathrm{osc}(\varphi, B(y, c))\right] \mathrm{d}y.$$

Proof of Lemma 4.4. Let $x \in \mathbb{T}^N$. For $y \in B(x,b)$ we have $B(x,a) \subset B(y,c)$. Consequently, we have

$$\operatorname{Esup}(\varphi, B(x,a)) \leqslant \operatorname{Esup}(\varphi, B(y,c)) \leqslant \varphi(y) + \operatorname{osc}(\varphi, B(y,c))$$

μ almost everywhere. Now average. ■

Lemma 4.5 (OSC_α continuously injects into $L^\infty(\mu)$)**.** *For all $\varphi \in \mathrm{OSC}_\alpha$, we have*

$$\|\varphi\|_{L^\infty(\mu)} \leqslant \frac{\max\{1, {\varepsilon_*}^\alpha\}}{\xi_N {\varepsilon_*}^N} \|\varphi\|_{\alpha,\varepsilon_*}.$$

Proof of Lemma 4.5. Using Lemma 4.4 with $a > 0$ and $b = \varepsilon_* - a$ gives

$$\|\varphi\|_{L^\infty(\mu)} \leqslant \frac{\max\{1, {\varepsilon_*}^\alpha\}}{\xi_N (\varepsilon_* - a)^N} \|\varphi\|_{\alpha,\varepsilon_*}.$$

Now let $a \to 0$. ■

Proposition 4.6 (Properties of osc)**.** *Let φ, $(\varphi_i)_{i=1}^M$, and ψ be elements of $L^\infty(\mu)$ such that $\psi \geqslant 0$. Let S be a Borel subset of $\mathbb{T}^N$.*

(O1)

$$\operatorname{osc}\left(\sum_{i=1}^M \varphi_i, S\right) \leqslant \sum_{i=1}^M \operatorname{osc}(\varphi_i, S).$$

(O2) *For all $a > 0$ and $x \in \mathbb{T}^N$ we have*

$$\begin{aligned}\operatorname{osc}(\varphi \mathbf{1}_S, B(x,a)) &\leqslant \operatorname{osc}(\varphi, S \cap B(x,a)) \cdot \mathbf{1}_{S \setminus B_a(\partial S)}(x) + 2 \cdot \operatorname{Esup}(|\varphi|, B(x,a) \cap S) \cdot \mathbf{1}_{B_a(\partial S)}(x) \\ &\leqslant \operatorname{osc}(\varphi, S \cap B(x,a)) \cdot \mathbf{1}_S(x) + 2 \cdot \operatorname{Esup}(|\varphi|, B(x,a) \cap S) \cdot \mathbf{1}_{B_a(\partial S)}(x).\end{aligned}$$

(O3) $\operatorname{osc}(\varphi\psi, S) \leqslant \operatorname{osc}(\varphi, S) \operatorname{Esup}(\psi, S) + \operatorname{osc}(\psi, S) \operatorname{Einf}(|\varphi|, S)$.

Proof of Proposition 4.6. (O1) is immediate.

For (O2), we check the first inequality (the second only gives a more convenient expression). It is easy to see that it holds whether $x \in S \setminus B_a(\partial S)$, $x \in B_a(\partial S)$ or x is in none of these two sets. For the second case use that $\operatorname{osc}(\varphi, S) \leqslant 2 \operatorname{Esup}(|\varphi|, S)$.

For (O3), if φ is nonnegative μ almost everywhere, then

$$\begin{aligned}\operatorname{osc}(\varphi\psi, S) &\leqslant \operatorname{Esup}(\varphi, S) \operatorname{Esup}(\psi, S) - \operatorname{Einf}(\varphi, S) \operatorname{Einf}(\psi, S) \\ &= \operatorname{Esup}(\psi, S)\big(\operatorname{Esup}(\varphi, S) - \operatorname{Einf}(\varphi, S)\big) + \operatorname{Einf}(\varphi, S)\big(\operatorname{Esup}(\psi, S) - \operatorname{Einf}(\psi, S)\big) \\ &= \operatorname{Esup}(\psi, S) \operatorname{osc}(\varphi, S) + \operatorname{Einf}(\varphi, S) \operatorname{osc}(\psi, S).\end{aligned}$$

If φ is nonpositive μ almost everywhere, argue similarly using $\operatorname{osc}(\varphi\psi, S) = \operatorname{osc}(-\varphi\psi, S)$. Otherwise, we have

$$\begin{aligned}\operatorname{Esup}(\varphi\psi, S) - \operatorname{Einf}(\varphi\psi, S) &= \operatorname{Esup}(\varphi\psi, S) + \operatorname{Esup}(-\varphi\psi, S) \\ &\leqslant \operatorname{Esup}(\psi, S)(\operatorname{Esup}(\varphi, S) + \operatorname{Esup}(-\varphi, S)) = \operatorname{Esup}(\psi, S) \operatorname{osc}(\varphi, S),\end{aligned}$$

so (O3) is proven. ■

4.3. **Proof of Theorem 4.2 (Lasota-Yorke inequality).** Recall that in our setting

$$\mathcal{P}_f(\varphi) = \sum_i \frac{\varphi}{|\det Df|} \circ (f|_{U_i})^{-1} \cdot \mathbf{1}_{f(U_i)} \quad \text{a.e.}$$

Assume f satisfies (PE1)-(PE5), $\varphi \in \mathrm{OSC}_\alpha$, and $\varepsilon \leqslant \varepsilon_*$. Using (O1) and (O2), for μ almost every $x \in \mathbb{T}^N$, we have

$$
\begin{aligned}
\operatorname{osc}(\mathcal{P}_f\varphi, B(x,\varepsilon)) &\leqslant \sum_{i=1}^{M} \operatorname{osc}\left(\left(\frac{\varphi}{|\det(Df)|}\circ (f|_{U_i})^{-1}\right)\mathbf{1}_{f(U_i)}, B(x,\varepsilon)\right)\\
&\leqslant \sum_{i=1}^{M} \operatorname{osc}\left(\frac{\varphi}{|\det(Df)|}\circ (f|_{U_i})^{-1}, f(U_i)\cap B(x,\varepsilon)\right)\mathbf{1}_{f(U_i)}(x)\\
&\qquad + 2\left(\operatorname{Esup}\left(\left|\frac{\varphi}{|\det(Df)|}\circ (f|_{U_i})^{-1}\right|, f(U_i)\cap B(x,\varepsilon)\right)\right)\mathbf{1}_{B_\varepsilon(\partial\, f(U_i))}(x)\\
&\leqslant \sum_{i=1}^{M} \operatorname{osc}\left(\frac{\varphi}{|\det(Df)|}, U_i \cap (f|_{U_i})^{-1}(B(x,\varepsilon))\right)\mathbf{1}_{f(U_i)}(x)\\
&\qquad + 2\left(\operatorname{Esup}\left(\frac{|\varphi|}{|\det(Df)|}, U_i \cap (f|_{U_i})^{-1}(B(x,\varepsilon))\right)\right)\mathbf{1}_{B_\varepsilon(\partial\, f(U_i))}(x).
\end{aligned}
\tag{4.9}
$$

We will check that the L^1 norm of the right side of (4.9) is bounded by

$$
\gamma_S\,|\varphi|_{\alpha,\varepsilon_*}\,\varepsilon^\alpha + K_S\|\varphi\|_{L^1(\mu)}\varepsilon^\alpha \tag{4.10}
$$

with γ_S and K_S as stated in (4.2) and (4.3).

We estimate the two components of the right side of (4.9) separately. For the first component, define

$$
R_i^{(1)}(x) := \operatorname{osc}\left(\varphi|\det(Df)|^{-1}, U_i\cap (f|_{U_i})^{-1}(B(x,\varepsilon))\right).
$$

For $x \in f(U_i)$, setting $z_i := (f|_{U_i})^{-1}(x)$ and using (PE4), we have

$$
R_i^{(1)}(x) \leqslant \operatorname{osc}(\varphi|\det(Df)|^{-1}, U_i\cap B(z_i, s\varepsilon)).
$$

(O3) implies that for μ almost every $x \in \mathbb{T}^N$ we have[13]

$$
\begin{aligned}
R_i^{(1)}(x) &\leqslant \operatorname{osc}(\varphi, U_i\cap B(z_i,s\varepsilon))\operatorname{Esup}(|\det(Df)|^{-1}, U_i\cap B(z_i,s\varepsilon))\\
&\qquad + \operatorname{osc}(|\det(Df)|^{-1}, U_i\cap B(z_i,s\varepsilon))\operatorname{Einf}(|\varphi|, U_i\cap B(z_i,s\varepsilon))\\
&\leqslant (1+c_{\det\mathrm{H}}(s\varepsilon)^\alpha)\,|\det(Df(z_i))|^{-1}\operatorname{osc}(\varphi, U_i\cap B(z_i,s\varepsilon))\\
&\quad + 2c_{\det\mathrm{H}}(s\varepsilon)^\alpha|\varphi(z_i)|\cdot|\det(Df(z_i))|^{-1}.
\end{aligned}
$$

For Lebesgue almost every $x \in \mathbb{T}^N$ the first component of the right side of (4.9) therefore satisfies

$$
\sum_{i=1}^{M} R_i^{(1)}\mathbf{1}_{f(U_i)}(x) \leqslant (1+c_{\det\mathrm{H}}(s\varepsilon)^\alpha)\,\mathcal{P}_f\big(\operatorname{osc}(\varphi, B(\cdot, s\varepsilon))\big) + 2c_{\det\mathrm{H}}(s\varepsilon)^\alpha\mathcal{P}_f|\varphi|.
$$

Integrating over $\mathbb{T}^N$ and using (2.5) yields

$$
\int_{\mathbb{T}^N}\sum_{i=1}^{M} R_i^{(1)}\mathbf{1}_{f(U_i)}\,\mathrm{d}\mu \leqslant (1+c_{\det\mathrm{H}}(s\varepsilon)^\alpha)\,|\varphi|_{\alpha,\varepsilon_*}(s\varepsilon)^\alpha + 2c_{\det\mathrm{H}}(s\varepsilon)^\alpha\|\varphi\|_{L^1(\mu)}. \tag{4.11}
$$

For the second component of the right side of (4.9) we use the extensions $f_{(i)}$ to V_i. Define

$$
R_i^{(2)}(x) = \operatorname{Esup}\left(\frac{|\varphi|}{|\det(Df)|}, U_i\cap (f|_{U_i})^{-1}(B(x,\varepsilon))\right)\cdot\mathbf{1}_{B_\varepsilon(\partial\, f(U_i))}(x).
$$

[13]We use that $\operatorname{Einf}(|\varphi|, U_i\cap B(z_i,s\varepsilon)) \leqslant |\varphi(z_i)|$ for a.e. $x = f(z_i)$, $z_i\in U_i$.

and let [14] $z_i := f_{(i)}^{-1}(x) \in V_i$ for $x \in B_\varepsilon(\partial\, f(U_i))$. Using the regularity of $\det(Df_{(i)})$, we have

$$R_i^{(2)}(x) \leqslant (1 + c_{\mathrm{detH}}(s\varepsilon)^\alpha)\,\mathrm{Esup}\,(|\varphi|, B(z_i, s\varepsilon))\,|\det(Df_{(i)}(z_i))|^{-1} \cdot \mathbf{1}_{B_\varepsilon(\partial\, f(U_i))}(x).$$

Integrating, changing variables $z_i = f_{(i)}^{-1}(x)$ and using Lemma 4.4 with $a = s\varepsilon$, $b = (1-s)\varepsilon_*$, and $c = \varepsilon_*$ yields

$$\begin{aligned}
&(1 + c_{\mathrm{detH}}(s\varepsilon)^\alpha)^{-1} \int_{\mathbb{T}^N} R_i^{(2)}(x)\,\mathrm{d}\mu(x) \\
&\quad\leqslant \int_{\mathbb{T}^N} \mathrm{Esup}(|\varphi|, B(z_i, s\varepsilon))|\det(Df_{(i)}(z_i))|^{-1} \cdot \mathbf{1}_{B_\varepsilon(\partial\, f(U_i))}(x)\,\mathrm{d}\mu(x) \\
&\quad= \int_{V_i} \mathrm{Esup}(|\varphi|, B(z_i, s\varepsilon)) \cdot \mathbf{1}_{B_\varepsilon(\partial\, f(U_i))}(f_{(i)}(z_i))\,\mathrm{d}\mu(z_i) \\
&\quad\leqslant \int_{V_i} \mathbf{1}_{B_\varepsilon(\partial\, f(U_i)}(f_{(i)}(z)) \times \\
&\qquad\qquad \left[\frac{1}{\mu(B(z,(1-s)\varepsilon_*))} \int_{B(z,(1-s)\varepsilon_*)} \big[|\varphi(\zeta)| + \mathrm{osc}(|\varphi|, B(\zeta, \varepsilon_*))\big]\,\mathrm{d}\mu(\zeta)\right] \mathrm{d}\mu(z) \\
&\quad= \int_{\mathbb{T}^N} \big[|\varphi(\zeta)| + \mathrm{osc}(|\varphi|, B(\zeta,\varepsilon_*))\big] \left[\int_{V_i} \frac{\mathbf{1}_{f_{(i)}^{-1}(B_\varepsilon(\partial f(U_i)))}(z) \cdot \mathbf{1}_{B(\zeta,(1-s)\varepsilon_*)}(z)}{\mu(B(z,(1-s)\varepsilon_*))}\,\mathrm{d}\mu(z)\right] \mathrm{d}\mu(\zeta) \\
&\quad= \int_{\mathbb{T}^N} \big[|\varphi(\zeta)| + \mathrm{osc}(|\varphi|, B(\zeta,\varepsilon_*))\big] \times \frac{\mu\big(f_{(i)}^{-1}(B_\varepsilon(\partial f(U_i))) \cap B(\zeta,(1-s)\varepsilon_*)\big)}{\mu(B(\zeta,(1-s)\varepsilon_*))}\,\mathrm{d}\mu(\zeta).
\end{aligned}$$

We arrive at the estimate

$$(1 + c_{\mathrm{detH}}(s\varepsilon)^\alpha)^{-1} \int_{\mathbb{T}^N} \sum_{i=1}^M R_i^{(2)}(x)\,\mathrm{d}\mu(x) \leqslant \rho(f,\varepsilon,\varepsilon_*)\big(\|\varphi\|_{L^1(\mu)} + |\varphi|_{\alpha,\varepsilon_*}\,{\varepsilon_*}^\alpha\big). \tag{4.12}$$

Combining estimates (4.11) and (4.12), we have

$$\begin{aligned}
\int_{\mathbb{T}^N} \mathrm{osc}(\mathcal{P}_f\varphi, B(x,\varepsilon))\,\mathrm{d}\mu(x) &\leqslant (1 + c_{\mathrm{detH}} s^\alpha\varepsilon^\alpha)\,|\varphi|_{\alpha,\varepsilon_*}\, s^\alpha\varepsilon^\alpha + 2c_{\mathrm{detH}} s^\alpha\varepsilon^\alpha\|\varphi\|_{L^1(\mu)} \\
&\quad + 2\,(1 + c_{\mathrm{detH}} s^\alpha\varepsilon^\alpha)\,\rho(f,\varepsilon,\varepsilon_*)\big(\|\varphi\|_{L^1(\mu)} + |\varphi|_{\alpha,\varepsilon_*}\,{\varepsilon_*}^\alpha\big).
\end{aligned}$$

Then inequality (4.10) holds with

$$\begin{aligned}
\gamma_S &= (1 + c_{\mathrm{detH}} s^\alpha{\varepsilon_*}^\alpha)\left(s^\alpha + 2\cdot\sup_{\varepsilon\leqslant\varepsilon_*}\frac{\rho(f,\varepsilon,\varepsilon_*)}{\varepsilon^\alpha}{\varepsilon_*}^\alpha\right) = (1 + c_{\mathrm{detH}} s^\alpha{\varepsilon_*}^\alpha)\,\eta(f,\varepsilon_*) \\
K_S &= 2c_{\mathrm{detH}} s^\alpha + 2\,(1 + c_{\mathrm{detH}} s^\alpha{\varepsilon_*}^\alpha)\left(\sup_{\varepsilon\leqslant\varepsilon_*}\frac{\rho(f,\varepsilon,\varepsilon_*)}{\varepsilon^\alpha}\right)
\end{aligned}$$

as claimed. ■

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