

# ON A SEMIGROUP PROBLEM

VIOREL NITICA\*

Department of Mathematics  
West Chester University of Pennsylvania  
West Chester, PA 19383, USA and  
Institute of Mathematics, Romanian Academy  
P.O. Box 1-764, RO-70700 Bucharest, Romania  
vnitica@wcupa.edu

ANDREI TÖRÖK

Department of Mathematics  
University of Houston  
Houston, TX 77204-3308, USA and  
Institute of Mathematics, Romanian Academy  
P.O. Box 1-764, RO-70700 Bucharest, Romania  
torok@math.uh.edu

ABSTRACT. If  $S$  is a semigroup in  $\mathbb{R}^n$  that is not separated by a linear functional, then it is known that the closure of  $S$  is a group. We investigate a similar statement in an infinite dimensional topological vector space  $X$ . We show that if  $X$  is an infinite dimensional Banach space, then there exists a semigroup  $S \subset X$ , not separated by the continuous functionals supported by the closed linear span of  $S$ , for which the closure of the semigroup is not a group. If  $X$  is an infinite dimensional Fréchet space, then the closure of a semigroup that is not separated is always a group if and only if  $X$  is  $\mathbb{R}^\omega$ , the countably infinite direct product of lines. Other infinite dimensional topological vector spaces, such as  $\mathbb{R}^\infty$ , the countably infinite direct sum of lines, are discussed. The Semigroup Problem has applications to the study of certain dynamical systems, in particular for the construction of topologically transitive extensions of hyperbolic systems. Some examples are shown in the paper.

## 1. INTRODUCTION

The topological vector spaces considered here are over the field  $\mathbb{R}$  of real numbers.

The goal of this paper is to investigate the following basic problem:

**Semigroup Problem.** *Let  $X$  be a Hausdorff topological vector space and let  $S \subset X$  be a semigroup. Let  $X_0$  be the closure of the linear span of  $S$ . Assume that  $S$  is not separated by any continuous linear functional in the dual of  $X_0$ , that is, for any  $\phi \in (X_0)^* \setminus \{0\}$  there exists  $x_1, x_2 \in S$  such that  $\phi(x_1) > 0$  and  $\phi(x_2) < 0$ . Does it follow that the closure of  $S$  is a group?*

The problem was studied so far in the finite dimensional setting. Take  $\mathbb{R}^n$  endowed with the Euclidean topology. If  $S$  is semigroup in  $\mathbb{R}^n$  not separated by any

---

1991 *Mathematics Subject Classification.* Primary: 37B, 37D; Secondary: 58B..

*Key words and phrases.* Semigroup, group, Banach space, Fréchet space, separation by functionals, topological transitivity, extension, hyperbolic dynamical system, infinite direct product of lines, infinite direct sum of lines.

VN was partially supported by Simons Foundation Grant 208729. AT was partially supported by Simons Foundation Grant 239583.

\* Corresponding author: Viorel Nitica.

linear functional then it is known that the closure of  $S$  is a group [16, 17]. A similar problem, in which separation by linear functionals is replaced by separation by certain maximal semigroups, was investigated for several classes of finite dimensional non-compact Lie groups such as Euclidean groups [16], nilpotent groups [20] and solvable groups [14].

While the Semigroup Problem is of independent interest, it is also relevant for the study of certain dynamical systems, in particular for construction of topologically transitive extensions of hyperbolic systems. A large collection of results about transitivity of extensions of hyperbolic dynamical systems with fiber a finite dimensional Lie group were recently obtained. An extended up to date review of the available material can be found in [19]. Much less is known if the fiber is an infinite dimensional topological group. The present paper shows that certain results from the finite dimensional setup are difficult to extend.

We describe now the structure of the paper. In Section 2 we show that if  $X$  is  $\mathbb{R}^\omega$ , the countably infinite direct product of lines, then the Semigroup Problem has a positive answer. In Section 3 we show that if  $X$  is  $\mathbb{R}^\infty$ , the countably infinite direct sum of lines, then the Semigroup Problem has a negative answer. The example we describe in Section 3 is crucial for the rest of the paper. In Section 4 we show that if  $X$  is an infinite dimensional Banach space, then there are sets  $S \subset X$  for which the Semigroup Problem has a negative answer. In Section 5 we show that if  $X$  is an infinite dimensional Fréchet space different from  $\mathbb{R}^\omega$ , then there are sets  $S \subset X$  for which the Semigroup Problem has a negative answer. In Section 6 we investigate topological transitivity of extensions with fiber  $\mathbb{R}^\omega$  and  $\mathbb{R}^\infty$ . In Section 7 we discuss further results and additional directions of research.

## 2. THE CASE $X = \mathbb{R}^\omega$

We first recall the following result, proved in [16],[17].

**Theorem 2.1.** *Assume that the semigroup  $S \subset \mathbb{R}^n$  is not separated by any non-zero linear functional. Then the closure of  $S$  is a group.*

We review the structure of the space  $\mathbb{R}^\omega = \prod_{n=1}^{\infty} \mathbb{R}$ . An element  $x \in \mathbb{R}^\omega$  is an infinite sequence of real numbers  $x = (x_n)_{n=1}^{\infty}$ . The topology is the usual Tychonoff product topology. The space has a structure of Fréchet space with the countable family of seminorms  $\|x\|_n = |x_n|$ . As such, it is metrizable. An invariant metric given by:

$$(1) \quad d_{\mathbb{R}^\omega}((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$

A sequence  $(x(k))_{k=1}^{\infty} \subset \mathbb{R}^\omega$  is convergent if and only if it converges on components. A continuous linear functional  $\phi \in (\mathbb{R}^\omega)^*$  can be identified with a sequence of real numbers with finite support  $\phi = (\phi_n)_{n=1}^{\infty}$ , that is there exists an integer  $N(\phi)$  such that  $\phi_n = 0$  if  $n > N(\phi)$ . The action of a functional  $\phi$  on a sequence  $x$  is given by  $\phi(x) = \sum_{n=1}^{\infty} \phi_n x_n$ .

**Theorem 2.2.** *Let  $S \subset \mathbb{R}^\omega$  be a semigroup that is not separated by any non-zero continuous linear functional. Then the closure of  $S$  is a group.*

*Proof.* It follows from above that all continuous linear functionals  $\phi$  on  $\mathbb{R}^\omega$  are of the form  $\phi = \psi \circ \pi_n$  for some positive integer  $n$  with  $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$  a linear functional and  $\pi_n : \mathbb{R}^\omega \rightarrow \mathbb{R}^n$  the canonical projection on the first  $n$  components.

It follows that if  $S$  is not separated by continuous linear functionals, then neither is  $\pi_n(S) \subset \mathbb{R}^n$ ,  $n \in \mathbb{N}$ . Hence Theorem 2.1 implies that the closure of  $\pi_n(S)$  in  $\mathbb{R}^n$  is a group.

Take now a point  $x$  in the closure of  $S$ . Then  $\pi_n(x)$  is a limit point of  $\pi_n(S)$  so  $-\pi_n(x)$  is also a limit; therefore one can find for each positive integer  $n$  a point  $x(n) \in S$  such that  $\pi_n(x(n))$  is within  $\frac{1}{n}$  on each coordinate from  $-\pi_n(x)$ . This means that the sequence  $(x(n))_{n=1}^\infty$  converges to  $-x$ .

Therefore the closure of  $S$  is a group.  $\square$

### 3. THE CASE $X = \mathbb{R}^\infty$

We start with general facts about  $\mathbb{R}^\infty$ , the countably infinite direct sum of copies of  $\mathbb{R}$ . A good reference to use here is [7]. The elements in  $\mathbb{R}^\infty$  are infinite sequences  $(h_n)_{n=1}^\infty$  of real numbers such that  $h_n = 0$  for all but a finite number of entries. We consider the finite dimensional groups  $\mathbb{R}^n$  embedded in  $\mathbb{R}^\infty$  on the first  $n$  coordinates. The topology we use on  $\mathbb{R}^\infty$  is the so-called *box* or *rectangular topology* and it is finer than the Tychonoff product topology. It is the induced topology from the topology defined on the infinite product  $\mathbb{R}^\omega$  by the basis consisting of the collection of all infinite products  $\prod_i U_i$ , where  $U_i$  is an open set in the  $i$ -th component of  $\mathbb{R}^\omega$ . Endowed with this topology  $\mathbb{R}^\infty$  becomes a topological group, as well as a separable topological vector space [7, Proposition 2], and the induced topology on each  $\mathbb{R}^n$  is the usual product topology. Any linear map  $\phi : \mathbb{R}^\infty \rightarrow \mathbb{R}$  is a continuous linear functional [7, Proposition 2]. A continuous linear functional  $\phi$  can be identified with a sequence of real numbers  $\phi = (\phi_n)_{n=1}^\infty$ . The action of a functional  $\phi$  on a sequence  $h$  is given by  $\phi(x) = \sum_{n=1}^\infty \phi_n h_n$ . The space  $\mathbb{R}^\infty$  has a structure of countable strict inductive limit of Fréchet spaces, or *LF-space* [28, Chapter 13]. As such, it is complete, but not metrizable. Hence it is not a Fréchet space.

We consider the semigroup  $S \subset \mathbb{R}^\infty$  generated by elements  $s(k)$  of type:

$$(2) \quad s(k)_n = \begin{cases} 1, & \text{if } n = k, \\ p \in \mathbb{Z}, & \text{if } 1 \leq n \leq k-1, \\ 0, & \text{if } n > k. \end{cases}$$

So  $s(k)$  is zero beyond the  $k$ -th entry, the  $k$ -th entry is 1 and the first  $k-1$  entries are arbitrary integers.

**Proposition 3.1.** *The semigroup  $S \subset \mathbb{R}^\infty$  is not separated by any non-zero continuous linear functional.*

*Proof.* Let  $\phi = (\phi_n)_{n=1}^\infty$  be a linear functional. We have the following cases, depending on the initial non-zero entries in  $\phi$ :

Case 1.  $\phi_k \neq 0$  and  $\phi_n = 0$  if  $n \neq k$ .

Consider any element  $x \in \mathbb{R}^\infty$  of type  $s(k)$  given by  $x_k = 1$  and the element  $y \in \mathbb{R}^\infty$  of type  $s(k+1)$  given by  $y_{k+1} = 1, y_k = p, y_n = 0, n \notin \{k, k+1\}$ . Then  $\phi(x) = \phi_k, \phi(y) = \phi_k p$  with  $p \in \mathbb{Z}$  arbitrary, so  $\phi$  does not separate  $S$ .

Case 2.  $\phi$  has at least two non-zero entries, with the first two being,  $\phi_{k_1}$  and  $\phi_{k_2}$ ,  $k_1 < k_2$ .

Consider the element  $x \in \mathbb{R}^\infty$  of type  $s(k_2)$  given by  $x_{k_2} = 1, x_n = 0, n \neq k_2$  and the element  $y \in \mathbb{R}^\infty$  of type  $s(k_2)$  given by  $y_{k_2} = 1, y_{k_1} = p, y_n = 0, n \notin \{k_1, k_2\}$ . Then  $\phi(x) = \phi_{k_2}, \phi(y) = \phi_{k_1} p + \phi_{k_2}$  with  $p \in \mathbb{Z}$  arbitrary, so  $\phi$  does not separate  $S$ .  $\square$

The next proposition shows that the answer to the Semigroup Problem for  $X = \mathbb{R}^\infty$  is negative.

**Proposition 3.2.** *The element  $0 \in \mathbb{R}^\infty$  does not belong to the closure of  $S$ .*

*Proof.* Any element in  $S$  is a finite linear combination with positive integer coefficients of elements of type  $s(k)$ . As such, the leading non-zero coefficient is always a positive integer. A neighborhood  $U$  of 0 that is a countable product of intervals  $(-.5, .5)$  does not contain any element from  $S$ , so 0 is not in the closure of  $S$ .  $\square$

Under additional assumptions, the Semigroup Problem for  $\mathbb{R}^\infty$  has a positive answer. If  $S \subset \mathbb{R}^\infty$  we denote by  $S_n$  the set of elements in  $S$  with the support included in the first  $n$  components.

**Theorem 3.1.** *Let  $S \subset \mathbb{R}^\infty$  be a semigroup. Assume that there exists an increasing sequence  $(n_i)_{i=1}^\infty$  of positive integers such that  $S_{n_i}$  is not separated by any non-zero continuous linear functional for all  $i$ . Then the closure of  $S$  is a group.*

*Proof.* Let  $g$  be a limit point of the semigroup  $S$ . Note that  $g$  has all but a finite number of entries zero, so there exists  $n_0$  such that  $g_i = 0$  for  $i \geq n_0$ . Due to our assumption there exists  $m_0 \geq n_0$  such that  $S_{m_0}$  is not separated. Then the set  $S_{m_0} \cup \{g\}$  is not separated either, and it follows from Proposition 2.1 that the closure of the semigroup generated by it is a subgroup in the closure of  $S$ . We conclude that  $g^{-1}$  belongs to the closure of  $S$ .  $\square$

#### 4. THE CASE WHERE $X$ IS A BANACH SPACE

Let  $(X, \|\cdot\|)$  be an infinite dimensional Banach space with a complete norm. We recall several basic facts that are needed for our result. They can be found in the classical Banach spaces monograph of Lindenstrauss and Tzafriri [13].

A sequence  $(x_n)_{n=1}^\infty \subset X$  is called a *Schauder basis* of  $X$  if for every  $x \in X$  there exists a unique sequence of scalars  $(a_n)_{n=1}^\infty$  such that  $x = \sum_{n=1}^\infty a_n x_n$ . A sequence  $(x_n)_{n=1}^\infty \subset X$  which is a Schauder basis of its closed linear span is called a *basic sequence*. It is a standard result attributed to Mazur that any Banach space has a basic sequence for a closed subspace. We can assume that the basic sequence  $(x_n)_{n=1}^\infty$  is normalized, that is  $\|x_n\| = 1$ .

A Banach space  $X$  with a Schauder basis  $(x_n)_{n=1}^\infty \subset X$  can be considered a sequence space by identifying  $x = \sum_{n=1}^\infty a_n x_n$  with the sequence  $(a_1, a_2, a_3, \dots)$ . The sequence space contains all sequences with finite support.

If  $(x_n)_{n=1}^\infty \subset X$  is a Schauder basis of  $X$ , then the linear operators  $P_n : X \rightarrow X$  given by  $P_n(\sum_{k=1}^\infty a_k x_k) = \sum_{k=1}^n a_k x_k$  are uniformly bounded. One can always pass to a different norm on  $X$  for which the uniformity constant is 1.

If  $X$  is a Banach space, denote by  $X^*$  its dual. Let  $X$  be a Banach space with a Schauder basis  $(x_n)_{n=1}^\infty \subset X$ . For every integer  $n$  the linear functional  $x_n^* : X \rightarrow \mathbb{R}$  defined by  $x_n^*(\sum_{k=1}^\infty a_k x_k) = a_n$  is a continuous linear functional. For any functional  $x^* \in X^*$  and  $x = \sum_{k=1}^\infty a_k x_k \in X$  one has  $x^*(x) = \sum_{k=1}^\infty a_k x^*(x_k)$ , that is, we can associate to  $x^*$  the sequence of scalars

$$(3) \quad (x^*(x_1), x^*(x_2), x^*(x_3), \dots).$$

The next theorem shows that the answer to the Semigroup Problem for any Banach space is negative.

**Theorem 4.1.** *Let  $X$  be an infinite dimensional Banach space. Then  $X$  contains a semigroup  $S$  that is not separated by any non-zero continuous linear functional (supported on the closure of the linear span of  $S$ ) and which does not contain 0 in its closure.*

*Proof.* Let  $(x_n)_{n=1}^{\infty}$  be a basic sequence in  $X$  and let  $X_0$  be the closed linear subspace spanned by the basic sequence in  $X$ . Using the identification of  $X_0$  with a sequence space, we construct the semigroup  $S$  using the same construction as in Section 3. Using the sequences of scalars associated to the functionals in  $X_0^*$  (3), one can carry over the proof of Proposition 3.1 and show that  $S$  is not separated by any functional in  $X_0^*$ .

It remains to show that 0 is not in the closure of  $S$ . Any element in  $S$  is a finite linear combination with positive integer coefficients of elements of type  $s(k)$ . As such, the leading non-zero coefficient of each element in  $S$  is always a positive integer. We proceed by contradiction. Let  $(x(n))_{n=1}^{\infty} \subset X_0$  be a sequence of elements in  $S$  that converges to 0. Using the identification with a sequence space, let  $x(n) = (x(n)_k)_k$ . Using that the projections  $P_n$  are uniformly bounded by 1 and that the Schauder basis is normalized, one has:

$$(4) \quad |x(n)_k| = \|P_k(x(n)) - P_{k-1}(x(n))\| \leq 2\|x(n)\| \quad \text{for all } k, n.$$

Now  $(\|x(n)\|)_{n=1}^{\infty}$  converges to 0 gives a contradiction with  $|x(n)_k| \geq 1$  for  $k$  the index of the leading non-zero coefficient of  $x(n)$ .  $\square$

## 5. THE CASE WHERE $X$ IS A FRÉCHET SPACE

Let  $X$  be an infinite dimensional Fréchet space. We recall some standard facts that can be found in [22] and [24]. A Fréchet space is a locally convex space that is complete with respect to a translation invariant metric. The topology can be defined using a countable increasing sequence of seminorms  $(p_n)_{n=1}^{\infty}$ . If a Fréchet space admits a continuous norm, we can take all the seminorms to be norms  $\|\cdot\|_n$  by adding the continuous norm to each of them.

A sequence  $(x_n)_{n=1}^{\infty}$  is a *Schauder basis* in  $X$  if every  $x \in X$  has a unique series expansion  $x = \sum_{n=1}^{\infty} a_n x_n$ , where  $a_n$  are scalars. If  $X$  has a basis  $(x_k)_{k=1}^{\infty}$  and a norm  $\|\cdot\|_0$ , we can assume that  $\|x_k\|_0 = 1$  for all  $k$ . In conjunction with the above, we can assume  $\|x_k\|_n \geq 1$  for all  $n$  and  $k$ . If for every  $n \in \mathbb{N}$ , there is some  $C > 0$  and  $q \in \mathbb{N}$  such that  $|a_k| \|x_k\|_n = \|a_k x_k\|_n \leq C \|x\|_q$ , for all  $x \in X$  and  $k \in \mathbb{N}$ , then the basis is *equicontinuous* [22, 10.1.2]. According to [22, Theorem 10.1.2], every Schauder basis in a Fréchet space is equicontinuous. As in the case for Banach spaces, the existence of a basis allows to identify  $X$  with a sequence space that contains all sequences with finite support.

Let  $X^*$  be the dual of  $X$  and  $(x_n)_{n=1}^{\infty} \subset X$  be a Schauder basis for  $X$ . For any continuous functional  $x^* \in X^*$  and  $x = \sum_{k=1}^{\infty} a_k x_k$  one has  $x^*(x) = \sum_{k=1}^{\infty} a_k x^*(x_k)$ , that is, we can associate to  $x^*$  the sequence of scalars

$$(5) \quad (x^*(x_1), x^*(x_2), x^*(x_3), \dots).$$

The next theorem answers the Semigroup Problem for a Fréchet space.

**Theorem 5.1.** *Let  $X$  be an infinite dimensional Fréchet space. Then either  $X$  is isomorphic to  $\mathbb{R}^{\omega}$  or there exists a semigroup  $S \subset X$ , not separated by any non-zero continuous linear functional, such that  $S$  does not have 0 in its closure. In particular, the Semigroup Problem has a positive answer if and only if  $X$  is isomorphic to  $\mathbb{R}^{\omega}$ .*

*Proof.* A structure theorem of Bessaga, Peczyński and Rolewicz [4] shows that any Fréchet space is either isomorphic to a product of a Banach space and  $\mathbb{R}^\omega$  or contains a closed subspace which is topologically isomorphic to an infinite dimensional nuclear Fréchet space with Schauder basis and a continuous norm.

If  $X$  is isomorphic to a product of a Banach space and  $\mathbb{R}^\omega$ , and the Banach space is infinite dimensional, then Theorem 4.1 implies that the Semigroup Problem has a negative answer.

Assume now that  $X$  contains a closed subspace  $X_0$  which is topologically isomorphic to an infinite dimensional nuclear Fréchet space with Schauder basis  $(x_n)_{n=1}^\infty$  and a continuous norm. Using the identification of  $X_0$  with a sequence space, we construct the semigroup  $S$  using the same construction as in Section 3. Using the sequences of scalars associated to the functionals in  $X_0^*$ , one can carry over the proof of Proposition 3.1 and show that  $S$  is not separated by any functional in  $X_0^*$ .

It remains to show that  $0$  is not in the closure of  $S$ . Any element in  $S$  is a finite linear combination with positive integer coefficients of elements of type  $s(k)$ . The leading non-zero coefficient of each element in  $S$  is always a positive integer. We proceed by contradiction. Let  $(x(n))_{n=1}^\infty \subset X_0$  be a sequence of elements in  $S$  that converges to  $0$ . Using the identification with a sequence space, let  $x(n) = (x(n)_k)_k$ . Using that the Schauder basis is equicontinuous, there exist  $C > 0$  and  $q$  such that:

$$(6) \quad |x(n)_k| \leq |x(n)_k| \|x_k\|_n \leq C \|x(n)\|_q,$$

for all  $n, k$ .

Now as  $(\|x(n)\|_q)_{n=1}^\infty$  converges to  $0$ , this gives a contradiction with  $|x(n)_k| \geq 1$  for  $k$  the index of the leading non-zero coefficient of  $x(n)$ .  $\square$

## 6. APPLICATIONS

As mentioned in the introduction, the Semigroup Problem is of interest due to applications to the construction and classification of topologically transitive extensions of hyperbolic systems.

We introduce some necessary terminology.

Let  $(X, d_X)$  be a compact metric topological space,  $f : X \rightarrow X$  a continuous map and  $(G, +, d_G)$  an Abelian topological group. A function  $\beta : X \rightarrow G$  will be referred to as a *cocycle*. Given  $f, \beta, G$  we consider the *skew product* (or *extension*)  $f_\beta : X \times G \rightarrow X \times G$  given by

$$f_\beta(x, g) = (f(x), g + \beta(x)), x \in X, g \in G.$$

We will refer to  $X$  as the *base* of the skew product and to  $G$  as the *fiber* of the skew product. One has:

$$(7) \quad f_\beta^n(x, g) = (f^n x, g + \beta(x) + \beta(f^2 x) + \cdots + \beta(f^{n-1} x)), x \in X, g \in G.$$

The extension  $f_\beta$  is called *topologically transitive* if for every pair of non-empty open sets  $U$  and  $V$  in  $X$ , there is a non-negative integer  $n$  such that  $f^n(U) \cap V \neq \emptyset$ . The extension  $f_\beta$  is called *weak topologically mixing* if  $f_\beta \times f_\beta$  is topologically transitive. We observe that if the group  $G$  is not metrizable, the notion of topological transitivity we use is weaker than having a dense positive semi-orbit  $\{f_\beta^n(x_0, g_0), n \in \mathbb{N}\}$ .

Let  $C(X, G)$  be the space of continuous functions from  $X$  to  $G$  endowed with the compact-open topology, or, if  $(G, d_G)$  is a metric space, endowed with the metric  $d_{C^0}(\beta_1, \beta_2) = \sup_{x \in X} d_G(\beta_1(x), \beta_2(x))$ . An extension  $f_\beta$  is called  *$C^0$ -stably transitive* ( *$C^0$ -stably topologically mixing*) if the cocycle  $\beta$  has an open neighborhood

$\mathcal{V} \subset C(X, G)$  such that for any  $\beta' \in \mathcal{V}$  the extension  $f_{\beta'}$  is transitive (topologically mixing).

Assume in addition that  $(G, d_G)$  is a metric space. Let  $C^\alpha(X, G)$  be the space of Hölder functions from  $X$  to  $G$  with the metric

$$d_{C^\alpha}(\beta_1, \beta_2) = d_{C^0}(\beta_1, \beta_2) + \sup_{x, y \in X, x \neq y} \frac{d_G(\beta_1(x), \beta_2(x))}{d_X(x, y)^\alpha}.$$

An extension  $f_\beta$  is called  $C^\alpha$ -stably transitive ( $C^\alpha$ -stably topologically mixing) if the cocycle  $\beta$  has an open neighborhood  $\mathcal{V} \subset C^\alpha(X, G)$  such that for any  $\beta' \in \mathcal{V}$  the extension  $f_{\beta'}$  is transitive (topologically mixing).

Two cocycles  $\beta_1, \beta_2$  are said to be *cohomologous over  $f$*  if there exists a function  $P : X \rightarrow G$ , called the *transfer map*, such that  $\beta_1 = P + \beta_2 - P \circ f$ . If  $x \in X$  and  $x, fx, \dots, f^n x = x$ , is a periodic orbit for  $f$  we call the sum  $\beta(n, x) := \sum_{i=0}^{n-1} \beta(f^i x)$  the *weight* of  $\beta$  over the periodic orbit. If  $f$  and  $\beta$  are given, we denote by  $\mathcal{L}_{f, \beta}$  the set of weights of  $\beta$  over all periodic orbits of  $f$  and refer to this set as the *periodic data*. If  $G$  has a structure of real topological vector space, then the set  $\mathcal{L}_{f, \beta}$  is said to satisfy the *inseparability hypothesis* if it is not separated by any continuous functional  $\phi : G \rightarrow \mathbb{R}$ . That is, for any continuous functional  $\phi \neq 0$  there exists  $g_1, g_2 \in \mathcal{L}_{f, \beta}$  such that  $\phi(g_1) > 0$  and  $\phi(g_2) < 0$ .

**Definition 6.1.** Let  $X$  be a smooth compact manifold and  $f : X \rightarrow X$  be a  $C^1$  diffeomorphism. The diffeomorphism  $f$  is called *Anosov* if there exists a continuous  $Df$ -invariant splitting of the tangent bundle  $TX = E^s \oplus E^u$  and constants  $0 < \lambda < 1$  and  $C > 0$  such that:

$$(8) \quad \begin{aligned} \|Df^n(v)\| &\leq C\lambda^n \|v\|, v \in E^s, \\ \|Df^{-n}(v)\| &\leq C\lambda^n \|v\|, v \in E^u, \end{aligned}$$

for all  $n \geq 0$ .

If  $f : X \rightarrow X$  is a diffeomorphism of a smooth compact manifold, we denote by  $f_* : H_1(X, \mathbb{R}) \rightarrow H_1(X, \mathbb{R})$  the induced (matrix) action on real homology.

The following theorem is proved in [21].

**Theorem 6.2.** *Let  $X$  be a smooth compact manifold and  $f : X \rightarrow X$  be a transitive Anosov diffeomorphism for which  $f_*$  does not have 1 as an eigenvalue. Let  $\beta : X \rightarrow \mathbb{R}^n$  be a Hölder cocycle. Then the following are equivalent:*

- (1) *The extension  $f_\beta$  is topologically transitive.*
- (2) *The extension  $f_\beta$  is  $C^0$ -stably topologically transitive.*
- (3) *The periodic data  $\mathcal{L}_{f, \beta}$  is not separated by any hyperplane.*
- (4) *There exist orbits which are unbounded in both the positive and negative sense, that is, there exist  $x, y \in X$ , such that for all  $N > 0$ , there exist  $n, m \geq 0$  such that  $\beta(n, x) \geq N$  and  $\beta(m, y) \leq -N$ .*
- (5) *The extension  $f_\beta$  is weak topologically mixing.*
- (6) *The extension  $f_\beta$  is  $C^0$ -stably weak topologically mixing.*
- (7) *The cocycle  $\beta$  is not cohomologous to one taking values in a half space of a hyperplane passing through the origin.*

It is desirable to extend Theorem 6.2 to infinite dimensional topological groups. Two attempts are made by Rosengarten, Reich [23] and Silverman, Miller [27]. In [23] the fiber is  $\mathbb{R}^\omega$  and it is shown that topological transitivity of the extension and inseparability of  $\mathcal{L}_{f, \beta}$  are still equivalent, but stable topological transitivity is not

valid for any Hölder extension. In [27] it is shown that if the fiber is  $L^2([0, 1])$ , then there are inseparable Hölder extensions that are not topologically transitive.

Theorem 2.2 can also be used to extend Theorem 6.2 if the fiber is  $\mathbb{R}^\omega$ . This is an alternative proof of [23]. The advantage of the new proof is that it follows, almost verbatim, the proof in [21], and it is independent of the Baire category argument used in [23].

**Theorem 6.3.** *Let  $X$  be a smooth compact manifold and  $f : X \rightarrow X$  be a transitive Anosov diffeomorphism for which  $f_*$  does not have 1 as an eigenvalue. Let  $\beta : X \rightarrow \mathbb{R}^\omega$  be a Hölder cocycle. Then the following are equivalent:*

- (1) *The extension  $f_\beta$  is topologically transitive.*
- (2) *The periodic data  $\mathcal{L}_{f,\beta}$  is not separated by any continuous functional  $\phi : \mathbb{R}^\omega \rightarrow \mathbb{R}$ .*
- (3) *For any continuous functional  $\phi : \mathbb{R}^\omega \rightarrow \mathbb{R}$  and for any  $M > 0$  there exist  $g, g'$  elements in the periodic data such that  $\phi(g) > M, \phi(g') < -M$ .*
- (4) *The extension  $f_\beta$  is weak topologically mixing.*
- (5) *The cocycle  $\beta$  is not cohomologous to one having the range separated by a continuous functional.*

*Proof.* We recall [18] that every closed subgroup of  $\mathbb{R}^\omega$  has the form  $\mathbb{R}^n \oplus \mathbb{Z}^m$ , where  $m, n$  are non-negative integers or  $\omega$ .

The rest of the proof is similar to [21] and consists of the following main steps:

- (1) Show that the group generated by  $\mathcal{L}_{f,\beta}$  is dense in  $\mathbb{R}^\omega$ .

The fact that the value 1 is not in the spectrum of  $f_* : H_1(X, \mathbb{R}) \rightarrow H_1(X, \mathbb{R})$  is equivalent to the following property: if for a real valued cocycle  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  the associated periodic data  $L_{f,\beta}$  belongs to a lattice  $a\mathbb{Z}$ , then  $f$  is cohomologous to a constant. See [6, see page 27] and also [21, Appendix]. If the group generated by  $\mathcal{L}_{f,\beta}$  is non-trivial and not dense, then there exists a projection on a line that gives a non-trivial cocycle with discrete periodic data, which leads to a contradiction.

- (2) Show that the density of the group generated by  $\mathcal{L}_{f,\beta}$  and inseparability of  $\mathcal{L}_{f,\beta}$  implies that the semigroup generated by  $\mathcal{L}_{f,\beta}$  is dense in  $\mathbb{R}^\omega$ .

This is provided by the solution of the Semigroup Problem for  $\mathbb{R}^\omega$ , Theorem 2.2.

- (3) Use that the semigroup generated by  $\mathcal{L}_{f,\beta}$  is dense in  $\mathbb{R}^\omega$  to prove topological transitivity for  $f_\beta$ .

This construction is standard and follows using shadowing of periodic orbits as in [21].

The rest of the implications above follow as in [21]. □

It is shown in [23] that for fiber  $\mathbb{R}^\omega$ , contrary to what happens for finite dimensional fiber, transitivity of an extension does not imply stably transitivity in either  $C^0$  or  $C^\alpha$  topology. This seems to be a general phenomenon for extensions of maps  $f$  of compact spaces if the fiber  $G$  is an infinite dimensional Abelian topological vector space. If  $X$  is a smooth manifold, we can consider also  $C^k, k \geq 1$ , topologies. More examples pointing in this direction appear in [23]. We leave as an open question the statement and the proof of a general result.

**Example 6.1.** We briefly describe how to construct Lipschitz extensions  $f_\beta$  of Anosov diffeomorphisms,  $\beta : X \rightarrow \mathbb{R}^\omega$ , for which the periodic data  $\mathcal{L}_{f,\beta}$  is not



separated. The construction is borrowed from [23], to which we refer for the missing details. One can apply Theorem 6.3 to these examples and conclude that they are topologically transitive.

A (strict) orthant of  $\mathbb{R}^n$  is the set of all vectors for which each component has a specified sign. For example, the set of vectors with all of the first  $(n-1)$  components strictly positive and the  $n$ -th component strictly negative is an orthant. One observes that any set  $S$  in  $\mathbb{R}^n$  that contains a point in each one of the  $2^n$  orthants cannot be separated by functionals. Indeed, for any vector  $v \in \mathbb{R}^n, v \neq 0$ , one can find an element  $p_1$  in  $S$  with the signs of all components equal to the signs of the respective components in  $v$  and one can find an element  $p_2$  in  $S$  with the signs of all components opposite to the signs of the respective components in  $v$ . The elements  $p_1, p_2$  are in different half-spaces relative to the the hyperplane orthogonal to  $v$ .

It is easy to show, using Anosov closing lemma, that for any transitive extension  $f_\beta$ , where  $\beta : X \rightarrow \mathbb{R}^n$  is a Hölder map, the periodic data  $\mathcal{L}_{f,\beta}$  is dense in  $\mathbb{R}^n$ . See for example [12, Corollary 6.4.17] and [21, Lemma 8].

The cocycle  $\beta : X \rightarrow \mathbb{R}^\omega$  is constructed inductively. Assume that  $\beta_n : X \rightarrow \mathbb{R}^n$ , the projection of  $\beta$  on the first  $n$ -components, is already known and the periodic data of  $f_{\beta_n}$  cannot be separated in  $\mathbb{R}^n$ . In particular, due to Theorem 6.2,  $f_{\beta_n}$  is transitive, and due to the observation above, it has the periodic data dense in  $\mathbb{R}^n$ . We also assume that

$$\sum_{k=1}^n \frac{1}{2^k} \cdot \frac{|\beta(x)_k - \beta(y)_k|}{1 + |\beta(x)_k - \beta(y)_k|} \leq \left( \sum_{k=1}^n \frac{1}{2^k} \right) d_X(x, y), \quad x, y \in X.$$

The induction step does not change  $\beta_n$ . Define the  $n+1$  component of  $\beta$  such that for  $\beta_{n+1} : X \rightarrow \mathbb{R}^{n+1}$ , the projection of  $\beta$  on the first  $n+1$ -components, the periodic data of  $f_{\beta_{n+1}}$  has points in all  $2^{n+1}$  orthants in  $\mathbb{R}^{n+1}$ . This can be done because the projection of the periodic data on  $\mathbb{R}^n$  is dense. Moreover, we can require that:

$$\sum_{k=1}^{n+1} \frac{1}{2^k} \cdot \frac{|\beta(x)_k - \beta(y)_k|}{1 + |\beta(x)_k - \beta(y)_k|} \leq \left( \sum_{k=1}^{n+1} \frac{1}{2^k} \right) d_X(x, y), \quad x, y \in X.$$

At the end of the induction process,  $\beta$  is Lipschitz and the periodic data for all  $f_{\beta_n}$  is not separated. As the linear functionals in  $(\mathbb{R}^\omega)^*$  have finite support, it follows that the periodic data for  $f_\beta$  is not separated.  $\square$

Even under stronger separation assumptions, the results in Theorem 6.3 cannot be obtained for fiber  $\mathbb{R}^\infty$ . This follows from the following proposition that characterizes compact sets in  $\mathbb{R}^\infty$ . We supply a proof for reader convenience.

**Proposition 6.1.** *Any compact set  $K \subset \mathbb{R}^\infty$  is supported on a finite number of coordinates, that is, there exists a positive integer  $n$  and a subspace  $\mathbb{R}^n$  in the sequence of subspaces that defines  $\mathbb{R}^\infty$  such that  $K \subset \mathbb{R}^n$ .*

*Proof.* We proceed by contradiction and assume that  $K$  is supported on an infinite number of coordinates. Then we can extract from  $K$  a sequence  $S = (x_n)_{n=1}^\infty$  such that the sequence of highest non-zero coordinates  $m_n$  satisfies  $m_1 < m_2 < m_3 < \dots < m_n < \dots$ . We will show that  $S$  has no accumulation point, a contradiction.

Indeed, if  $z \in \mathbb{R}^\infty$  is the accumulation point of  $S$ , then  $z$  has finite support, say of length  $N$ . We can choose a neighborhood  $U = \prod_{i=1}^\infty U_i, U_i \in \mathbb{R}$ , of  $z$  that contains only a finite number of elements in  $S$  by choosing the size of the open intervals  $U_{m_n}$  smaller than the size of  $|x_n(m_n)|$ .  $\square$

Proposition 6.1 has the following consequence for extensions with fiber  $\mathbb{R}^\infty$ .

**Corollary 6.1.** *Assume that  $X$  is a compact space and  $f : X \rightarrow X$  is a topologically transitive homeomorphism. Then no continuous cocycle  $\beta : X \rightarrow \mathbb{R}^\infty$  can give a topologically transitive extension  $f_\beta$ .*

## 7. FURTHER RESULTS AND ADDITIONAL DIRECTIONS OF STUDY

The proofs of Theorems 4.1 and 5.1 require the existence of an equicontinuous Schauder basis. Equicontinuity means that the expansion operators  $P_k(x) = \sum_{i=1}^k a_i x_i$  are equicontinuous. A general result in this direction is the Weak Basis Theorem [8, Section 14.3]: Let  $X$  be a locally convex space that is complete, bornological, and strictly webbed. Then every basis is an equicontinuous (Schauder) basis. Beside the Banach and Fréchet spaces, the class of  $LF$ -spaces (countable strict inductive limit of Fréchet spaces) satisfies the assumptions of the Weak Basis Theorem. Nevertheless, as exemplified by the case of  $\mathbb{R}^\omega$ , even if the space is Fréchet, equicontinuity of the basis does not imply equicontinuity of the coefficient functionals, that is, of  $f_k(x) = a_k$ . Equicontinuity of the coefficient functionals follows if the basis is a regular sequence, as introduced in [9].

We leave the discussion of the Semigroup Problem in the general case of the  $LF$ -space as an open problem for the reader and focus our attention on a particular class of  $LF$ -spaces frequently appearing in analysis.

Important examples of  $LF$ -spaces can be introduced as follows [28, page 131]. Denote by  $F(\mathbb{R})$  one of the following spaces:

$$C^k(\mathbb{R}), 0 \leq k < \infty; \quad C^\infty(\mathbb{R}); \quad L^p(\mathbb{R}), 1 \leq p \leq \infty.$$

Let  $K \subset \mathbb{R}$  be a compact set and denote by  $F_c(K)$  the set of functions  $f$  with support in  $K$ . This is always a Fréchet space. For a given sequence of compact sets  $K_1 \subset K_2 \subset \dots \subset K_n \subset \dots$  with  $\cup_{n=1}^\infty K_n = \mathbb{R}$ , let  $F_c(\mathbb{R})$  be the union of all  $F_c(K_n)$  endowed with the inductive limit of the topologies of Fréchet spaces  $F_c(K_n)$ .

It is known that spaces  $F_c(\mathbb{R})$  contain as a closed subspace the Schwartz space  $\mathcal{S}(\mathbb{R})$  of smooth functions that have fast decay at infinity. All these spaces can be identified with sequence spaces, in a way consistent with their inclusions, see e.g. [3].

As  $\mathcal{S}(\mathbb{R})$  is a Fréchet nuclear space with continuous norm and Schauder basis, given by the Hermitian functions [8, page 319], one can apply the argument in the proof of Theorem 5.1 to find a semigroup  $S \subset \mathcal{S}(\mathbb{R}) \subset F_c(\mathbb{R})$  that is not separated by any non-zero functional with support in  $\mathcal{S}(\mathbb{R})$  and for which the closure is not a group. This shows that the Semigroup Problem has a negative answer for all spaces  $F_c(\mathbb{R})$ .

For the following class of spaces we can generalize the result in Proposition 6.1. A Fréchet-Montel space is a barrelled topological vector space where every closed and bounded set is compact. The Schwartz space  $\mathcal{S}(\mathbb{R})$  is a Fréchet-Montel complete spaces. For complete  $LF$ -spaces that are inductive limits of Fréchet-Montel spaces one can also show that there are no topologically transitive extensions of topologically transitive homeomorphisms of compact spaces. This follows from the fact that such spaces  $E = \text{ind}_n E_n$  are *compactly regular*, that is, any compact set in  $E$  is embedded in one of the subspaces  $E_n$ . See Theorems 2.7 and 3.3 in [29].

For  $F$ -spaces, that is, complete metrizable topological vector space, the weak basis theorem is not always true [25]. Also, there exist  $F$ -spaces without a basic

sequence [11]. Our method for solving the Semigroup Problem does not work for this class of spaces, so the problem is open.

Another feature that may prevent a positive answer for the Semigroup Problem for certain  $F$ -spaces is the lack of enough continuous functionals, e.g., in the spaces  $L^p([0, 1])$ ,  $0 < p < 1$ , the only continuous functional is the trivial one [24, page 37]. In this case the Semigroup Problem simply says that the closure of any semigroup is a group. For the particular example of  $L^p([0, 1])$ ,  $0 < p < 1$ , choosing a semigroup with a single non-zero generator, one can easily construct a counterexample. Other way to settle the problem in this case is to use the fact that  $L^p([0, 1])$ ,  $0 < p < 1$ , contains a subspace isomorphic to  $\ell_2$  [10, Theorem 6] and use the fact that the Semigroup Problem has a negative answer for Hilbert spaces.

The negative answers for the general Semigroup Problem that we discuss in this paper suggests the addition of more assumptions about the set  $S$ , besides non-separability, that may lead to positive answers. For Banach spaces some results in this direction can be found in [5].

Finally, we would like to mention that Sidorov [26] proved that for any topologically transitive homeomorphism of a complete metric separable space, in particular of a compact metric space, and for any separable Banach space in the fiber, there exists topologically transitive continuous extensions. The results in this paper show the existence of topologically transitive continuous extensions for fiber  $\mathbb{R}^\omega$  (Theorem 6.3) and non-existence of topologically transitive continuous extensions if the fiber is  $\mathbb{R}^\infty$  (Corollary 6.1) or one of the compactly regular spaces mentioned above. An obvious obstruction for the existence of topologically transitive extensions (over a compact space) is that the fiber has to be a separable (Hausdorff) topological vector space. Examples of fibers that can be eliminated due to this obstruction are  $\ell^\infty(\mathbb{Z}^+)$  and the space of functions of bounded variation  $BV([0, 1])$ . For the last example see e.g. [1].

All of these leave open the question of characterizing the family of separable infinite dimensional topological vector spaces that can be fibers for topologically transitive continuous extensions with the action in the base a topologically transitive homeomorphism of a compact space. In this generality the problem is open even if the action in the base is a basic hyperbolic set.

If we extend the class of maps we study from the class of (linear) extensions to the class of fibered maps of type  $f_\Phi : X \times G \rightarrow X \times G$  given by  $f_\Phi(x, g) = (f(x), \Phi(x, g))$ , where  $\Phi : X \times G \rightarrow G$ , then more can be said. Indeed, it is known from a result of Anderson [2] that any infinite dimensional separable Fréchet space  $F$  is homeomorphic via a homeomorphism  $\phi_F$  to  $\mathbb{R}^\omega$ . This allow us to carry over, via a conjugacy by the map  $Id_X \times \phi_F$ , the result from Theorem 6.3 and construct (continuous) topologically transitive fibered maps with fiber any separable Fréchet space. It is also known [15] that infinite dimensional separable  $LF$ -spaces are homeomorphic to  $l_2, \mathbb{R}^\infty$  or  $l_2 \times \mathbb{R}^\infty$ . This allows to construct (continuous) topologically transitive fibered maps with fiber any infinite-dimensional separable  $LF$ -space homeomorphic to  $l_2$ . We note that the result in Proposition 6.1 does not preclude the existence of topologically transitive fibered maps with fiber  $\mathbb{R}^\infty$ . Nevertheless, even if such a result is possible, this still leave open the problem if the fiber is  $l_2 \times \mathbb{R}^\infty$ .

In more generality, for any topologically transitive action in the base, we do not know if the class of fibers for which there exist topologically transitive extensions (or fibered maps) is closed under direct product.

## REFERENCES

- [1] C. R. Adams, The space of functions of bounded variation and certain general spaces, *Trans. Amer. Math. Soc.*, **40** (1936), 421–438.
- [2] R. D. Anderson, Hilbert space is homeomorphic to the countable infinite product of lines, *Bull. Amer. Math. Soc.* **72** (1966), 515–519. MR 32 #8298.
- [3] C. Bargetz, Completing the Valdivia–Vogt tables of sequence-space representations of spaces of smooth functions and distributions, *Monatshefte für Mathematik*, **177** (2015), 1–14.
- [4] C. Bessaga, A. Pełczyński and S. Rolewicz, On diametral approximative dimension and linear homogeneity of  $F$ -spaces, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **9** (1961), 677–683.
- [5] P. A. Borodin, Density of a semigroup in a Banach space, *Izvestiya: Mathematics*, **78** (2014), 1079–1104.
- [6] R. Bowen, *On Axiom A Diffeomorphisms*, Regional Conference Series in Mathematics, No. 35. American Mathematical Society Providence, R.I. 1978.
- [7] R. Brown, P. J. Higgins and S. A. Morris, Countable products and sums of lines and circles: their closed subgroups, quotients and duality properties, *Math. Proc. Camb. Phil. Soc.* **78** (1975), 19–32.
- [8] H. Jarchow, *Locally Convex Spaces*, Springer, 1981.
- [9] N. Kalton, Normalization properties of Schauder bases, *Proc. London Math. Soc.* **22** (1971), 91–105.
- [10] N. Kalton, The metric linear spaces  $L_p$  for  $0 < p < 1$ , *Contemporary Mathematics* **52** (1986), 55–69.
- [11] N. Kalton, The basic sequence problem, *Studia Mathematica*, **116** (1995) 168–187.
- [12] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, 1995.
- [13] J. Lindenstrauss and L. Tzafriri, *Classic Banach Spaces I,II*, Springer-Verlag, 1977.
- [14] K. Lui, V. Nitica and S. Venkatesh, The semigroup problem for central semidirect product of  $\mathbb{R}^n$  with  $\mathbb{R}^m$ , *Topology Proceedings* **45** (2015), 9–29.
- [15] P. Mankiewicz, On topological, Lipschitz, and uniform classification of LF-spaces, *Studia Math.* **52** (1974), 109–142.
- [16] I. Melbourne, V. Nitica and A. Török, Stable transitivity of certain noncompact extensions of hyperbolic systems, *Annales Henri Poincaré* **6** (2005), 725–746.
- [17] I. Melbourne, V. Nitica and A. Török, A note about stable transitivity of noncompact extensions of hyperbolic systems, *Discrete & Contin. Dynam. Systems* **14** (2006), 355–363.
- [18] S. A. Morris, Locally compact abelian groups and the variety of topological groups generated by the reals, *Proc. Amer. Math. Soc.* **34** (1972), 290–292.
- [19] V. Nitica and A. Török, Open and dense topological transitivity of extensions by non-compact fiber of hyperbolic systems: a review, *Axioms* **4** (2015), 84–101.
- [20] V. Nitica and A. Török, Stable transitivity of Heisenberg group extensions of hyperbolic systems, *Nonlinearity* **27** (2014), 661–683.
- [21] V. Nitica and M. Pollicott, Transitivity of Euclidean extensions of Anosov diffeomorphisms, *Ergodic Theory and Dynamical Systems* **25** (2005), 257–269.
- [22] A. Pietsch, *Nuclear Locally Convex Spaces*, Ergebnisse Der Mathematik und Ihrer Grensgebiete, Volume 66, Springer-Verlag, New York/Heidelberg/Berlin, 1972.
- [23] Z. Rosengarten and A. Reich, Transitivity of infinite dimensional extensions of Anosov diffeomorphisms, (2012) arxiv 1209.2183v1
- [24] W. Rudin, *Functional Analysis*, second edition, McGraw-Hill, 1991.
- [25] J.H. Shapiro, On the weak basis theorem in  $F$ -spaces, *Canadian J. Math.*, vol. XXVI, (1974) 1294–1300.
- [26] E. A. Sidorov, Topologically transitive cylindrical cascades. (Russian), *Mat. Zametki* **14** (1973), 441–452.
- [27] T. Silverman and S. M. Miller, Hilbert extensions of Anosov diffeomorphisms, (2013) <http://www.math.psu.edu/mass/reu/2013/mathfest/HilbertCounterexample.pdf>
- [28] F. Trèves, *Topological Vector Spaces, Distributions and Kernels*, Academic Press, New York, London, 1967.
- [29] J. Wengenroth, Acyclic inductive spectra of Fréchet spaces, *Studia Mathematica* **120** (1996), 247–258.