Homework 8: Feb 21, 2017

(Due in on Thursday, February 23)

Heat Conduction in a Thin Insulated Circular Ring

Consider heat flow through a thin insulated circular ring described by:

$$\begin{aligned} u_r &= ku_{xx}, \quad x \in (-L, L), t > 0, \\ u(-L,t) &= u(L,t), \quad t > 0, \\ u_x(-L,t) &= u_x(L,t), \quad t > 0, \\ u(x,0) &= f(x), \quad x \in (-L, L). \end{aligned}$$

(A) Write all separated solutions $u_n(x,t) = \phi_n(x)G_n(t)$. (State and solve the corresponding eigenvalue problem for $\phi(x)$ and an ODE for $G(t)$.)

(B) Write the general solution $u(x,t)$ defined by the separated solutions $u_n(x,t)$.

(C) Determine the particular solution satisfying the initial data $u(x,0) = f(x)$.
Hint: To determine the Fourier coefficients you must investigate and use the orthogonality properties of sines, cosines, and sines and cosines. Namely, you must first calculate the following integrals:

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx, \quad \int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx, \quad \int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx.$$

(D) By using what you've learned above, find the unique solution of the following initial boundary value problem for heat condition in a thin insulated circular ring:

$$\begin{aligned} u_r &= ku_{xx}, \quad x \in (-L, L), t > 0, \\ u(-L,t) &= u(L,t), \quad t > 0, \\ u_x(-L,t) &= u_x(L,t), \quad t > 0, \\ u(x,0) &= 2 + \cos\frac{3\pi x}{L} + \sin\frac{\pi x}{L}, \quad x \in (-L, L). \end{aligned}$$