Homework 9: March 2, 2017

Wave Equation

1. (Problem 4.2.1) (a) Using the wave equation

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + \rho_0(x)Q(x,t)$$

calculate the sagged equilibrium position $u_E(x)$ if $Q(x,t) = -g$. The boundary conditions are $u(0) = 0$, $u(L) = 0$.

(b) Show that $v(x,t) = u(x,t) - u_E(x)$ satisfies the homogeneous wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}.$$

2. (Problem 4.4.1-will cover material in class on March 7). Consider vibrating strings of uniform density ρ_0 and tension T_0.
(a) What are the natural frequencies of a vibrating string of length L fixed at both ends?
(b) What are the natural frequencies of a vibrating string of length H which is fixed at $x = 0$ and "free" at the other end? Sketch a few modes of vibration as in Fig. 4.4.1.

3. (Problem 4.4.2) Consider the vibrations of a uniform string that satisfies:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + \alpha u.$$

(a) Show that if $\alpha < 0$ the body force αu is restoring (toward $u = 0$). Show that if $\alpha > 0$, the body force αu tends to push the string further away from its unperturbed position $u = 0$.
(b) Do the separation of variables when α and c^2 are constant. Analyze the resulting time-dependent ordinary differential equation.
(c) Solve the initial boundary value problem if $\alpha < 0$ with the following initial and boundary conditions:

$$u(0,t) = 0,\ u(L,t) = 0,\ u(x,0) = 0,\ \frac{\partial u}{\partial t}(x,0) = f(x).$$

What are the frequencies of vibration?

HOMEWORK 9 (Answer Key) ①

① (a) $u_E(x)$ SATISFIES: $\begin{cases} \dfrac{T_0}{\rho_0}\dfrac{d^2u}{dx^2} + Q(x,t) = 0 \\ u(0)=0\ ,\ u(L)=0 \end{cases}$, where $Q = -g$.

Thus, $u_E(x)$ SATISFIES $\boxed{\dfrac{d^2u}{dx^2} = \dfrac{\rho_0}{T_0} g} \Rightarrow \boxed{u_E(x) = \dfrac{\rho_0}{T_0} g \dfrac{x^2}{2} + C_1 x + C_2}$

From boundary conditions: $\begin{cases} u_E(0) = 0 \Rightarrow \boxed{C_2 = 0} \\ u_E(L) = 0 \Rightarrow \dfrac{\rho_0}{T_0} g \dfrac{L^2}{2} + C_1 L = 0 \Rightarrow \boxed{C_1 = -\dfrac{\rho_0}{T_0} g \dfrac{L}{2}} \end{cases}$

Thus: $\boxed{u_E(x) = \dfrac{\rho_0}{T_0}\dfrac{g}{2}x^2 - \dfrac{\rho_0}{T_0}\dfrac{g}{2}xL = \dfrac{\rho_0}{T_0}\dfrac{g}{2}x(x-L)}$

(b) Verify that $v_{tt} = c^2 v_{xx}$ where $v = u - u_E$.

Since $u_E(x)$ does not depend on t, we have $v_{tt} = u_{tt}$.

Since $u_{tt} = \dfrac{T_0}{\rho_0} u_{xx} + Q(x,t)$ we have: $v_{tt} = u_{tt} = \dfrac{T_0}{\rho_0} u_{xx} + Q$.

Now, recall that $u_E(x)$ satisfies $\dfrac{T_0}{\rho_0}(u_E)_{xx} + Q = 0$. From here we can write $Q = -\dfrac{T_0}{\rho_0}(u_E)_{xx}$.

Now, we can go back to v_{tt} and write it as:

$$v_{tt} = u_{tt} = \frac{T_0}{\rho_0} u_{xx} + Q = \frac{T_0}{\rho_0} u_{xx} - \frac{T_0}{\rho_0}(u_E)_{xx} = \underbrace{\left(\frac{T_0}{\rho_0}\right)}_{=c^2}\left(u_{xx} - (u_E)_{xx}\right)$$

Thus: $\underline{\underline{v_{tt}}} = c^2\left(u_{xx} - (u_E)_{xx}\right) = c^2\big(\underbrace{u - u_E}_{v}\big)_{xx} \underset{v = u - u_E}{=} \underline{\underline{c^2 v_{xx}}}$

② (a) $u(x,t) = \displaystyle\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n \cos\frac{n\pi c t}{L} + B_n \sin\frac{n\pi c t}{L}\right)$

NATURAL FREQUENCIES OF OSCILLATIONS

Natural frequencies of oscillation are $\boxed{\dfrac{n\pi c}{L},\ n = 1, 2, 3, \ldots}$

(b) Eigenvalue problem: $\begin{cases} \phi''(x) + \lambda\phi(x) = 0 \\ \phi(0) = 0,\ \phi'(H) = 0 \end{cases}$

Solution: $\phi(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

$\phi(0) = 0 \Rightarrow A = 0 \Rightarrow \boxed{\phi(x) = B\sin\sqrt{\lambda}\,x}$

$\phi'(x) = B\sqrt{\lambda}\cos\sqrt{\lambda}x$. $\phi'(H) = B\sqrt{\lambda}\cos\sqrt{\lambda}H = 0 \Rightarrow$ $\boxed{\sqrt{\lambda}H = \dfrac{(2k-1)\pi}{2}\quad k = 1, 2, 3}$

Thus:
$$\boxed{\begin{aligned}\lambda_n &= \left(\frac{(2k-1)\pi}{2H}\right)^2, \quad k=1,2,3,\ldots\\ \phi_n(x) &= \sin\frac{(2k-1)\pi}{2H}\end{aligned}}$$

GENERAL SOLUTION:
$$u(x,t) = \sum_{k=1}^{\infty} \sin\frac{(2k-1)\pi}{2H}\left(A_n \cos\frac{(2k-1)\pi c t}{2H} + B_n \sin\frac{(2k-1)\pi c t}{2H}\right)$$

OR

$$u(x,t) = \sum_{k=1}^{\infty} \sin\left(k-\tfrac{1}{2}\right)\frac{\pi}{H}\left(A_n \cos\frac{(k-\frac{1}{2})\pi c}{H}t + B_n \sin\frac{(k-\frac{1}{2})\pi c}{H}t\right)$$

NATURAL FREQ. OF OSCILLATION

Natural frequency of oscillations are: $\boxed{\frac{(k-\frac{1}{2})\pi c}{H}, \quad k=1,2,3,\ldots}$

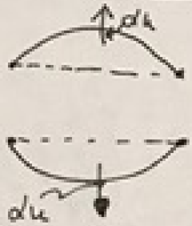
(3) $u_{tt} = c^2 u_{xx} + \alpha u$

(a) $\boxed{\text{For } \alpha < 0:}$ When $u>0$ (displ. above equilibrium position), the sign of the body force αu is <u>negative</u>, trying to <u>restore</u> the equilibrium position.

When $u<0$ (displacement is below equilibrium) the sign of the body force αu is <u>positive</u>, trying again to <u>restore</u> the equilibrium position.

$\Rightarrow$ RESTORING FORCE

$\boxed{\text{For } \alpha > 0:}$



When $u>0$, the sign of αu is <u>positive</u>, pushing the string <u>away</u> from equilibrium.
When $u<0$, the sign of αu is <u>negative</u>, again pushing the string <u>away</u> from equilibrium.

(b) $u(x,t) = \phi(x)h(t) \Rightarrow h''\phi = c^2\phi'' h + \alpha\phi h \quad / \div c^2\phi h$

$$\Rightarrow \frac{h''}{c^2 h} = \frac{\phi''}{\phi} + \frac{\alpha}{c^2}$$

$$\frac{h''}{c^2 h} - \frac{\alpha}{c^2} = \frac{\phi''}{\phi} = -\lambda$$

$$\frac{h''(t)}{c^2 h(t)} - \frac{\alpha}{c^2} = -\lambda \Rightarrow \frac{h''(t)}{h(t)} - \alpha + \lambda c^2 = 0$$

$$\Rightarrow h''(t) + (\lambda c^2 - \alpha)h(t) = 0 \quad , \quad h(t) = e^{rt} \Rightarrow$$

$$r^2 + (\lambda c^2 - \alpha) = 0$$
$$r^2 = \alpha - \lambda c^2$$
$$r = \pm\sqrt{\alpha - \lambda c^2}$$

If $-\lambda c^2 + \alpha < 0 \Rightarrow \boxed{h(t) = A\cos\sqrt{\lambda c^2 - \alpha}\, t + B\sin\sqrt{\lambda c^2 - \alpha}\, t}$

If $-\lambda c^2 + \alpha > 0 \Rightarrow \boxed{h(t) = A e^{\sqrt{\alpha - \lambda c^2}\, t} + B e^{-\sqrt{\alpha - \lambda c^2}\, t}}$

(c) $\boxed{\alpha < 0} \Rightarrow -\lambda c^2 + \alpha < 0 \Rightarrow \boxed{h(t) = A\cos\sqrt{\lambda c^2 - \alpha}\, t + B\sin\sqrt{\lambda c^2 - \alpha}\, t}$

$\phi_n(x) = \sin\frac{n\pi x}{L}$, $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, 3, \dots$

$$h_n(x) = A_n\cos\sqrt{\left(\frac{n\pi c}{L}\right)^2 - \alpha}\; t + B_n\sin\sqrt{\left(\frac{n\pi c}{L}\right)^2 - \alpha}\; t$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n\cos\sqrt{\left(\frac{n\pi c}{L}\right)^2 - \alpha}\; t + B_n\sin\sqrt{\left(\frac{n\pi c}{L}\right)^2 - \alpha}\; t\right)}$$

$u(x,0) = 0 \Rightarrow \boxed{A_n = 0 \text{ for all } n.}$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \sin\sqrt{\left(\frac{n\pi c}{L}\right)^2 - \alpha}\; t}$$

$$u_t(x,t) = \sum_{n=1}^{\infty} B_n \cdot \sqrt{\left(\frac{n\pi c}{L}\right)^2 - \alpha}\; \cos\left(\sqrt{\left(\frac{n\pi c}{L}\right)^2 - \alpha}\; t\right) \cdot \sin\left(\frac{n\pi x}{L}\right)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \sqrt{\left(\frac{n\pi c}{L}\right)^2 - \alpha}\; \sin\frac{n\pi x}{L} = f(x)$$

$$\Rightarrow \quad B_n \sqrt{\left(\frac{n\pi c}{L}\right)^2 - \alpha} = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$$

$$\Rightarrow \quad \boxed{B_n = \frac{2}{L\sqrt{\left(\frac{n\pi c}{L}\right)^2 - \alpha}} \int_0^L f(x) \sin\frac{n\pi x}{L}\, dx} \quad (*)$$

Solution: $\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n \frac{\sin n\pi x}{L} \sin\sqrt{\left(\frac{n\pi c}{L}\right)^2 - \alpha}\; t \quad \text{where } B_n \text{ are given by } (*)}$

Frequencies of vibration are $\sqrt{\left(\frac{n\pi c}{L}\right)^2 - \alpha}$.