NAME: ____

Midterm Exam 2, April 13, 2017

1. (a) (5 pts) Write D'Alambert formula for the solution of the wave equation $u_{tt} = c^2 u_{xx}$ with initial data $u(x,0) = f(x)$, $u_t(x,0) = g(x)$.

(b) (10 pts) Solve the following initial-value problem for the wave equation:

$$u_{tt} = 4u_{xx},\ x \in \mathbb{R},\ t > 0,$$

with initial displacement $u(x,0) = \cos x$ and initial velocity $u_t(x,0) = \sin x$.

(c) (5 pts) What is the domain of dependence of the point $(x,t) = (3,3)$? Sketch it.

(d) (5 pts) What is the domain of influence of the point $(x,0) = (0,0)$? Sketch it.

2. (a) (15 pts) What is the Sturm-Liouville eigenvalue problem that results from solving the following initial-boundary value problem for the heat equation with non-constant coefficients:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(x^3 \frac{\partial u}{\partial x}\right), x \in (0,L), T > 0,$$

and with the homogeneous Neumann boundary data $u_x(0,t) = 0$, $u_x(L,t) = 0$.

(b) (10 pts) What is the Rayleigh quotient for that eigenvalue problem, and the sign of the corresponding eigenvalues?
(*Recall:* Rayleigh quotient

$$\lambda = \frac{[-p\phi d\phi/dx]|_a^b + \int_a^B [p(d\phi/dx)^2 - q\phi^2]dx}{\int_a^b \phi^2 \sigma dx},\ \text{for}\ \frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0.)$$

3. (25 pts) Consider a damped string fixed at the end points, described by the following PDE:

$$u_{tt} = 4u_{xx} - u_t.$$

Derive the expression for the total energy of the string. Is the total energy of this damped string conserved? Prove your statement.

4. (25 pts) Find the solution of the Laplace's equation in 2D:

$$\triangle u = 0,\ x \in (0,3),\ y \in (0,1),$$

subject to the following boundary data:

$$\begin{aligned} &u(x,0) = 0,\quad x \in (0,3) \\ &u(x,1) = 0,\ x \in (0,3) \\ &u(0,y) = \sin \pi y,\ y \in (0,1) \\ &u(3,y) = 0,\ y \in (0,1). \end{aligned}$$