NAME (please print): ____________________

Quiz 6, March 30, 2017

1. True of false:

(a) (10 pts) Every solution of the wave equation $u_{tt} = c^2 u_{xx}$ can be written as a sum $u(x,t) = F(x - ct) + G(x + ct)$ of a forward and a backward moving wave.

(b) (10 pts) There are always finitely many eigenvalues of Sturm-Liouville eigenvalue problems: $\lambda_1 < \lambda_2 < \cdots < \lambda_N$.

(c) (10 pts) The eigenfunctions associated with different eigenvalues in Sturm-Liouville eigenvalue problems are orthogonal.

2. (a) (15 pts) Solve $u_{tt} = 4u_{xx}$ for $x \in \mathbb{R}$, $t > 0$, with the initial data $u(x,0) = \sin x$ and $u_t(x,0) = 2\cos x$.

(b) (15 pts) Sketch and specify the **Domain of dependence** of point $(x,t) = (1,2)$ and the **Range of influence** of point $(x_0, 0) = (1, 0)$.

3. (a) (20 pts) Derive the expression for the rate of change of the total energy of a vibrating linearly elastic string of length L, modeled by $u_{tt} = c^2 u_{xx}$.

(b) (10 pts) Suppose that the string is fixed at both ends $x = 0$ and $x = L$. What is the rate of change of the total energy of the string? Is the energy conserved?

(c) (10 pts) For the boundary data in (b), find the total energy of the string at any time t if initially the string had velocity $u_t(x,0) = 1$, and its displacement was $u(x,0) = \sin \frac{\pi x}{L}$?

4. (EXTRA 20 pts) "Solve" the non-constant coefficient heat equation with energy source, described by

$$c(x)\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}\right) + u, \quad x \in (0,L), t > 0$$

describing heat flow in a non-homogeneous rod of length L, with boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$. Suppose that the initial temperature of the rod is

$$u(x,0) = f(x),\ x \in (0,L).$$

Write the solution of this problem in terms of the generalized Fourier series by assuming that the corresponding eigenfunctions of the corresponding Sturm-Liouville eigenvalue problem, denoted $\phi_n(x)$, are known and are complete. Specify how to calculate the coefficients in the generalized Fourier series.

(*Recall:* Sturm-Liouville eigenvalue problem, general form:

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0,\ x \in (a,b), \quad \text{with } p(x) > 0, \sigma(x) > 0$$

with boundary conditions

$$\begin{aligned} \beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) &= 0, \\ \beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) &= 0. \end{aligned}$$

The eigenfunctions are orthogonal in the sense that $\int_a^b \phi_n(x)\phi_m(x)\sigma(x)dx = 0$ for $n \neq m$.)