EXISTENCE OF A UNIQUE SOLUTION TO A NONLINEAR MOVING-BOUNDARY PROBLEM OF MIXED TYPE ARISING IN MODELING BLOOD FLOW

SUNČICA ČANIĆ*, ANDRO MIKELIĆ†, TAE-BEOM KIM‡, AND GIOVANNA GUIDOBONI§

Abstract. We prove the (local) existence of a unique mild solution to a nonlinear moving-boundary problem of a mixed hyperbolic-degenerate parabolic type arising in modeling blood flow through compliant (viscoelastic) arteries.

Key words. Moving boundary problem, PDE of mixed type, Hyperbolic-degenerate parabolic problem, Blood flow, Compliant arteries.

AMS(MOS) subject classifications. 35M10, 35G25, 35K65, 35Q99, 76D03, 76D08, 76D99.

1. Introduction.

This work was motivated by a study of blood flow in compliant arteries. In medium-to-large arteries blood can be modeled by the Navier-Stokes equations for an incompressible, viscous Newtonian fluid, while the arterial walls behave as a viscoelastic material [2, 3, 4]. To study the coupled fluid-structure interaction (FSI) problem we derived in [6] a leading-order, closed, effective model for the benchmark problem in blood flow: the pressure-driven FSI problem defined on a time-dependent cylindrical domain $\Omega(t)$ with small aspect ratio $\varepsilon = R/L$ (see Fig. 1) and axially symmetric flow. Kelvin-Voigt linearly viscoelastic cylindrical membrane and Kelvin-Voigt linearly viscoelastic cylindrical Koiter shell were used in [6] to model the viscoelastic behavior of arterial walls.

The leading-order problem derived in [6] defines a nonlinear, moving-boundary problem for a system of partial differential equations of mixed hyperbolic-parabolic type in two space dimensions. The resulting problem is a hydrostatic approximation of the full FSI problem between the Navier-Stokes equations for an incompressible, viscous Newtonian fluid and the Kelvin-Voigt linearly viscoelastic cylindrical membrane or Koiter shell

* Department of Mathematics, University of Houston, Houston, Texas 77204-3476.
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† Université de Lyon, Lyon, F-69003, Université Lyon 1, Institut Camille Jordan, UMR 5208, Département de Mathématiques, 43, Bd du 11 nov. 1918, 69622 Villeurbanne Cedex, France (`Andro.Mikelic@univ-lyon1.fr`).
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‡ Department of Mathematics, University of Houston, Houston, Texas 77204-3476.
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§ Department of Mathematics, University of Houston, Houston, Texas 77204-3476.
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model. The problem is given in terms of two unknown functions: the axial component of the fluid velocity, $v_z = v_z(r, z, t)$, and the radial displacement of the arterial wall, $\eta = \eta(z, t)$:

$$\frac{\partial (R+\eta)^2}{\partial t} + \frac{\partial}{\partial z}\int_0^{R+\eta} 2 r v_z dr = 0, \quad 0 < z < L,\ 0 < t < T, \tag{1.1}$$

$$\varrho_F \frac{\partial v_z}{\partial t} - \mu_F \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right) = -\frac{\partial p}{\partial z}, \quad (r,z) \in \Omega(t),\ 0 < t < T, \tag{1.2}$$

with

$$p - p_{\mathrm{ref}} = \left(\frac{hE}{R(1-\sigma^2)}K + p_{\mathrm{ref}}\right)\frac{\eta}{R} + \frac{hC_v}{R^2}K\frac{\partial \eta}{\partial t}, \qquad 0 < z < L,\ 0 < t < T, \tag{1.3}$$

where $K = 1$ for the membrane and $K = 1 + \frac{h^2}{12R^2}$ for the Koiter shell.

Here ϱ_F is the fluid density, μ_F is the fluid dynamic viscosity coefficient, p is the fluid pressure with p_{ref} denoting the pressure at which the domain reference configuration is assumed (straight cylinder of radius R). The constants describing the structure properties are the Young's modulus of elasticity E, the Poisson ratio σ, the wall thickness h, and the structure viscoelasticity constant C_v. Typical values for these constants can be found, e.g., in [6, 8]. Problem (1.1)–(1.3) is defined on $\Omega(t)$, $0 < t < T$, where

$$\Omega(t) = \big\{(r,z):\ 0 \le r < R + \eta(z,t),\ 0 < z < L,\ \text{so that } (r\cos\vartheta, r\sin\vartheta, z) \in I\!R^3 \text{ for } \vartheta \in [0, 2\pi) \text{ defines a cylinder in } I\!R^3\big\}. \tag{1.4}$$

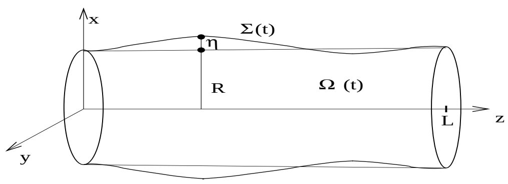

FIG. 1. *Deformed domain* $\Omega(t)$.

Problem (1.1)–(1.3) is supplemented by the following initial and boundary conditions

$$v_z(0,z,t) - \text{ bounded}, v_z(R+\eta(z,t), z, t) = 0, v_z(r,z,0) = v_z^0(r,z), \tag{1.5}$$

$$\eta(z,0) = \eta^0(z), p(0,t) = P_0(t), p(L,t) = P_L(t), \tag{1.6}$$

describing pressure-driven fluid flow in a compliant cylinder $\Omega(t)$, with no-slip boundary conditions at the lateral boundary of $\Omega(t)$.

Equation (1.1) is derived from the conservation of mass condition for incompressible fluids, while equation (1.2) is the leading-order hydrostatic approximation of the balance of axial momentum. It was shown in [7] that (1.1)–(1.3) plus its ε-correction (not shown here) satisfy the original fluid-structure interaction problem to the ε^2 accuracy.

Problem (1.1)–(1.6) in a nonlinear, initial-boundary-value problem of hyperbolic-parabolic type. The hyperbolic waves described by the quasi-linear transport equation (1.1) may develop shock waves in the axial direction z giving rise to discontinuities in the displacement of the arterial wall (lateral boundary), not typically observed in cardiovascular flow of healthy humans [5]. The second equation, (1.2), could potentially smooth out the steep wave fronts due to the fluid viscosity effects. Unfortunately, the fluid viscosity in the axial direction in not present in (1.2) since the corresponding terms are negligible (higher order) in comparison with the diffusion in the radial direction [6]. This gives rise to a problem with degenerate/anisotropic diffusion in the momentum equation (1.2) which presents various difficulties in the proof of the existence of a solution. However, due to the time-differentiated term in equation (1.3) coming from the viscoelasticity of arterial walls, the sharp wave fronts in the displacement of the arterial walls will be smoothed out, giving rise to a solution of (1.1)–(1.6) which is physiologically reasonable. More precisely, we will prove in this manuscript the existence of a unique **mild solution** to problem (1.1)–(1.6) with sufficient regularity in the axial direction allowing solutions with no shock formation. As we shall see in the proof, the dominant smoothing of shock fronts in the displacement of the arterial walls is provided by the viscoelastic term in the pressure-displacement relationship (1.3) describing the arterial wall properties.

This reduced problem has many interesting features. It captures the main properties of fluid-structure interaction in blood flow with physiologically reasonable equations and data [8], while allowing fast numerical computations and a relatively simple analysis related to its well-posedness.

Within the past ten years there has been considerable progress in the analysis of fluid-structure interaction problems between an incompressible, viscous fluid and an elastic or viscoelastic structure. All the results that are related to an elastic structure interacting with a viscous, incompressible fluid have been obtained under the assumption that the structure is entirely immersed in the fluid, see e.g., [9, 10, 12]. To our knowledge, there have been no results showing existence of a solution to a fluid-structure interaction problem where an elastic structure is a part of the fluid boundary, which is the case, for example, in modeling blood flow through elastic arteries. Often times additional ad hoc terms of viscoelastic nature are added to the vessel wall model to provide stability and convergence of the underlying numerical algorithm [16, 18], or to provide enough regularity in the proof of the existence of a solution as in [11, 10, 14, 21]. In [11, 9] terms describing bending (flexion) rigidity were added to provide smoothing mechanisms for the evolution of the structure displacement.

The novelty of the present paper is in considering a problem with the viscoelastic smoothing in the structure equation described by the lowest possible time derivative appearing in the physiologically relevant equations allowing the use of measurements data to describe the viscoelastic arterial wall properties.

2. The Nonlinear Problem on a Fixed Domain. We begin by mapping the moving-boundary problem (1.1)–(1.6) onto a fixed domain. At the same time we will be introducing the non-dimensional variables to derive the corresponding nonlinear problem defined on a fixed domain in non-dimensional form.

To simplify notation we introduce

$$\gamma(z,t) := R + \eta(z,t).$$

Introduce the mapping $r \mapsto \dfrac{r}{\gamma} =: \tilde{r}$ which maps $\Omega(t)$ onto the fixed domain $\Omega := (0,1) \times (0,L) \times (0,T)$. In addition, consider the following scalings of the independent and dependent variables

$$z = L\tilde{z}, t = \tau\tilde{t}, v_z = V\tilde{v_z}, \eta = N\tilde{\eta}, V = \frac{L}{\tau}, \gamma = R\tilde{\gamma} \text{ where } \tilde{\gamma} = 1 + \frac{N}{R}\tilde{\eta}.$$

Also, denote $\tilde{T} = T/\tau$. With these transformations, the problem is now defined on the scaled fixed domain

$$\begin{aligned}\tilde{\Omega} = &\big\{(\tilde{r},\tilde{z}) : \tilde{r} \in (0,1), \tilde{z} \in (0,1), \text{ so that } (\tilde{r}\cos\vartheta, \tilde{r}\sin\vartheta, \tilde{z}),\\ &\vartheta \in [0,2\pi), \text{defines a cylinder in } I\!R^3\big\}.\end{aligned} \tag{2.1}$$

The corresponding **nonlinear, fixed-boundary problem** in non-dimensional form then reads: for $0 < \tilde{t} < \tilde{T}$ find $\tilde{\gamma}(\tilde{z},\tilde{t})$ and $\tilde{v}_z(\tilde{r},\tilde{z},\tilde{t})$ so that

$$\tilde{\gamma}\frac{\partial\tilde{\gamma}}{\partial\tilde{t}} + \frac{\partial}{\partial\tilde{z}}\int_0^1 \tilde{\gamma}^2\tilde{v}_z\tilde{r}\,d\tilde{r} = 0, \quad 0 < \tilde{z} < 1, \tag{2.2}$$

$$\begin{aligned}\frac{\partial\tilde{v}_z}{\partial\tilde{t}} - C_1\frac{1}{\tilde{\gamma}^2}\frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}\frac{\partial\tilde{v}_z}{\partial\tilde{r}}\right) - \frac{\tilde{r}}{\tilde{\gamma}}\frac{\partial\tilde{\gamma}}{\partial\tilde{t}}\frac{\partial\tilde{v}_z}{\partial\tilde{r}} &= -C_2\frac{\partial\tilde{\gamma}}{\partial\tilde{z}} - C_3\frac{\partial^2\tilde{\gamma}}{\partial\tilde{z}\partial\tilde{t}},\\ &(\tilde{r},\tilde{z}) \in \tilde{\Omega},\end{aligned} \tag{2.3}$$

with

$$\begin{cases}\tilde{\gamma}(0,\tilde{t}) = \tilde{\gamma}_0(\tilde{t}),\ \tilde{\gamma}(1,\tilde{t}) = \tilde{\gamma}_L(\tilde{t}),\ \tilde{\gamma}(\tilde{z},0) = \tilde{\gamma}^0(\tilde{z}),\\ \tilde{v}_z(1,\tilde{z},\tilde{t}) = 0, \tilde{v}_z(\tilde{r},\tilde{z},\tilde{t}=0) = \tilde{v}_z^0(\tilde{r},\tilde{z}),\ \big|\tilde{v}_z(0,\tilde{z},\tilde{t})\big| < +\infty,\end{cases} \tag{2.4}$$

where

$$C_1 = \frac{\mu_F \tau}{\rho_F R^2},\ C_2 = \left(\frac{Eh}{(1-\sigma^2)R}K + p_{ref}\right)\frac{1}{V^2\rho_F},\ C_3 = \frac{hC_vK}{RLV\rho_F}. \quad (2.5)$$

The inlet and outlet data $\tilde{\gamma}_0$ and $\tilde{\gamma}_L$ are obtained from the pressure data $P_0(t)$ and $P_L(t)$ given in (1.6), by integrating the pressure-displacement relationship (1.3) with respect to t, and then transforming the result into the non-dimensional form. Thus, $\tilde{\gamma}_0$ and $\tilde{\gamma}_L$ are scaled by R. They describe the inlet and outlet fluctuations of the domain radius around the "reference" domain radius $\tilde{r} = 1$.

PROPOSITION 2.1. *For the initial data $\tilde{\gamma}^0 = R = 1$, $\tilde{v}_z^0 = 0$ and boundary data $\tilde{\gamma}_0 = \tilde{\gamma}_L = R = 1$ problem* (2.2)–(2.5) *has a solution $\tilde{\gamma} = R = 1$, $\tilde{v}_z = 0$.*

We will show below, by using the Implicit Function Theorem, that problem (2.2)–(2.5) has a unique "mild" solution whenever the initial and boundary data are "close" to those listed in Proposition 2.1, namely, whenever the initial and boundary displacement from the reference radius $r = R = 1$ is small and whenever the initial velocity v_z^0 is close to zero.

In the rest of the manuscript we will be working with the non-dimensional form of the problem. To simplify notation, superscript "wiggle" that denotes the non-dimensional variables, will now be dropped, and this nomenclature will continue throughout the rest of the manuscript. Also, whenever R is used in the remainder of the paper, it refers to $R = 1$. Domain Ω below corresponds to the fixed, scaled domain, defined in (2.2).

3. Mild solution of the nonlinear problem. We will consider solutions of problem (2.2)–(2.5) with the initial and boundary data corresponding to the following function spaces:

$$\gamma^0 \in H^1(0,1),\ v_z^0 \in H^1_{0,0}(\Omega, r),\ \gamma_0, \gamma_L \in H^2(0,T). \quad (3.1)$$

Here $H^1_{0,0}(\Omega, r) = \big\{w \in L^2(\Omega, r) : \dfrac{\partial w}{\partial r} \in L^2(\Omega, r),\ < w > \in H^1(0,1),$ $w|_{r=1} = 0,\ | \ w|_{r=0} \ | < +\infty\big\}$, where $< w > := \int_0^1 wrdr$. The norm on $H^1_{0,0}(\Omega, r)$ is given by:

$$\|w\|^2_{H^1_{0,0}(\Omega,r)} = \int_\Omega \left(|w|^2 + \left|\frac{\partial w}{\partial r}\right|^2\right) rdrdz + \int_0^1 \left|\frac{\partial}{\partial z}\int_0^1 w\ rdr\right|^2 dz.$$

(The norm on $L^2(\Omega, r)$ is given by $\|f\|^2_{L^2(\Omega,r)} = \int_\Omega f\ rdrdz$.)

Thus, we define the space of data Λ to be

$$\Lambda = H^1(0,1) \times H^1_{0,0}(\Omega, r) \times (H^2(0,T))^2. \quad (3.2)$$

In order to define mild solution of problem (2.2)–(2.5) we introduce the following solution spaces

$$X_v := \big\{v \in L^2(0,T; H^1_{0,0}(\Omega, r)) \cap L^\infty(0,T; L^2(\Omega, r)) \mid \partial_t v \in L^2(0,T; L^2(\Omega, r)),$$
$$\Delta_r v \in L^2(0,T; L^2(\Omega, r)), \partial^2_{z,z} < v > \in L^2(0,T; L^2(0,1))\big\},$$

corresponding to the velocity space, and

$$X_\gamma := \left\{ \gamma \in H^1(0,T;H^1(0,1)) \mid \partial_t \gamma \in L^\infty(0,T;L^2(0,1)) \right\},$$

corresponding to the space of displacements.

DEFINITION 3.1. *Suppose that the initial data* $\gamma^0 \in H^1(0,1)$, $v_z^0 \in H^1_{0,0}(\Omega, r)$ *and that the boundary data* $(\gamma_0, \gamma_L) \in (H^2(0,T))^2$. *Function* $(\gamma, v_z) \in X_\gamma \times X_v$ *is called a mild solution of problem* (2.2)–(2.5) *if* (2.2)–(2.5) *holds for a.a.* $z \in (0,1), r \in [0,1)$ *and* $t \in (0,T)$.

4. Existence of a mild solution.

4.1. The framework. We aim at using the Implicit Function Theorem of Hildebrandt and Graves [22] to prove the (local) existence of a mild solution in a neighborhood of the solution stated in Proposition 2.1.

THEOREM 4.1. **(Implicit Function Theorem [22])** *Suppose that:*

- $F : U(\lambda_0, x_0) \subset \Lambda \times X \to Z$ *is defined on an open neighborhood* $U(\lambda_0, x_0)$ *and* $F(\lambda_0, x_0) = 0$, *where* Λ, X, Z *are Banach spaces.*
- F_x *exists as a Frechét partial derivative on* $U(\lambda_0, x_0)$ *and* $F_x(\lambda_0, x_0) : X \to Z$ *is bijective,*
- F *and* F_x *are continuous at* (λ_0, x_0).

Then the following are true:

- *Existence and uniqueness: There exist positive numbers* δ_0 *and* δ *such that for every* $\lambda \in \Lambda$ *satisfying* $\|\lambda - \lambda_0\| \le \delta_0$ *there is exactly one* $x \in X$ *for which* $\|x - x_0\| \le \delta$ *and* $F(\lambda, x(\lambda)) = 0$.
- *Continuity: If* F *is continuous in a neighborhood of* (λ_0, x_0), *than* x *is continuous in a neighborhood of* λ_0.

4.2. The mapping F. To define F we first remark that we will consider the conservation of mass equation (2.2) as a condition which will be satisfied for all possible solution candidates (γ, v_z). More precisely, when considering the continuity of F and F_x and when showing the bijective property of F_x will be "perturbing" our function F by a small source term f only in the balance of momentum equation, and not in the conservation of mass equation, preserving the conservation of mass property identically for all possible solutions, which is physically reasonable.

First notice that the conservation of mass equation (2.2) can be rewritten, after dividing (2.2) by γ, as a linear operator in γ, $\mathcal{L}_{<v_z>}(\gamma^0, \gamma_0, \gamma_L)$, which to each given $< v_z >$ and initial and boundary data γ^0, γ_0 and γ_L associates the (unique) solution $\gamma \in X_\gamma$ of the following linear transport problem:

$$\frac{\partial \gamma}{\partial t} + 2 < v_z > \frac{\partial \gamma}{\partial z} + \gamma \frac{\partial < v_z >}{\partial z} = 0, \tag{4.1}$$

with $\gamma(0,t) = \gamma_0(t)$ whenever $< v_z >$ is positive, $\gamma(1,t) = \gamma_L(t)$ whenever $< v_z >$ is negative, and $\gamma(z,0) = \gamma^0(z)$.

Define F via the momentum equation (2.3) where $\gamma \in X_\gamma$ in equation (2.3) is obtained from the conservation of mass “condition” (4.1).

DEFINITION 4.1. **(Mapping F)** *Let* $Z := L^2(0,T;L^2(\Omega,r))$. *Define mapping* $F : U((R,0,R,R),0) \subset \Lambda \times X_v \to Z$, *which associates to each* $((\gamma^0, v_z^0, \gamma_0, \gamma_L), v_z) \in U((R,0,R,R),0)$ *an* $f \in Z$

$$F : ((\gamma^0, v_z^0, \gamma_0, \gamma_L), v_z) \mapsto f \tag{4.2}$$

such that

$$\begin{cases} F((\gamma^0, v_z^0, \gamma_0, \gamma_L), v_z) := \dfrac{\partial v_z}{\partial t} - C_1 \dfrac{1}{\gamma^2}\Delta_r v_z - \dfrac{r}{\gamma}\dfrac{\partial \gamma}{\partial t}\dfrac{\partial v_z}{\partial r} \\ \qquad\qquad\qquad\qquad + C_2 \dfrac{\partial \gamma}{\partial z} + C_3 \dfrac{\partial^2 \gamma}{\partial z \partial t}, \\ v_z(1,z,t) = 0, v_z(r,z,t=0) = v_z^0(r,z), \ |v_z(0,z,t)| < +\infty, \end{cases} \tag{4.3}$$

where $\gamma \in X_\gamma$ *depends on* v_z *and is given as a solution of*

$$\begin{cases} \dfrac{\partial \gamma}{\partial t} + 2 < v_z > \dfrac{\partial \gamma}{\partial z} + \gamma \dfrac{\partial < v_z >}{\partial z} = 0, \\ \gamma(0,t) = \gamma_0(t), \ \gamma(1,t) = \gamma_L(t), \ \gamma(z,0) = \gamma^0(z). \end{cases} \tag{4.4}$$

Denote by $(\lambda_0, x_0) = ((R,0,R,R),0)$. Then we see, by Proposition 2.1, that $F(\lambda_0, x_0) = 0$.

We will be using the Implicit Function Theorem to show the existence of a unique mild solution $(\gamma, v_z) \in X_\gamma \times X_v$ of (2.2)–(2.5) for each set of data $\lambda = (\gamma^0, v_z^0, \gamma_0, \gamma_L)$ in a neighborhood of $\lambda_0 = (R,0,R,R)$, by considering small perturbations (λ, v_z) of the zero set $(\lambda_0, 0)$ of the mapping F, given by the balance of momentum equation (2.3), in which $\gamma \in X_\gamma$ satisfies the mass conservation condition (2.2).

PROPOSITION 4.1. *Mapping* F *is continuous at* (λ_0, x_0).

The proof is a direct consequence of the form of (4.3) and of the continuous dependence of the solution γ of (4.1) on the coefficients depending on $< v_z >$ and on the initial and boundary data.

4.3. The Frechét Derivative of F. Introduce perturbation of v_z around $\hat{v}_z$ as follows:

$$v_z = \hat{v}_z + \delta w_z, \quad \delta > 0.$$

Define $\hat{\gamma}$ via $\hat{v}_z$ as the solution of (4.4) corresponding to $v_z = \hat{v}_z$. Then the Frechét derivative of F with respect to $x = v_z$, evaluated at $((\hat{\gamma}^0, \hat{v}_z^0, \hat{\gamma}_0, \hat{\gamma}_L), \hat{v}_z)$ is a mapping

$$F_x((\hat{\gamma}^0, \hat{v}_z^0, \hat{\gamma}_0, \hat{\gamma}_L), \hat{v}_z) : X_v \to Z$$

defined by

$$F_x((\hat{\gamma}^0,\hat{v}_z^0,\hat{\gamma}_0,\hat{\gamma}_L),\hat{v}_z)w_z := \frac{\partial w_z}{\partial t} - C_1\frac{1}{\hat{\gamma}^2}\Delta_r w_z + C_1\frac{2}{\hat{\gamma}^3}\eta\Delta_r\hat{v}_z$$
$$-\frac{r}{\hat{\gamma}}\frac{\partial\eta}{\partial t}\frac{\partial\hat{v}_z}{\partial r} - \frac{r}{\hat{\gamma}}\frac{\partial\hat{\gamma}}{\partial t}\frac{\partial w_z}{\partial r} + \frac{r}{\hat{\gamma}^2}\eta\frac{\partial\hat{\gamma}}{\partial t}\frac{\partial\hat{v}_z}{\partial r} + C_2\frac{\partial\eta}{\partial z} + C_3\frac{\partial^2\eta}{\partial z\partial t}, \tag{4.5}$$

where η is given as a solution of

$$\begin{cases} \dfrac{\partial\eta}{\partial t} + 2<\hat{v}_z>\dfrac{\partial\eta}{\partial z} + 2<w_z>\dfrac{\partial\hat{\gamma}}{\partial z} + \hat{\gamma}\dfrac{\partial<w_z>}{\partial z} + \eta\dfrac{\partial<\hat{v}_z>}{\partial z} = 0, \\ \eta(0,t)=0,\ \eta(1,t)=0,\ \eta(z,0)=0, \end{cases} \tag{4.6}$$

with

$$w_z(1,z,t)=0, w_z(r,z,0)=0,\ w_z(0,z,t) - \textit{bounded}. \tag{4.7}$$

By the similar arguments as those used for continuity of the mapping F one can see that the following is true.

THEOREM 4.2. *The Frechét derivative F_x is a continuous mapping from X_v to Z.*

Next we need to show that the Frechét derivative, evaluated at (λ_0, x_0), is a bijection. From (4.6)–(4.7) we see that the Frechét derivative evaluated at $(\lambda_0,x_0)=((R,0,R,R),0)$ is given by the following

$$\begin{cases} F_x((R,0,R,R),0)w_z := \dfrac{\partial w_z}{\partial t} - C_1\dfrac{1}{R^2}\Delta_r w_z + C_2\dfrac{\partial\eta}{\partial z} + C_3\dfrac{\partial^2\eta}{\partial z\partial t}, \\ w_z(1,z,t)=0, w_z(r,z,0)=0,\ w_z(0,z,t) - \textit{bounded}, \end{cases} \tag{4.8}$$

where η, which depends on w_z, satisfies

$$\begin{cases} \dfrac{\partial\eta}{\partial t} + R\dfrac{\partial}{\partial z} < w_z >= 0, \\ \eta(0,t)=0 \text{ whenever } < w_z > \text{ positive}, \\ \eta(1,t)=0 \text{ whenever } < w_z > \text{ negative}, \\ \eta(z,0)=0. \end{cases} \tag{4.9}$$

THEOREM 4.3. *The Frechét derivative defined by* (4.8)–(4.9) *is a bijection from X_v to Z.*

Theorem 4.3 is a consequence of the following result: for every $f \in L^2(0,T;L^2(\Omega,r))$ and $(\eta^0,w_z^0,\eta_0,\eta_L)\in\Lambda$ there exists a unique function $(\eta,w_z)\in X_\gamma\times X_v$ satisfying for a.e. $0<z<1, 0\le r<1, 0\le t\le T$

$$\frac{\partial\eta}{\partial t} + R\frac{\partial}{\partial z} < w_z >= 0, \tag{4.10}$$
$$\frac{\partial w_z}{\partial t} - \frac{C_1}{R^2}\,\triangle_r\, w_z + C_2\frac{\partial\eta}{\partial z} + C_3\frac{\partial^2\eta}{\partial z\partial t} = f(r,z,t), \tag{4.11}$$

with

$$\begin{cases} \eta(0,t) = 0, \ \eta(1,t) = 0, \ \eta(z,0) = 0 \\ w_z(1,z,t) = 0, \ w_z(0,z,t) - \text{bounded}, \ w_z(r,z,0) = 0. \end{cases} \tag{4.12}$$

(Equation (4.10) implies that the boundary conditions for η are equivalent to the homogeneous Neumann condition for $< w_z >$ at $z = 0, 1$.) In fact, we will show a slightly more general result (general data):

THEOREM 4.4. *Let $f \in L^2(0,T; L^2(\Omega, r))$ and $\eta_0, \eta_L \in H^2(0,T)$, $\eta^0 \in H^1(0,1)$ and $w_z^0 \in H^1_{0,0}(\Omega, r)$. Then, there exists a unique function $(\eta, w_z) \in X_\gamma \times X_v$ satisfying for a.e. $0 < z < 1, 0 \le r < 1, 0 < t \le T$*

$$\frac{\partial \eta}{\partial t} + R\frac{\partial}{\partial z} < w_z >= 0, \tag{4.13}$$

$$\frac{\partial w_z}{\partial t} - \frac{C_1}{R^2} \triangle_r \, w_z + C_2\frac{\partial \eta}{\partial z} + C_3\frac{\partial^2 \eta}{\partial z \partial t} = f(r,z,t), \tag{4.14}$$

with

$$\begin{cases} \eta(0,t) = \eta_0(t), \ \eta(1,t) = \eta_L(t), \ \eta(z,0) = \eta^0(z) \\ w_z(1,z,t) = 0, \ w_z(0,z,t) - \text{bounded}, \ w_z(r,z,0) = w_z^0(r,z). \end{cases} \tag{4.15}$$

This result motivated the choice of the parameter space Λ for the existence of a unique mild solution to the nonlinear problem (2.2)–(2.4).

To prove this result we proceed in two steps:

1. Show the existence of a unique *weak solution* to (4.13), (4.14) and (4.15).
2. Obtain energy estimates which provide higher regularity of the weak solution, giving rise to the *mild solution* $(\eta, w_z) \in X_\gamma \times X_v$.

STEP 1. **Existence of a unique weak solution of (4.13)–(4.15).**

Introduce the function $\bar\eta$ which satisfies the homogeneous boundary data at $z = 0$ and $z = 1$: $\bar\eta = \eta(z,t) - ((\eta_L(t) - \eta_0(t))z + \eta_0(t))$. Problem (4.13)–(4.15) can then be rewritten in terms of $\bar\eta$ as follows

$$\frac{\partial \bar\eta}{\partial t} + R\frac{\partial}{\partial z} < w_z >= -g_1, \tag{4.16}$$

$$\frac{\partial w_z}{\partial t} - \frac{C_1}{R^2} \triangle_r \, w_z + C_2\frac{\partial \bar\eta}{\partial z} + C_3\frac{\partial^2 \bar\eta}{\partial z \partial t} = f - g_2. \tag{4.17}$$

with

$$\begin{cases} \bar\eta(0,t)=0, \ \bar\eta(1,t)=0, \ \bar\eta(z,0)=(\eta_L(0)-\eta_0(0))z + \eta_0(0)=\bar\eta^0(z), \\ w_z(1,z,t) = 0, \ w_z(0,z,t) - \text{bounded}, \ w_z(r,z,0) = w^0(r,z), \end{cases} \tag{4.18}$$

where

$$\begin{cases} g_1(z,t) = ((\eta_L'(t) - \eta_0'(t))z + \eta_0'(t)), \\ g_2(r,z,t) = C_2(\eta_L(t) - \eta_0(t)) + C_3(\eta_L'(t) - \eta_0'(t)). \end{cases} \tag{4.19}$$

To define a weak solution introduce the following function spaces

$$\Gamma = H^1(0,T : L^2(0,1)), \tag{4.20}$$

$$V = \{w \in L^2(0,T : H^1_{0,0}(\Omega,r)) : \frac{\partial w}{\partial t} \in L^2(0,T : H^{-1}_{0,0}(\Omega,r))\}. \tag{4.21}$$

Definition 4.2. *We say that* $(\bar\eta, w_z) \in \Gamma \times V$ *is a weak solution of* (4.16)–(4.19) *provided that for all* $\varphi \in H^1_0(0,1)$ *and* $\xi \in H^1_{0,0}(\Omega,r)$

$$\int_0^1 \frac{\partial \bar\eta}{\partial t}\varphi\, dz - R\int_0^1 < w_z > \frac{\partial \varphi}{\partial z}\, dz = -\int_0^1 g_1\varphi\, dz \tag{4.22}$$

$$\begin{gathered}\int_\Omega \frac{\partial w_z}{\partial t}\xi\; rdrdz + \frac{C_1}{R^2}\int_\Omega \frac{\partial w_z}{\partial r}\frac{\partial \xi}{\partial r}\; rdrdz - C_2\int_0^1 \bar\eta\; \frac{\partial}{\partial z} < \xi > \; dz \\ -C_3\int_0^1 \frac{\partial \bar\eta}{\partial t}\;\frac{\partial}{\partial z} < \xi > \; dz = \int_0^1 f\varphi\; dz - \int_0^1 g_2\varphi\; dz,\end{gathered} \tag{4.23}$$

for a.e. $0 \le t \le T$, *and satisfying* $\bar\eta(z,0) = \bar\eta^0(z),\ w_z(r,z,0) = w^0_z(r,z)$.

We first show that for the boundary data η_0 and η_L in $H^1(0,T)$ and for the initial data $\eta^0 \in L^2(0,1)$, $w^0_z \in L^2(\Omega,r)$, there exists a unique weak solution of (4.16)–(4.19).

Notice that the weak formulation of the problem reflects lack of regularity in the z-direction due to the parabolic degeneracy in the momentum equation (4.23) and due to the hyperbolic nature of the averaged conservation of mass equation (4.22). This will introduce various difficulties in the proof of the existence of a unique weak solution which we state next.

Theorem 4.5. *Let* $f \in L^2(0,T;L^2(\Omega,r))$. *Assume that the initial data* $\bar\eta^0$ *and* w^0_z *satisfy* $\bar\eta^0 \in L^2(0,1)$ *and* $w^0_z \in L^2(\Omega,r)$ *and that the boundary data* $\eta_0(t)$ *and* $\eta_L(t)$ *satisfy* $\eta_0, \eta_1 \in H^1(0,T)$. *Then there exists a unique weak solution* $(\bar\eta, w_z) \in \Gamma \times V$ *of* (4.16)–(4.19).

Proof. The proof is an application of the Galerkin method combined with the nontrivial energy estimates to deal with the lack of regularity in the z-direction. We present the proof in the following four steps:

1. Construction of the Galerkin approximations.
2. Uniform energy estimates.
3. Weak convergence of a sub-sequence of Galerkin approximations to a solution using compactness arguments.
4. Uniqueness of the weak solution.

Construction of the Galerkin Approximations: Let $\{\phi_k\}_{k=1}^\infty$ be the smooth functions which are orthogonal in $H^1_0(0,1)$, orthonormal in $L^2(0,1)$ and span the solution space for $\bar\eta$. Furthermore, let $\{w_k\}_{k=1}^\infty$ be the smooth functions which satisfy $w_k|_{r=1} = 0$, and are orthonormal in $L^2(\Omega,r)$ and span the solution space for the velocity w_z. Introduce the function space $C^k_{0,0}(\Omega) = \left\{v \in C^k(\Omega) : v|_{r=1} = 0\right\}$, for any $k = 0,1,...,\infty$.

Fix positive integers m and n. We look for the functions $\bar{\eta}_m : [0,T] \to C_0^\infty(0,1)$ and $w_{z_n} : [0,T] \to C_{0,0}^\infty(\Omega)$ of the form

$$\bar{\eta}_m(t) = \sum_{i=1}^{m} d_i^m(t)\,\phi_i, \quad w_{z_n}(t) = \sum_{j=1}^{n} l_j^n(t)\,w_j, \tag{4.24}$$

where the coefficient functions d_h^m and l_k^n are chosen so that the functions $\bar{\eta}_m$ and w_{z_n} satisfy the weak formulation (4.22)–(4.23) of the linear problem (4.16)–(4.19), projected onto the finite dimensional subspaces spanned by $\{\phi_i\}$ and $\{w_j\}$ respectively:

$$\int_0^1 \frac{\partial \bar{\eta}_m}{\partial t}\phi_h\, dz - R\int_0^1 < w_{z_n} > \frac{\partial \phi_h}{\partial z}\, dz = -\int_0^1 g_1\phi_h\, dz \tag{4.25}$$

$$\begin{aligned}\int_\Omega \frac{\partial w_{z_n}}{\partial t} w_k\; rdrdz + \frac{C_1}{R^2}\int_\Omega \frac{\partial w_{z_n}}{\partial r}\frac{\partial w_k}{\partial r} rdrdz - C_2\int_0^1 \bar{\eta}_m \frac{\partial}{\partial z}< w_k > dz \\ - C_3\int_0^1 \frac{\partial \bar{\eta}_m}{\partial t}\,\frac{\partial}{\partial z} < w_k > \; dz = \int_\Omega f w_k\; dz - \int_\Omega g_2 w_k\; rdrdz\end{aligned} \tag{4.26}$$

for a.e $0 \le t \le T$, $h = 1, \cdots, m$ and $k = 1, \cdots, n$, and

$$\begin{cases} d_h^m(0) = \displaystyle\int_0^1 \bar{\eta}^0(z)\phi_h(z)dz, \\ l_k^n(0) = \displaystyle\int_\Omega w_z^0 w_k rdrdz. \end{cases} \tag{4.27}$$

The existence of the coefficient functions satisfying these requirements is guaranteed by the following Lemma.

Lemma 4.1. *Assume that $f \in L^2(0,T; L^2(\Omega, r))$. For each $m = 1, 2, ...$ and $n = 1, 2, ...$ there exist unique functions $\bar{\eta}_m$ and w_{z_n} of the form (4.24), satisfying* (4.25)–(4.27). *Moreover*

$$(\bar{\eta}_m, v_{w_n}) \in C^1(0, T : C_0^\infty(0,1)) \times C^1(0, T : C_{0,0}^\infty(\Omega)).$$

Proof. To simplify notation, let us first introduce the following vector functions:

$$d^m(t) = \begin{pmatrix} d_1^m(t) \\ \vdots \\ d_m^m(t) \end{pmatrix}, \; l^n(t) = \begin{pmatrix} l_1^n(t) \\ \vdots \\ l_n^n(t) \end{pmatrix}, \; Y(t) = \begin{pmatrix} d^m(t) \\ l^n(t)) \end{pmatrix} \tag{4.28}$$

Then, equation (4.25) written in matrix form reads:

$$A_1 d^{m'}(t) + A_2 l^n(t) = S_1(t), \tag{4.29}$$

where A_1 is an $m \times m$ matrix, A_2 an $m \times n$ matrix and S_1 an $m \times 1$ matrix defined by the following: $[A_1]_{h,i} = (\phi_i, \phi_h)_{L^2(0,1)} = \delta_{h,i}$, $[A_2]_{h,i} = -R\left(< w_j >, \frac{\partial \phi_h}{\partial z}\right)_{L^2(0,1)}$, $[S_1(t)]_{h,1} = (g_1, \phi_h)_{L^2(0,1)}$, where $h, i = 1, ..., m$ and $j = 1, ..., n$. Similarly, equation (4.26) written in matrix form reads:

$$B_1 l^{n'}(t) + B_2 l^n(t) - B_3 d^m(t) - B_4 d^{m'}(t) = S_2(t), \tag{4.30}$$

where B_1 and B_2 are $n \times n$ matrices, B_3 and B_4 are $n \times m$ matrices, and $S_2(t)$ is an $n \times 1$ matrix defined by the following: $[B_1]_{k,j} = (w_j, w_k)_{L^2(\Omega,r)} = \delta_{k,j}$, $[B_2]_{k,j} = \frac{C_1}{R^2}\left(\frac{\partial w_j}{\partial r}, \frac{\partial w_k}{\partial r}\right)_{L^2(\Omega,r)}$, $[B_3]_{k,i} = C_2\left(\frac{\partial <w_k>}{\partial z}, \phi_i\right)_{L^2(0,1)}$, $[B_4]_{k,i} = C_3\left(\frac{\partial <w_k>}{\partial z}, \phi_i\right)_{L^2(0,1)}$, $[S_2(t)]_{k,1} = (f - g_2, w_k)_{L^2(\Omega,r)}$, where $k, j = 1, ..., n$ and $i = 1, ..., m$.

Equations (4.29) and (4.30) can be written together as the following system

$$\begin{cases} AY'(t) + BY(t) = S(t), \\ Y(0) = \begin{pmatrix} d^m(0) \\ l_k^n(0) \end{pmatrix}, \end{cases} \tag{4.31}$$

where Y is defined in (4.28) and

$$A = \begin{pmatrix} A_1^{m\times m} & 0^{m\times n} \\ -B_4^{n\times m} & B_1^{n\times n} \end{pmatrix}_{(m+n)\times(m+n)},$$

$$B = \begin{pmatrix} 0^{m\times m} & A_2^{m\times n} \\ -B_3^{n\times m} & B_2^{n\times n} \end{pmatrix}_{(m+n)\times(m+n)}.$$

Function S is an $(m+n)\times 1$ matrix which incorporates the initial and boundary data obtained from the right hand-sides of (4.29) and (4.30).

To guarantee the existence of a solution $Y(t)$ of appropriate regularity first notice that linear independence of the sets $\{\phi_1, \cdots, \phi_m\}$ and $\{w_1, \cdots, w_n\}$ guarantees that the matrix A is nonsingular. Additionally, since the coefficient matrices are constant, there exists a unique C^1 function $Y(t) = (d^m(t), l^n(t))$ satisfying (4.31). Moreover $(\bar\eta_m, w_{z_n})$, defined via $d^m(t)$ and $l^n(t)$ in (4.24), respectively, solves (4.25)–(4.27) for all $0 \le t \le T$, thus $(\bar\eta_m, w_{z_n}) \in C^1(0, T : C_0^\infty(0,1)) \times C^1(0, T : C_{0,0}^\infty(\Omega))$. This completes the proof of Lemma 4.1. $\square$

Energy Estimate: We continue our proof of the existence of a weak solution to (4.16)–(4.19) by obtaining an energy estimate for $\bar\eta_m$ and w_{z_n} which is uniform in m and n. The estimate will bound the L^2-norms of $\bar\eta_m$ and w_{z_n}, the L^2-norms of $\frac{\partial w_{z_n}}{\partial r}$ and $\frac{\partial \bar\eta_m}{\partial t}$, and the $L^2(0,T; H_{0,0}^{-1}(\Omega, r))$-norm of $\frac{\partial w_{z_n}}{\partial t}$, in terms of the initial and boundary data and the coefficients of (4.16)–(4.18). Notice again the lack of information about the smoothness in z of the functions $\bar\eta$ and w_z.

THEOREM 4.6. *There exists a constant C depending on $1/R, T, C_2$ and C_3, such that*

$$\sup_{0\le t\le T}\left[\|w_{z_n}\|^2_{L^2(\Omega,r)}+\frac{C_2}{R}\|\bar\eta_m\|^2_{L^2(0,1)}\right]+\frac{2C_1}{R^2}\left\|\frac{\partial w_{z_n}}{\partial r}\right\|^2_{L^2(0,T;L^2(\Omega,r))}$$
$$+\frac{C_3}{R}\left\|\frac{\partial\bar\eta_m}{\partial t}\right\|^2_{L^2(0,T;L^2(0,1))}+\left\|\frac{\partial w_{z_n}}{\partial t}\right\|^2_{L^2(0,T;H^{-1}_{0,0}(\Omega,r))}\le C\Big[\|\bar\eta^0\|^2_{L^2(0,1)}$$
$$+\left\|w^0_z\right\|^2_{L^2(\Omega,r)}+\|f\|^2_{L^2(0,T;L^2(\Omega,r))}+\|\eta_L-\eta_0\|^2_{H^1(0,T)}+\|\eta_0\|^2_{H^1(0,T)}\Big]$$

Furthermore, $\dfrac{\partial}{\partial z}\displaystyle\int_0^1 w_{z_n}rdr\in L^2(0,T;L^2(0,1))$, *and its* $L^2(0,T;L^2(0,1))$*-norm is bounded by the right hand-side of the above energy estimate.*

Proof. We aim at using the Gronwall's inequality. However, due to the lack of smoothness in z, it is impossible to control the terms with the z-derivative of $\bar\eta_m$. To deal with this problem, we manipulate the conservation of mass and balance of momentum equations in order to cancel the unwanted terms. The remaining terms, which we will estimate in terms of the data, will be those appearing in the estimate above.

We begin by first multiplying (4.26) by l^n_k and summing $k=1,\cdots,n$ to find

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|w_{z_n}|^2\,rdrdz+\frac{C_1}{R^2}\int_\Omega\left|\frac{\partial w_{z_n}}{\partial r}\right|^2 rdrdz-\underbrace{C_2\int_0^1\bar\eta_m\frac{\partial}{\partial z}<w_{z_n}>dz}_{(i)}$$
$$-\underbrace{C_3\int_0^1\frac{\partial\bar\eta_m}{\partial t}\frac{\partial}{\partial z}<w_{z_n}>dz}_{(ii)}=\int_\Omega f w_{z_n}rdrdz-\int_\Omega g_2w_{z_n}rdrdz. \tag{4.32}$$

Multiply (4.25) by d^m_h and sum $h=1,\cdots,m$ to find

$$\frac{1}{2}\frac{d}{dt}\int_0^1|\bar\eta_m|^2\,dz-\underbrace{R\int_0^1<w_{z_n}>\frac{\partial\bar\eta_m}{\partial z}\,dz}_{(i)}=-\int_0^1 g_1\bar\eta_m\,dz. \tag{4.33}$$

Multiply (4.25) by $\dot d^m_h$ and sum $h=1,\cdots,m$ to find

$$\int_0^1\left|\frac{\partial\bar\eta_m}{\partial t}\right|^2\,dz-\underbrace{R\int_0^1<w_{z_n}>\frac{\partial^2\bar\eta_m}{\partial t\partial z}\,dz}_{(ii)}=-\int_0^1 g_1\frac{\partial\bar\eta_m}{\partial t}\,dz. \tag{4.34}$$

Multiply equation (4.33) by $\frac{C_2}{R}$ and (4.34) by $\frac{C_3}{R}$ and add the two resulting equations to equation (4.32) to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left[\int_\Omega |w_{z_n}|^2\ rdrdz + \frac{C_2}{R}\int_0^1 |\bar{\eta}_m|^2\ dz\right] + \frac{C_1}{R^2}\int_\Omega \left|\frac{\partial w_{z_n}}{\partial r}\right|^2\ rdrdz \\ + \frac{C_3}{R}\int_0^1 \left|\frac{\partial \bar{\eta}_m}{\partial t}\right|^2\ dz = \int_\Omega f w_{z_n}\ rdrdz - \int_\Omega g_2 w_{z_n}\ rdrdz \\ - \frac{C_2}{R}\int_0^1 g_1\bar{\eta}_m\ dz - \frac{C_3}{R}\int_0^1 g_1\frac{\partial \bar{\eta}_m}{\partial t}\ dz.\end{aligned} \tag{4.35}$$

We can see that the terms denoted by (i) and (ii), which we cannot control, canceled out. By using the Cauchy inequality to estimate the right hand-side of (4.35) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left[\|w_{z_n}\|^2_{L^2(\Omega,r)} + \frac{C_2}{R}\|\bar{\eta}_m\|^2_{L^2(0,1)}\right] + \frac{C_1}{R^2}\left\|\frac{\partial w_{z_n}}{\partial r}\right\|^2_{L^2(\Omega,r)} \\ + \frac{C_3}{2R}\left\|\frac{\partial \bar{\eta}_m}{\partial t}\right\|^2_{L^2(0,1)} \le \left\|f_2 + \frac{g_2}{4} + \frac{C_2 + C_3}{2R}g_1\right\|^2_{L^2(\Omega,r)} \\ + \frac{1}{2}\|w_{z_n}\|^2_{L^2(\Omega,r)} + \frac{C_2}{2R}\|\bar{\eta}_m\|^2_{L^2(0,1)}.\end{aligned}$$

We are now in a position to apply the differential form of the Gronwall's inequality to conclude that there exists a constant $C > 0$ depending on T, C_2, C_3 and $1/R$ such that

$$\begin{aligned}\sup_{0\le t\le T}\left[\|w_{z_n}\|^2_{L^2(\Omega,r)} + \frac{C_2}{R}\|\bar{\eta}_m\|^2_{L^2(0,1)}\right] + \frac{2C_1}{R^2}\int_0^T \left\|\frac{\partial w_{z_n}}{\partial r}\right\|^2_{L^2(\Omega,r)} dt \\ + \frac{C_3}{2R}\int_0^T \left\|\frac{\partial \bar{\eta}_m}{\partial t}\right\|^2_{L^2(0,1)} dt \le C\left[\|\bar{\eta}^0\|^2_{L^2(0,1)} + \left\|w_z^0\right\|^2_{L^2(\Omega,r)}\right. \\ \left. + \|f\|^2_{L^2(0,T:L^2(\Omega,r))} + \|\eta_1 - \eta_0\|^2_{H^1(0,T)} + \|\eta_0\|_{H_0^1(0,T)}\right].\end{aligned} \tag{4.36}$$

We conclude the proof by showing that $\dfrac{\partial w_{z_n}}{\partial t} \in L^2(0,T; H^{-1}_{0,0}(\Omega,r))$, and that $\partial v_{z_n}/\partial t$ satisfies the estimate stated in Theorem 4.6.

Fix $\nu \in H^1_{0,0}(\Omega,r)$ such that $\|\nu\|_{H^1_{0,0}(\Omega,r)} \le 1$. Since $C^\infty_{0,0}(\Omega)$ is dense in $H^1_{0,0}(\Omega,r)$, we can write $\nu = \nu_1 + \nu_2$, where $\nu_1 \in$ span $\{w_j\}_{j=1}^n$ and $(\nu_2, w_j)_{L^2(\Omega,r)} = 0$ for $j = 1,\cdots,n$. Then (4.24) and (4.26) imply

$$\begin{aligned}\left|\int_\Omega \frac{\partial w_{z_n}}{\partial t}\nu\ rdrdz\right| \le \left[\frac{C_1}{R}\left\|\frac{\partial w_n}{\partial r}\right\|_{L^2(\Omega,r)} + C_2\|\bar{\eta}_m\|_{L^2}\right. \\ \left. + C_3\left\|\frac{\partial \bar{\eta}_m}{\partial t}\right\|_{L^2} + \|f\|_{L^2} + \|g_2\|_{L^2}\right]\|\nu\|_{H^1_{0,0}(\Omega,r)},\ a.e.\ 0 \le t \le T.\end{aligned}$$

Thus, since $\|\nu_1\|_{H^1_{0,0}(\Omega,r)} \le 1$, by using the energy estimate (4.36), we find that there exists a constant $\tilde{C}$ depending on $T, 1/R, C_2, C_3$ such that

$$\int_0^T \left\| \frac{\partial w_{z_n}}{\partial t} \right\|^2_{H^{-1}_{0,0}(\Omega,r)} dt \leq \tilde{C} \left[\|\bar{\eta}^0\|^2_{L^2(0,1)} + \left\| w_z^0 \right\|^2_{L^2(\Omega,r)} \right. \\ \left. + \|f\|^2_{L^2(0,T;L^2(\Omega,r))} + \|\eta_1 - \eta_0\|^2_{H^1(0,T)} + \|\eta_0\|^2_{H^1_0(0,T)} \right]. \tag{4.37}$$

This concludes the proof of Theorem 4.6. ∎

It is interesting to notice that the coefficient of the vessel wall viscosity, C_3, governs the estimate for the time-derivative of the structure displacement, which is to be expected. Thus, our estimate shows how the structure viscoelasticity regularizes the time evolution of the structure.

Also, notice that the right hand-side of the energy estimate incorporates the initial data for both the structure displacement and the structure velocity, but the boundary data for only the structure displacement. This is a consequence of the parabolic degeneracy in the balance of momentum equation and is an interesting feature of this reduced, effective model.

Weak convergence to a solution: We use the uniform energy estimate, presented in Theorem 4.6, to conclude that there exist convergent subsequences that converge weakly to the functions which satisfy (4.16)–(4.19) in the weak sense. This is a standard approach except for the fact that we need to deal with the weighted L^2-norms in Ω, with the singular weight r that is present due to the axial symmetry of the problem. We deal with this technical obstacle by using Lemma 4.58 on page 120 in [1], with $p = 2$ and $\nu = 1$.

By the energy estimate stated in Theorem 4.6 we see that the sequence $\{\bar{\eta}_m\}_{m=1}^{\infty}$ is bounded in $H^1(0,T;L^2(0,1))$. Similarly, $\{w_{z_n}\}_{n=1}^{\infty}$ is bounded in $L^2(0,T;H^1_{0,0}(\Omega,r))$ and that $\partial w_{z_n}/\partial t$ is bounded in $L^2(0,T;H^{-1}_{0,0}(\Omega,r))$. Therefore, there exist convergent subsequences $\left\{\bar{\eta}_{m_j}\right\}_{m_j=1}^{\infty}$ and $\left\{w_{z_{n_j}}\right\}_{n_j=1}^{\infty}$ such that

$$\begin{cases} \eta_{m_j} \rightharpoonup \eta & \text{weakly in } H^1(0,T;L^2(0,1)), \\ w_{z_{n_j}} \rightharpoonup w_z & \text{weakly in } L^2(0,T;L^2(\Omega,r)), \\ \dfrac{\partial w_{z_{n_j}}}{\partial r} \rightharpoonup \dfrac{\partial w_z}{\partial r} & \text{weakly in } L^2(0,T;L^2(\Omega,r)), \\ \dfrac{\partial w_{z_{n_j}}}{\partial t} \rightharpoonup \dfrac{\partial w_z}{\partial t} & \text{weakly in } L^2(0,T;H^{-1}_{0,0}(\Omega,r)). \end{cases} \tag{4.38}$$

What is left to show is that the limiting functions satisfy (4.16)–(4.19) in the weak sense and that the limiting functions satisfy the initial data. This requires relatively standard arguments which can be found in, e.g., [13]. For details of this calculation, please see [19]. Similar arguments have been used also in [20].

Uniqueness: Uniqueness of the weak solution is a direct consequence of the linearity of the problem.

This completes the proof of Theorem 4.5. ∎

This proof completes the first step in the proof of Theorem 4.4. What is left to show is that the weak solution of (4.13) and (4.15) has higher regularity and that, in fact, it belongs to the space $X_\gamma \times X_v$.

COROLLARY 4.1. *The energy estimate stated in Theorem 4.6 implies that, in fact,* $\bar\eta \in L^\infty(0,T;L^2(0,1)) \cap H^1(0,T;L^2(0,1))$, $w_z \in L^2(0,T;H^1_{0,0}(\Omega,r)) \cap L^\infty(0,T;L^2(\Omega,r))$ *with* $\frac{\partial w_z}{\partial t} \in L^2(0,T;H^{-1}_{0,0}(\Omega,r))$.

STEP 2. **Higher regularity of the weak solution to (4.13)-(4.15).**

To show that our weak solution $(\bar\eta, w_z)$ is actually in $X_\gamma \times X_v$ we proceed in two steps. First we show that the sequence $\{\frac{\partial w_{z_n}}{\partial t}\}_{n=1}^\infty$ is bounded in $L^2(0,T;L^2(\Omega,r))$, and then, using this information, we show that $(\bar\eta, w_z) \in X_\gamma \times X_v$. To show this improved regularity property of our weak solution we need to assume, as usual, some higher regularity of the initial and boundary data. The precise assumptions are given below.

THEOREM 4.7. *(Improved Regularity: Part I) Suppose that the boundary data* $\eta_1, \eta_0 \in H^2(0,T)$ *and the initial data* $\bar\eta^0 \in L^2(0,1)$, $w_z^0 \in H^1_{0,0}(\Omega,r)$. *Then the weak solution* $(\bar\eta, w_z) \in \Gamma \times V$ *satisfies* $\dfrac{\partial\bar\eta}{\partial t} \in L^\infty(0,T:L^2(0,1))$, $\dfrac{\partial w_z}{\partial r} \in L^\infty(0,T:L^2(\Omega,r))$, $\dfrac{\partial w_z}{\partial t} \in L^2(0,T:L^2(\Omega,r))$.

Moreover, there exists a $C>0$, *depending on* $1/R, C_2, C_3, T$, *such that*

$$
\begin{aligned}
&\sup_{0\le t\le T}\left[\frac{C_3}{R}\left\|\frac{\partial\bar\eta}{\partial t}\right\|^2_{L^2(0,1)} + \frac{2C_1}{R^2}\left\|\frac{\partial w_z}{\partial r}\right\|^2_{L^2(\Omega,r)}\right]\\
&+2\int_0^T\left\|\frac{\partial w_z}{\partial t}\right\|^2_{L^2(\Omega,r)} ds \le C\Big(\|f\|^2_{L^2(0,T:L^2(\Omega,r))} \qquad (4.39)\\
&+\|\eta_1-\eta_0\|^2_{H^2(0,T)} + \|\eta_0\|^2_{H^2(0,T)} + \left\|\bar\eta^0\right\|^2_{L^2(0,1)} + \left\|w_z^0\right\|^2_{H^1_{0,0}(\Omega,r)}\Big).
\end{aligned}
$$

Proof. Again, we need to deal with the lack of regularity in the z direction by canceling the terms which we cannot control at this point. As before, we need to manipulate the conservation of mass equation and the conservation of momentum equation in such as way that, when they are added up, the unwanted terms cancel out and produce an equation whose terms on the right hand-side can be estimated using the Cauchy's and Young's inequalities. The energy estimate will then follow by an application of the Gronwall's inequality.

As in the previous proof, we begin by multiplying equation (4.26) by $\dot l_k^n(t)$, and sum over $k=1,\cdots,n$. Then we differentiate (4.25) with respect to t, multiply by $\dot d_h^m(t)$ and by $\frac{C_3}{R}$, and sum over $h=1,\cdots,m$. Finally, we differentiate (4.25) with respect to t, multiply by $d_h^m(t)$ and by $\frac{C_2}{R}$, and sum over $h=1,\cdots,m$. The resulting equations contain the unwanted terms which, when added up, cancel and produce the following equality:

$$\frac{1}{2}\frac{d}{dt}\left[\frac{C_3}{R}\left\|\frac{\partial\bar{\eta}_m}{\partial t}\right\|^2_{L^2(0,1)}+\frac{C_1}{R^2}\left\|\frac{\partial w_{z_n}}{\partial r}\right\|^2_{L^2(\Omega,r)}\right]+\left\|\frac{\partial w_{z_n}}{\partial t}\right\|^2_{L^2(\Omega,r)}$$
$$=\int_\Omega f\frac{\partial w_{z_n}}{\partial t}\,rdrdz-\int_\Omega g_2\frac{\partial w_{z_n}}{\partial t}\,rdrdz-\frac{C_2}{R}\int_0^1\frac{\partial g_1}{\partial t}\bar{\eta}_m\,dz \tag{4.40}$$
$$-\frac{C_3}{R}\int_0^1\frac{\partial g_1}{\partial t}\frac{\partial\bar{\eta}_m}{\partial t}\,dz+\frac{C_2}{R}\int_0^1\frac{\partial^2\bar{\eta}_m}{\partial t^2}\bar{\eta}_m\,dz.$$

Before we estimate the right hand-side of this equation, we will integrate the entire equation with respect to t in order to deal with the term on the right hand-side which contains the second derivative with respect to t of $\bar{\eta}_m$. We obtain

$$\frac{1}{2}\left[\frac{C_3}{R}\left\|\frac{\partial\bar{\eta}_m(t)}{\partial t}\right\|^2_{L^2(0,1)}+\frac{C_1}{R^2}\left\|\frac{\partial w_{z_n}(t)}{\partial r}\right\|^2_{L^2(\Omega,r)}\right]+\int_0^t\left\|\frac{\partial w_{z_n}}{\partial s}\right\|^2_{L^2(\Omega,r)}ds$$
$$\begin{aligned}&=\int_0^t\int_\Omega(f-g_2)\frac{\partial w_{z_n}}{\partial s}\,rdrdzds-\frac{C_2}{R}\int_0^t\int_0^1\frac{\partial g_1}{\partial s}\bar{\eta}_m\,dzds\\&\quad-\frac{C_3}{R}\int_0^t\int_0^1\frac{\partial g_1}{\partial s}\frac{\partial\bar{\eta}_m}{\partial s}\,dzds+\frac{C_2}{R}\int_0^t\int_0^1\frac{\partial^2\bar{\eta}_m}{\partial s^2}\bar{\eta}_m\,dzds\\&\quad+\frac{1}{2}\left[\frac{C_3}{R}\left\|\frac{\partial\bar{\eta}_m(0)}{\partial t}\right\|^2_{L^2(0,1)}+\frac{C_1}{R^2}\left\|\frac{\partial w_{z_n}(0)}{\partial r}\right\|^2_{L^2(\Omega,r)}\right].\end{aligned} \tag{4.41}$$

The first three terms on the right hand-side can be estimated by using the Cauchy inequality. To estimate the fourth term, we use integration by parts with respect to s to obtain

$$-\frac{C_2}{R}\int_0^t\int_0^1\frac{\partial^2\bar{\eta}_m}{\partial s^2}\bar{\eta}_m\,dzds=\frac{C_2}{R}\int_0^t\int_0^1\left|\frac{\partial\bar{\eta}_m}{\partial s}\right|^2\,dzds$$
$$-\frac{C_2}{R}\int_0^1\frac{\partial\bar{\eta}_m(z,t)}{\partial t}\bar{\eta}_m(z,t)\,dz+\frac{C_2}{R}\int_0^1\frac{\partial\bar{\eta}_m(z,0)}{\partial t}\bar{\eta}_m(z,0)\,dz$$

where $\displaystyle\int_0^1\frac{\partial\bar{\eta}_m(z,0)}{\partial t}\bar{\eta}_m(z,0)dz=-\int_0^1\left(g_1+R^2\frac{\partial<w_{z_n}>}{\partial z}\right)_{t=0}\bar{\eta}_m(z,0)dz.$ This implies

$$\left|\frac{C_2}{R}\int_0^t\int_0^1\frac{\partial^2\bar{\eta}_m}{\partial s^2}\bar{\eta}_m\,dzds\right|\le\tilde{K}\left(\left\|\frac{\partial\bar{\eta}_m}{\partial s}\right\|^2_{L^2(0,T:L^2(0,1))}+\|\bar{\eta}_m(t)\|^2_{L^2(0,1)}\right.$$
$$\left.+\left\|\bar{\eta}^0\right\|^2_{H^1(0,1)}+\left\|w_z^0\right\|^2_{H^1_{0,0}(\Omega,r)}+\|g_1(0)\|^2_{L^2(0,1)}\right)+\frac{C_3}{4R}\left\|\frac{\partial\bar{\eta}_m(t)}{\partial t}\right\|^2_{L^2(0,1)}$$

where $\tilde{K}>0$ depends on $C_2, C_3, 1/C_3, 1/R, R$.

The last two terms in (4.41) can be estimated by first using the conservation of mass equation (4.25) to obtain

$$\int_0^1 \left|\frac{\partial \bar\eta_m(0)}{\partial t}\right|^2 dz = -\int_0^1 \frac{\partial < w_{z_n}(0) >}{\partial z}\frac{\partial \bar\eta_m(0)}{\partial t} dz - \int_0^1 g_1(0)\frac{\partial \bar\eta_m(0)}{\partial t} dz.$$

and then the Cauchy's inequality so that:

$$\int_0^1 \left|\frac{\partial \bar\eta_m(0)}{\partial t}\right|^2 dz \le \left\|\frac{\partial}{\partial z} < w_z^0 >\right\|^2_{L^2(0,1)} + \|g_1(0)\|^2_{L^2(0,1)}$$

to obtain

$$\frac{1}{2}\left[\frac{C_3}{R}\left\|\frac{\partial \bar\eta_m(0)}{\partial t}\right\|^2_{L^2(0,1)} + \frac{C_1}{R^2}\left\|\frac{\partial w_{z_n}(0)}{\partial r}\right\|^2_{L^2(\Omega,r)}\right] \\ \le \tilde C\left(\|w_z^0\|^2_{H^1_{0,0}(\Omega,r)} + (\eta_1'(0) - \eta_0'(0))^2 + (\eta_0'(0))^2\right)$$

where $\tilde C > 0$ depends on $C_1, C_3, 1/R$.

By combining these estimates and by using the energy estimate stated in Theorem 4.6 we see that there exists a constant $C > 0$ depending on $C_1, C_2, C_3, 1/R, R$ such that

$$\frac{C_3}{4R}\left\|\frac{\partial \bar\eta_m(t)}{\partial t}\right\|^2_{L^2(0,1)} + \frac{C_1}{2R^2}\left\|\frac{\partial w_{z_n}(t)}{\partial r}\right\|^2_{L^2(\Omega,r)} + \frac{1}{2}\int_0^t \left\|\frac{\partial w_{z_n}}{\partial s}\right\|^2_{L^2(\Omega,r)} ds \\ \le C(\|f\|^2_{L^2(0,T:L^2(\Omega,r))} + \|\eta_1 - \eta_0\|^2_{H^2(0,T)} + \|\eta_0\|^2_{H^2(0,T)} + \left\|\bar\eta^0\right\|^2_{H^1(0,1)} \\ + \left\|w_z^0\right\|^2_{L^2_{0,0}(\Omega,r)} + |\eta_1'(0) - \eta_0'(0)|^2 + (\eta_0'(0))^2)$$

for a.e $0 \le t \le T$. Passing to the limit as $m \to \infty$ and $n \to \infty$ we recover the estimate (4.39). This completes the proof of Theorem 4.7. $\square$

Next we show that the weak solution $(\bar\eta, w_z)$ is, in fact, a *mild solution*, namely, that $(\bar\eta, w_z) \in X_\gamma \times X_v$ under some additional assumptions on the smoothness of the initial data. It is in this step that we can finally take control over certain derivatives with respect to z of our solution.

THEOREM 4.8 (Improved regularity: Part II). *Assume, in addition to the assumptions of Theorem 4.7, that the initial data $\bar\eta^0 \in H^1(0,1)$. Then the weak solution $\bar\eta$ satisfies $\bar\eta \in H^1(0,T;H^1(0,1))$. Furthermore, the following estimate holds:*

$$\sup_{0\le t\le T}\frac{C_2C_3}{12}\left\|\frac{\partial \bar\eta}{\partial z}\right\|^2_{L^2(0,1)} + \frac{C_3^2}{12}\left\|\frac{\partial^2 \bar\eta}{\partial t \partial z}\right\|^2_{L^2(0,T:L^2(0,1)} \\ \le C\Big(\|f\|^2_{L^2(0,T:L^2(\Omega,r))} + \|\eta_1 - \eta_0\|^2_{H^2(0,T)} \\ + \|\eta_0\|^2_{H^2(0,T)} + \left\|\bar\eta^0\right\|^2_{H^1(0,1)} + \left\|w_z^0\right\|^2_{H^1_{0,0}(\Omega)}\Big), \tag{4.42}$$

where C depends on $1/R, C_2, C_3, T$. This implies that, in fact,

$$\frac{\partial^2 < w_z >}{\partial z^2} \in L^2(0, T : L^2(0,1)), \quad \Delta_r w_z \in L^2(0, T : L^2(\Omega, r)). \quad (4.43)$$

Proof. The proof is based on the following idea. We will use the weak form of the momentum equation (4.26) to estimate $\partial\bar{\eta}/\partial z$ and $\partial^2\bar{\eta}/\partial z\partial t$. In order to obtain the desired estimate, we will substitute the test function w_k in the weak form of the momentum equation (4.26) by $(1-r)\frac{\partial\phi_k(z)}{\partial z} \in C^1_{0,0}(\Omega)$. We will then use the fact that

$$(1-r)\frac{\partial^2\bar{\eta}_m}{\partial z\partial t} = (1-r)\sum_{k=1}^{m} \dot{d}_k^m(t)\frac{\partial\phi_k(z)}{\partial z} = \sum_{k=1}^{m} \dot{d}_k^m(t)\underbrace{(1-r)\frac{\partial\phi_k(z)}{\partial z}}_{w_k(r,z)},$$

where

$$w_k(r,z) = (1-r)\frac{\partial\phi_k(z)}{\partial z} \in C^1_{0,0}(\Omega, r). \quad (4.44)$$

Notice that, without loss of generality, we could have used the space $C^1_{0,0}$ in the definition of the Galerkin approximation for the velocity, instead of the space $C^\infty_{0,0}$. Thus, everything obtained will hold assuming $w_k \in C^1_{0,0}$. This relaxed choice of the space for w_k is now important to obtain improved regularity.

We now proceed by substituting w_k in (4.26) with (4.44) and by multiplying equation (4.26) by $\dot{d}_k^m(t)$ and summing over $k = 1, ...m$ to obtain

$$\begin{aligned} &\int_\Omega \frac{\partial w_{z_n}}{\partial t}\frac{\partial^2\bar{\eta}_m}{\partial z\partial t}\,(1-r)rdrdz - \frac{C_1}{R^2}\int_\Omega \frac{\partial w_{z_n}}{\partial r}\frac{\partial^2\bar{\eta}_m}{\partial z\partial t}\,rdrdz \\ &+ C_2\int_\Omega \frac{\partial\bar{\eta}_m}{\partial z}\frac{\partial^2\bar{\eta}_m}{\partial z\partial t}\,(1-r)rdrdz + C_3\int_\Omega \left|\frac{\partial\bar{\eta}_m}{\partial z\partial t}\right|^2 (1-r)rdrdz \\ &\quad = \int_\Omega f\frac{\partial^2\bar{\eta}_m}{\partial z\partial t}\,(1-r)rdrdz - \int_\Omega g_2\frac{\partial^2\bar{\eta}_m}{\partial z\partial t}\,(1-r)rdrdz. \end{aligned} \quad (4.45)$$

Multiplying (4.45) by C_3 and integrating with respect to r where possible, we get

$$\begin{aligned} &\frac{C_2C_3}{12}\frac{d}{dt}\left\|\frac{\partial\bar{\eta}_m}{\partial z}\right\|^2_{L^2(0,1)} + \frac{C_3^2}{6}\left\|\frac{\partial^2\bar{\eta}_m}{\partial t\partial z}\right\|^2_{L^2(0,1)} \\ &= -C_3\int_\Omega \frac{\partial w_{z_n}}{\partial t}\frac{\partial^2\bar{\eta}_m}{\partial t\partial z}(1-r)rdrdz + \frac{C_1C_3}{R^2}\int_\Omega \frac{\partial w_{z_n}}{\partial r}\frac{\partial^2\bar{\eta}_m}{\partial t\partial z}rdrdz \\ &\quad + C_3\int_\Omega (f-g_2)\frac{\partial^2\bar{\eta}_m}{\partial t\partial z}(1-r)rdrdz. \end{aligned}$$

By applying the Cauchy inequality to the right hand-side, and then using the differential form of Gronwall's inequality, and by employing the improved regularity estimate (4.39) we obtain

$$\begin{aligned}
&\frac{C_2C_3}{12}\sup_{0\le t\le T}\left\|\frac{\partial\bar\eta_m}{\partial z}\right\|^2_{L^2(0,1)}+\frac{C_3^2}{12}\int_0^T\left\|\frac{\partial^2\bar\eta_m}{\partial t\partial z}\right\|^2_{L^2(0,1)}dt\\
&\le C(\|f\|^2_{L^2(0,T:L^2(\Omega,r))}+\|\eta_1-\eta_0\|^2_{H^2(0,T)}+\|\eta_0\|^2_{H^2(0,T)}+\left\|\bar\eta^0\right\|^2_{H^1(0,1)}\\
&\quad+\left\|w_z^0\right\|^2_{H^1_{0,0}(\Omega)}+|\eta_1'(0)-\eta_0'(0)|^2+(\eta_0'(0))^2).
\end{aligned}$$

Passing to limit we recover the desired estimate (4.42). Moreover since $\bar\eta\in H^1(0,T;H^1(0,1))$, from equations (4.16) and (4.17), we conclude (4.43). For details please see [19]. This concludes the proof of Theorem 4.8. ◻
With this proof we have completed the second step in showing that problem (4.13)–(4.15) has a unique mild solution. This result implies, in particular, that the Frechét derivative is a bijection from X_v to Z and, thus, completes the proof of Theorem 4.3.

Remark. An alternate proof for the problem with zero initial and boundary data can be obtained by using a (distributional) Laplace transform approach, the (complex) Lax-Milgram Lemma, the Paley-Wiener Theorem, and abstract elliptic regularity theory [15].)

The Implicit Function Theorem 4.1 now implies existence of a unique, mild solution to the nonlinear, moving boundary problem (1.1)–(1.6).

In order to state this result in terms of the pressure inlet and outlet boundary data as formulated in (1.6) we remark that the condition on the boundary data $\eta_0,\eta_L\in H^2(0,T)$ translates into the following condition in terms of the pressure data $P_0,P_L\in H^1(0,T)$. This is due to the pressure-displacement relationship (1.3). Thus, the parameter space Λ in terms of the pressure boundary data becomes $\tilde\Lambda:=H^1(0,L)\times H^1_{0,0}(\Omega,r)\times(H^1(0,T))^2$. We can now state our main result in terms of the pressure data:

Theorem 4.9 (**Main Result**). *Assume that the initial data η^0 for the displacement η from the reference cylinder of radius R, is in $H^1(0,L)$, and that the initial data v_z^0 for the axial component of the velocity is in $H^1_{0,0}(\Omega,r)$. Furthermore, suppose that the inlet and outlet pressure data $P_0(t)$ and $P_L(t)$ which correspond to the fluctuations around the reference pressure $p_{\rm ref}$, are such that $P_0,P_L\in H^1(0,T)$. Then, there exists a neighborhood $S\subset X_\gamma\times X_v$ around the solution $\eta=0,v_z=0$, and a neighborhood $D\subset\tilde\Lambda$ around the initial and boundary data $\eta^0=0,v_z^0=0,P_0=p_{\rm ref},P_L=p_{\rm ref}$ such that there exists exactly one mild solution $(\eta,v_z)\in S\subset X_\gamma\times X_v$ of (1.1)–(1.6) for each choice of the initial and boundary data $(\eta^0,v_z^0,P_0,P_L)\in D\subset\tilde\Lambda$.*

5. Conclusions. In this manuscript we proved the existence of a unique mild solution to a nonlinear moving-boundary problem of mixed hyperbolic-parabolic type arising in modeling blood flow through viscoelastic arteries. The result holds for small perturbations of the data around the reference cylinder of radius R and axial velocity equal to zero. Future research in this direction includes an extension of this result to the solutions obtained as small perturbations of flow in a cylinder of radius R with the axial velocity corresponding to the Womersley profile, assumed for time-periodic pressure gradients. This scenario corresponds more closely to the physiologically relevant blood flow conditions.

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