UH - Math 3330 - Dr. Heier - Sample Final Exam - Fall 2009

Print your NAME:

Solve all of the ten problems. Please show all work to support your solutions. Our policy is that if you show no supporting work, you will receive no credit. This is a closed book test. Please close your textbook, notebook, cell phone. No calculators are allowed in this test. Please do not start working before you are told to do so. The regular time allowed is 175 minutes.

Problem 1 ____________________/10 points

Problem 2 ____________________/10 points

Problem 3 ____________________/10 points

Problem 4 ____________________/10 points

Problem 5 ____________________/10 points

Problem 6 ____________________/10 points

Problem 7 ____________________/10 points

Problem 8 ____________________/10 points

Problem 9 ____________________/10 points

Problem 10 ____________________/10 points

Total ____________________/100 points

1. (a) (5 points) Prove $(A \cup B) \backslash C = (A \backslash C) \cup (B \backslash C)$.

(b) (5 points) Prove or disprove that $A \cup B = A \cup C$ implies $B = C$.

2. (a) (5 points) Let $x, y \in \mathbb{Z}$. Let $x \sim y$ if and only if $7 | 3x - 10y$. Prove that $\sim$ is an equivalence relation on $\mathbb{Z}$.

(b) (5 points) Let $A = \mathbb{Q} \backslash \{0\}$. Let $(a, b), (c, d) \in A \times A$. Let $(a, b) \sim (c, d)$ if and only if $ad = bc$. Prove that $\sim$ is an equivalence relation on $A \times A$.

3. (10 points) Prove by induction that for $n \in \mathbb{N}$ the following holds.

$$1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \ldots + n \cdot 2^n = (n-1)2^{n+1} + 2.$$

4. (a) (5 points) Find the value of $d = \gcd(200, 56)$.

(b) (5 points) Find m, n such that $d = m \cdot 200 + n \cdot 56$.

5. (a) (5 points) Find a solution $x \in \mathbb{N}$ with $0 \leq x < 42$ such that

$$55x \equiv 50 \mod 42.$$

(b) (5 points) Find the solution of the following system of equations in $\mathbb{Z}_{13}$:

$$\begin{aligned} [2][x] + [3][y] &= [6] \\ [2][x] + [2][y] &= [1]. \end{aligned}$$

6. (a) (5 points) Let a be an element of a group G. Prove that a and a^{-1} have the same order.

(b) (5 points) Let $f : G \to H$ be an epimorphism of groups. Let G be abelian. Prove that H is abelian.

7. (a) (5 points) Let f be the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 5 & 1 & 2 \end{pmatrix}$. Write f as a product of disjoint cycles.

(b) (4 points) Write f as a product of transpositions.

(c) (1 point) Is f even or odd?

8. (10 points) Prove that if H and K are normal subgroups of a group G such that $H \cap K = \{e\}$, then $hk = kh$ for all $h \in H,\ k \in K$.

9. (10 points) Prove that for a fixed element a in a ring R, the set $\{x \in R \,|\, ax = 0\}$ is a subring of R.

10. (10 points) Let R be a ring in which all elements x satisfy $x^2 = x$. Prove that are every element of R is equal to its own additive inverse. (Hint: Consider $(x + x)^2$.)