UH - Math 4377/6308 - Dr. Heier - Fall 2010
Sample Midterm Exam II
Time: 53 min

1. (a) (15 points) Let $W_1 = \{(a_1, a_2, a_1 - a_2) | a_1, a_2 \in \mathbb{R}\} \subset \mathbb{R}^3$. Let $W_2 = \{(b, -b, 0) | b \in \mathbb{R}\} \subset \mathbb{R}^3$. Is $W_1 \oplus W_2 = \mathbb{R}^3$? Justify your answer.

(b) (10 points) Determine if the following subset S of the vector space of 2×2 matrices with real entries is a subspace. Justify your answer carefully.

$$S = \left\{ \begin{pmatrix} a & b \\ c & a \cdot b \cdot c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

2. (a) (10 points) Find nullity and rank of

$$T : \mathbb{R}^5 \to \mathbb{R}^3, (a_1, a_2, a_3, a_4, a_5) \mapsto (a_1 + a_5, -a_1 + a_2 + a_3, 3a_1 - a_2 - a_3 + 2a_5).$$

(b) (15 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(a_1, a_2) = (a_1 + a_2, a_1 - 2a_2)$. Let $\beta = \{(1, 0), (0, 1)\}$ and $\gamma = \{(1, 2), (1, 1)\}$. Compute $[T \circ T]_\beta^\gamma$.

3. (a) (15 points) Let $T : V \to W$ be an invertible linear transformation. Prove that the inverse $T^{-1} : W \to V$ is linear.

(b) (10 points) Let $T : V \to V$ be a linear transformation. Let $T \circ T : V \to V$ be the zero transformation, i.e., for all $v \in V$, $T(T(v)) = \vec{0}$. Prove that T is not invertible.

4. Let f_1, f_2, f_3 be linear functionals on $\mathbb{R}^3$ defined by

$$f_1(x, y, z) = x - y \quad f_2(x, y, z) = x + y + z \quad f_3(x, y, z) = y - z.$$

(a) (10 points) Show that $\{f_1, f_2, f_3\}$ is a basis for $\mathbb{R}^{3*}$.

(b) (15 points) Find a basis for $\mathbb{R}^3$ that is dual to $\{f_1, f_2, f_3\}$.