UH - Math 4377/6308 - Dr. Heier - Fall 2012

Midterm Exam

Monday, October 22, 2012

Print your NAME: Solution

Solve all of the problems. Please show all work to support your solutions. Our policy is that if you show no supporting work, you will receive no credit. This is a closed book test. Please close your textbook, notebook, cell phone. No calculators are allowed in this test. Please do not start working before you are told to do so. The time allowed will be announced by the proctor.

Problem 1 ______________________________ /20 points

Problem 2 ______________________________ /10 points

Problem 3 ______________________________ /20 points

Problem 4 ______________________________ /20 points

Problem 5 ______________________________ /20 points

Problem 6 ______________________________ /10 points

Total ______________________________ /100 points

1a. (10 points) Let $\mathbb{N}^{\neq 0} = \{1, 2, 3, \ldots\}$ be the set of positive integers. Let $A = \mathbb{N}^{\neq 0} \times \mathbb{N}^{\neq 0}$. For $(a_1, a_2), (b_1, b_2) \in A$, let $(a_1, a_2) \sim (b_1, b_2)$ if and only if $a_1 \cdot b_2 = a_2 \cdot b_1$. Prove that $\sim$ is an equivalence relation.

1b. (5 points) Keep all definitions from part (a), except that now $(a_1, a_2) \sim (b_1, b_2)$ if and only if $a_1 \cdot b_2^2 = a_2 \cdot b_1^2$. Is this new relation also an equivalence relation? Prove your answer.

1c. (5 points) Let X, Y, Z be arbitrary sets. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Is the following statement true in general? If g is not one-to-one, then $g \circ f : X \to Z$ is not one-to-one. Prove your answer.

1a. Reflexive: $(a_1,a_2) \sim (a_1,a_2) \Leftrightarrow a_1 a_2 = a_2 a_1$ ✓

Symmetric: $(a_1,a_2) \sim (b_1,b_2) \Leftrightarrow a_1 b_2 = a_2 b_1$

$\Leftrightarrow b_1 a_2 = b_2 a_1$

$\Leftrightarrow (b_1,b_2) \sim (a_1,a_2)$

Transitive: $(a_1,a_2) \sim (b_1,b_2)$ and $(b_1,b_2) \sim (c_1,c_2)$

$\Rightarrow a_1 b_2 = a_2 b_1$ and $b_1 c_2 = b_2 c_1$

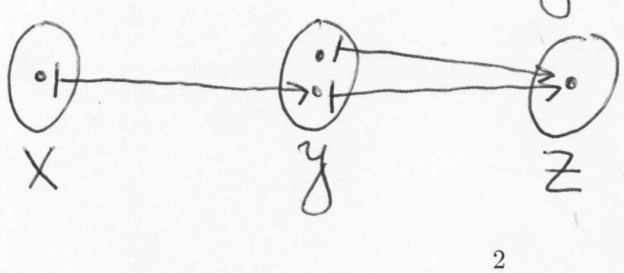
$\underset{c_2 \neq 0}{\Rightarrow} a_1 b_2 = a_2 \dfrac{b_2 c_1}{c_2} \underset{b_2 \neq 0}{\Rightarrow} a_1 c_2 = a_2 c_1$

$\Rightarrow (a_1,a_2) \sim (c_1,c_2)$

1b. No! $(1,2) \sim (1,2) \Leftrightarrow 1 \cdot 4 = 2 \cdot 1$ false

Thus, $\sim$ is not reflexive.

1c. The statement is not true. For a counterexample, take maps as indicated by the diagram:



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2a. (5 points) Determine if the following subset of $\mathbb{R}^2$ is a subspace:

$$S = \{(a_1, a_2) \in \mathbb{R}^2 : a_1 \cdot a_2 = 0\}.$$

2b. (5 points) Let V be a vector space. Let W_1, W_2 be subspaces of V. Complete the following sentence (do not prove your answer–just complete the sentence). The union $W_1 \cup W_2$ is a subspace of V if and only if ....

2a. Clearly: $(1,0), (0,1) \in S$.

But $(1,0) + (0,1) = (1,1) \notin S$

$\Rightarrow$ S not closed under "+"

$\Rightarrow$ S is not a subspace.

2b. $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

3a. (10 points) Find the condition(s) on a, b, c, d so that

$$(a, b, c, d) \in \text{span}\{(2,2,0,2), (2,1,1,2), (1,0,1,1), (-1,1,-2,-1)\}.$$

3b. (10 points) Find bases for the null space and range of

$T : \mathbb{R}^4 \to \mathbb{R}^3, (a_1, a_2, a_3, a_4) \mapsto (5a_1 - 3a_2 - 2a_3 - 5a_4, 2a_1 - a_2 - a_3 - 2a_4, -a_1 + a_3 + a_4).$

3a.

$$a_1 \begin{pmatrix} 2 \\ 2 \\ 0 \\ 2 \end{pmatrix} + a_2 \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix} + a_3 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + a_4 \begin{pmatrix} -1 \\ 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$\Leftrightarrow \begin{aligned} 2a_1 + 2a_2 + a_3 - a_4 &= a \\ 2a_1 + a_2 \qquad + a_4 &= b \\ a_2 + a_3 - 2a_4 &= c \\ 2a_1 + 2a_2 + a_3 - a_4 &= d \end{aligned}$$

$$\Leftrightarrow \begin{aligned} 2a_1 + 2a_2 + a_3 - a_4 &= a \\ -a_2 - a_3 + 2a_4 &= b - a \\ a_2 + a_3 - 2a_4 &= c \\ 0 &= d - a \end{aligned}$$

$$\Leftrightarrow \left.\begin{aligned} 2a_1 + 2a_2 + a_3 - a_4 &= a \\ -a_2 - a_3 + 2a_4 &= b - a \end{aligned}\right\} \text{echelon form}$$

$$\begin{aligned} 0 &= c + b - a \\ 0 &= d - a \end{aligned}$$

$\Rightarrow$ The conditions are $a = b + c$ and $a = d$.

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3b. $T(a_1, \ldots, a_4) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Leftrightarrow \quad \begin{aligned} 5a_1 - 3a_2 - 2a_3 - 5a_4 &= 0 \\ 2a_1 - a_2 - a_3 - 2a_4 &= 0 \\ -a_1 \qquad + a_3 + a_4 &= 0 \end{aligned}$$

$$\Leftrightarrow \quad \begin{aligned} -a_1 \qquad + a_3 + a_4 &= 0 \\ -a_2 + a_3 \qquad &= 0 \\ -3a_2 + 3a_3 \qquad &= 0 \end{aligned}$$

$$\Leftrightarrow \quad \begin{aligned} a_1 &= a_3 + a_4 \\ a_2 &= a_3 \end{aligned}$$

$$\Rightarrow N(T) = \left\{ \begin{pmatrix} a_3 + a_4 \\ a_3 \\ a_3 \\ a_4 \end{pmatrix} \,\middle|\, a_3, a_4 \in \mathbb{R} \right\}$$

$$= \left\{ a_3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + a_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \,\middle|\, a_3, a_4 \in \mathbb{R} \right\}$$

$\Rightarrow \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is basis for $N(T)$.

$T(1,0,0,0) = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$, $T(0,1,0,0) = \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix}$

These two vectors are clearly lin. indep. They form a basis for $R(T)$ due to $\underbrace{\text{nullity}(T)}_{=2} + \underbrace{\text{rank}(T)}_{\Rightarrow\ =2} = 4$

4a. (10 points) Let $T : \mathrm{Mat}_{2\times 2}(\mathbb{R}) \to \mathrm{Mat}_{2\times 2}(\mathbb{R}), T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a+d & 2c \\ d & a+c \end{pmatrix}$. Let $\beta = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$. Compute $[T]_\beta^\beta$.

4b. (10 points) Let $\{v_1, v_2, v_3, v_4\}$ be a basis for $\mathbb{R}^4$. Is $\{v_1 + v_2 + 2v_3, v_2 + v_3, v_1 + v_3, v_4\}$ also a basis for $\mathbb{R}^4$? Prove your answer.

4a. $T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} = 2 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$$\Rightarrow [T]_\beta^\beta = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

4b. Note that

$$v_1 + v_2 + 2v_3 = (v_2 + v_3) + v_1 + v_3$$

$\Rightarrow$ Not lin. indep. $\Rightarrow$ Not a basis.

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5a. (10 points) Let $n \geq 2$ and $0 < k < n$ be integers. Let V be a vector space and W_1 a subspace of V. Assume that $\dim V = n$ and $\dim W_1 = k$. Prove that there exists a subspace W_2 of V such that $W_1 \oplus W_2 = V$. What is the dimension of W_2? In this problem, you may cite any theorem from class.

5b. (10 points) Let V be a finite dimensional vector space. Let $T : V \to V$ be a linear transformation. Assume that $\operatorname{rank}(T) = \operatorname{rank}(T \circ T)$. Prove that $R(T) \cap N(T) = \{\vec{0}\}$.

(Hint for 5b: $R(T)$ denotes the range of T, $N(T)$ the null space of T. Use the dimension formula for the restriction of T to $R(T)$.)

5a. Let $\{v_1, \dots, v_k\}$ be a basis for W_1.

Replacement Thm $\Rightarrow \exists\, v_{k+1}, \dots, v_n \in V$:

$\{v_1, \dots, v_n\}$ is basis for V.

let $W_2 = \operatorname{span}\{v_{k+1}, \dots, v_n\}$.

- $W_1 \cap W_2 = \{\vec{0}\}$ is clear because $\{v_1, \dots, v_n\}$ is lin. indep.
- $W_1 + W_2 = V$ is clear because $\{v_1, \dots, v_n\}$ is a basis.

5b. let $\vec{0} \neq x \in R(T) \cap N(T)$.

let $S = T|_{R(T)} : R(T) \to V$. Dimension formula:

$$\underbrace{\text{nullity}(S)}_{\geq 1} + \text{rank}(S) = \dim R(T) = \underline{\text{rank}(T)}$$

b/c $x \in N(S)$

$\text{rank}(S) = \text{rank}(T \circ T) = \text{rank}(T)$ ← assumption

Contradiction!

6. (10 points) For the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and the ordered basis $\beta = \{(1,2),(4,9)\}$, find an invertible matrix Q such that $[L_A]_\beta = Q^{-1}AQ$. Then use the formula to find $[L_A]_\beta$ explicitly.

In class, we saw $Q = [Id]_\beta^{(\text{std. basis})}$.

$$\Rightarrow Q = \begin{pmatrix} 1 & 4 \\ 2 & 9 \end{pmatrix}.$$

$$\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2 & 9 & 0 & 1 \\ \hline 1 & 4 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ \hline 1 & 0 & 9 & -4 \\ 0 & 1 & -2 & 1 \end{array} \quad \Rightarrow Q^{-1} = \begin{pmatrix} 9 & -4 \\ -2 & 1 \end{pmatrix}$$

$$\Rightarrow [L_A]_\beta = \begin{pmatrix} 9 & -4 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & -4 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 13 \\ 3 & 13 \end{pmatrix}$$

$$= \begin{pmatrix} 15 & 65 \\ -3 & -13 \end{pmatrix}$$