Selected Solutions to HW 5

§1.6 #14 : Find bases for the following subspaces of F^5

$$W_1 = \{(a_1, a_2, a_3, a_4, a_5) \in F^5 : a_1 - a_3 - a_4 = 0\}$$

First, recognize which values are free to take any number in F without affecting $a_1 - a_3 - a_4 = 0$. We see that a_2 and a_5 are not present, so

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$ are vectors in the basis for F^5.

For the remaining vectors we must construct the basis vector in a way that satisfies the given equation.

- allowing a_3 to be zero $\Rightarrow a_1 = a_4 \Rightarrow$ $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ is a basis vector
- allowing a_1 to be zero $\Rightarrow a_3 = -a_4 \Rightarrow$ $\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$ is a basis vector.

We stop here b/c if we carry this scheme forward and allow $a_4 = 0$ then a basis vector could be $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ but this is a linear combination of $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$, so it must be scraped. $\therefore$ Basis$(W_1) = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}$

and $\dim(W_1) = 4$.

§ 1.6 #14 cont.

Using the same thought process as before, we have that a basis for W_2 is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ with $\dim W_2 = 2$

#5 : Let $G = \{(1,-1,0,1), (1,0,1,0), (1,2,2,2), (0,2,2,2)\}$

Let $L = \{(-1,4,2,0)\}$

(a). Show that G spans $\mathbb{R}^4$. Want to show

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 2 & 2 \\ 0 & 1 & 2 & 2 \\ 1 & 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

reduces to a matrix w/ full rank.

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & a_1 \\ 0 & 1 & 2 & 2 & a_3 \\ 0 & 1 & 3 & 2 & a_2+a_1 \\ 0 & -1 & 1 & 2 & a_4-a_1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & a_1 \\ 0 & 1 & 2 & 2 & a_3 \\ 0 & 0 & 1 & 0 & a_2+a_1-a_3 \\ 0 & 0 & 3 & 4 & a_4-a_1+a_3 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & a_1 \\ 0 & 1 & 2 & 2 & a_3 \\ 0 & 0 & 1 & 0 & a_2+a_1-a_3 \\ 0 & 0 & 0 & 4 & a_4-a_1+a_3-3(a_2+a_1-3) \end{array}\right)$$

We observe that this matrix is full rank $\iff$ each column has a pivot. The right hand side are numbers in the underlying field undergoing operations that put them back into the field (addition and scalar multiplication).

(b) Choose $H = \{(1,-1,0,1), (1,0,1,0), (0,2,2,2)\}$; $L = \{(-1,4,2,0)\}$.

reduce $H \cup L$ to see if all vectors are linearly independent.

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & a_1 \\ -1 & 0 & 2 & 4 & a_2 \\ 0 & 1 & 2 & 2 & a_3 \\ 1 & 0 & 2 & 0 & a_4 \end{array}\right) = \dots = \left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & a_1 \\ 0 & 1 & 2 & 2 & a_3 \\ 0 & 0 & 1 & 0 & a_2+a_1-a_3 \\ 0 & 0 & 0 & 4 & a_4-a_1+a_3-2(a_2+a_1-a_3) \end{array}\right)$$

By the Replacement Theorem $H \cup L$ generates V.

$\operatorname{Span} G = \operatorname{Span}(H \cup L)$

Again, we've reduced to a full rank matrix. $\therefore$ Independence of vectors.

§1.6 #29 Prove: If W_1 and W_2 are finite dimensional subspaces of V where V is a Vector space then W_1+W_2 is finite dimensional and that $\dim(W_1+W_2)=\dim(W_1)+\dim(W_2)-\dim(W_1\cap W_2)$.

(a)

let $Basis(W_1\cap W_2)=\{u_1,u_2,\ldots,u_k\}$ then $\dim(W_1\cap W_2)=k$

let $Basis(W_1)=\{u_1,u_2,\ldots,u_k,v_1,v_2,\ldots,v_m\}$ then $\dim(W_1)=k+m$

let $Basis(W_2)=\{u_1,u_2,\ldots,u_k,w_1,w_2,\ldots,w_p\}$ then $\dim(W_2)=k+p$

then $Basis(W_1+W_2)=\{u_1,\ldots,u_k,v_1,\ldots,v_m,w_1,\ldots,w_p\}$ then $\dim(W_1+W_2)=(k+m)+(k+p)-k$
$=k+m+p$.

Note: $W_1\subseteq W_1\cap W_2$.
$W_2\subseteq W_1\cap W_2$.
$\{v_1,\ldots,v_m\}\in W_1\setminus W_2$.
$\{w_1,\ldots,w_p\}\in W_2\setminus W_1$.

(b) W_1,W_2 finite dimensional subspaces and $V=W_1+W_2$.
Prove $V=W_1\oplus W_2$ iff $\dim(V)=\dim(W_1)+\dim(W_2)$.

($\Rightarrow$) assume $V=W_1\oplus W_2$.
$W_1\cap W_2=\{\}$ by assumption.
Since $V=W_1+W_2$ is given and by above we have $\dim(V)=\dim(W_1)+\dim(W_2)$ ∎

($\Leftarrow$) assume $\dim(V)=\dim(W_1)+\dim(W_2)$.
$\Rightarrow$ by above, part (a), we have $\dim(W_1\cap W_2)=\{\}$ and $V=W_1+W_2$ as given.
These together define the direct sum, so $\therefore V=W_1\oplus W_2$. ∎