Selected Solutions to HW8

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#2 § 2.4 Problem 4: Let $A, B \in M_{n \times n}$ and be invertible matrices. Prove that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

proof: A, B are given as invertible so $AA^{-1} = I_n = A^{-1}A$ and $BB^{-1} = I_n = B^{-1}B$.
If AB is invertible then $AB(AB)^{-1} = I_n$ should be true
and $(AB)^{-1}AB = I_n$ should be true.

- $AB(AB)^{-1} = I_n$; multiply by $B^{-1}A^{-1}$ on the left.

  $B^{-1}\underbrace{A^{-1}A}_{I_n}B(AB)^{-1} = B^{-1}A^{-1}$; use $A^{-1}A = B^{-1}B = I_n$

  $\therefore (AB)^{-1} = B^{-1}A^{-1}$

- $(AB)^{-1}AB = I_n$; multiply by $B^{-1}A^{-1}$ on the right.

  $(AB)^{-1}A\underbrace{BB^{-1}}_{I_n}A^{-1} = B^{-1}A^{-1}$; use $BB^{-1} = AA^{-1} = I_n$

  $\therefore (AB)^{-1} = B^{-1}A^{-1}$

See definition on pg 100. AB is invertible b/c $\exists X$ s.t. $ABX = I$ and $XAB = I$.
$X = B^{-1}A^{-1}$ and $B^{-1}A^{-1} = (AB)^{-1}$. ∎

#3 §2.4 Problem 7: Let A be an $(n\times n)$ matrix.

(a) Sps that $A^2=0$. Prove that A is not invertible.

(b) Sps that $AB=0$ for some nonzero $n\times n$ matrix B. Could A be invertible? Explain.

(a) Proof by Contradiction:

Sps that A is invertible. Then $\exists\, B\in M_{n\times n}$ s.t. $AB=I$. Recall $A^2=0 \Rightarrow A^2B=0$

$A^2B = A(AB)=0 \Rightarrow AI=0 \Rightarrow A=0$ which contradicts that A invertible

$\therefore A$ is NOT invertible.

(b) Proof by Contradiction:

Sps that A is invertible. Then $B=B$ can be seen as $B=(A^{-1}A)B$ or $B=A^{-1}(AB)$.

But $AB=0$ by assumption so $B=A^{-1}(0) \Rightarrow B=0$ which is a contradiction.

$\therefore A$ cannot be invertible.

#4 §2.4 Problem 16: Let B be $(n\times n)$ and invertible. $\Phi: M_{n\times n}(F)\to M_{n\times n}(F)$ by $\Phi(A)=B^{-1}AB$. Prove that Φ is an isomorphism.

Need to show two things about Φ: ① That it's linear ② That it's invertible.

① $\phi(a_1A_1+a_2A_2) = B^{-1}(a_1A_1+a_2A_2)B = B^{-1}(a_1A_1)B + B^{-1}(a_2A_2)B$

$\phi(a_1A_1+a_2A_2) = a_2B^{-1}A_2B + a_1B^{-1}A_1B = a_2\phi(A_2)+a_1\phi(A_1)$. $\therefore \phi$ is Linear.

② If ϕ is invertible then $\exists\, \Omega: M_{n\times n}(F)\to M_{n\times n}(F)$ and $\phi(\Omega(X))=X$ and $\Omega(\phi(Y))=Y$

Let $\Omega(A)=BAB^{-1}$. $\phi(\Omega(X))=\phi(BXB^{-1}) = B^{-1}(B X \overset{B\text{ invert.}}{B^{-1}})B = (B^{-1}B)X(B^{-1}B)=X$ ✓

$\Omega(\phi(Y))=\Omega(B^{-1}YB) = \underset{B\text{ invert.}}{BB^{-1}}YBB^{-1} = (BB^{-1})Y(BB^{-1})=Y$ ✓ $\therefore \phi$ invertible $\therefore \phi$ is an Isomorphism