UH - Math 7350 - Dr. Heier - Spring 2012
HW 1
Due 02/09/12, at the beginning of class.

Use regular sheets of paper, stapled together.
Don't forget to write your name on page 1.

1. (1 point) Give an example of a topological space which has all the properties of a topological manifold except that it is not Hausdorff. Discuss your example in detail.

2. (1 point) Give an example of a topological space which has all the properties of a topological manifold except that it is not second countable. Discuss your example in detail.

3. (1 point) Prove that $\mathbb{P}^n$ is Hausdorff and second countable.

4. (1 point) Consider the real line $\mathbb{R}$ as a topological manifold in the usual sense. Prove that there exist uncountably many distinct smooth structures on $\mathbb{R}$.

5. (1 point) Let M be a smooth manifold. Let $f : M \to \mathbb{R}^k$ be a smooth function as defined in class. Prove that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$ is smooth for *every* chart (U, φ) in the maximal atlas defining M.

6. (1 point) Let M be a smooth manifold of dimensional at least 1. Prove that C^∞ is an infinite dimensional vector space.

7. (1 point) Let $P : \mathbb{R}^{n+1}\backslash\{\vec{0}\} \to \mathbb{R}^{k+1}\backslash\{\vec{0}\}$ be a smooth map, and suppose that for some $d \in \mathbb{Z}$, $P(\lambda x) = \lambda^d P(x)$ for all $\lambda \in \mathbb{R}\backslash\{0\}$ and $x \in \mathbb{R}^{n+1}\backslash\{\vec{0}\}$. (Such a map is said to be homogenous of degree d.) Prove that the map $\tilde{P} : \mathbb{P}^n \to \mathbb{P}^k$ defined by $\tilde{P}([x]) := [P(x)]$ is well-defined and smooth.

8. (1 point) Let G be a smooth manifold with a group structure such that the map $G \times G \to G$ given by $(g, h) \mapsto gh^{-1}$ is smooth. Prove that G is a Lie group.

9. (1 point) Let M be a topological space with the property that for every open cover $\mathcal{X}$ of M, there exists a partition of unity subordinate to $\mathcal{X}$. Prove that X is paracompact.

10. Let M be a smooth manifold, let $B \subset M$ be a closed subset, and let $\delta : M \to \mathbb{R}$ be a positive continuous function.

(a) (0.5 points) Using a partition of unity, prove that there is a smooth function $\tilde{\delta} : M \to \mathbb{R}$ such that $0 < \tilde{\delta}(x) < \delta(x)$ for all $x \in M$.

(b) (0.5 points) Prove that there is a continuous function $\psi : M \to \mathbb{R}$ that is smooth and positive on $M\backslash B$, identically equal to zero on B, and satisfies $\psi(x) < \delta(x)$ everywhere on M. Hint: Consider $1/(1 + f)$, where $f : M\backslash B \to \mathbb{R}$ is a positive exhaustion function.