UH - Math 3330 - Dr. Heier - Sample Final Exam
Spring 2013

Print your NAME: Solution

Solve all of the ten problems. Please show all work to support your solutions. Our policy is that if you show no supporting work, you will receive no credit. This is a closed book test. Please close your textbook, notebook, cell phone. No calculators are allowed in this test. Please do not start working before you are told to do so. The regular time allowed is 175 minutes.

Problem 1 ______________________	/10 points
Problem 2 ______________________	/10 points
Problem 3 ______________________	/10 points
Problem 4 ______________________	/10 points
Problem 5 ______________________	/10 points
Problem 6 ______________________	/10 points
Problem 7 ______________________	/10 points
Problem 8 ______________________	/10 points
Problem 9 ______________________	/10 points
Problem 10 ______________________	/10 points
Total ______________________	/100 points

1. (a) (5 points) Prove $(A \cup B) \backslash C = (A \backslash C) \cup (B \backslash C)$.

(b) (5 points) Prove or disprove that $A \cup B = A \cup C$ implies $B = C$.

a) "$\subset$" Let $x \in (A \cup B) \backslash C$.

$\Rightarrow$ $(x \in A$ or $x \in B)$ and $x \notin C$

$\Rightarrow$ $(x \in A$ and $x \notin C)$ or $(x \in B$ and $x \notin C)$

$\Rightarrow$ $x \in A \backslash C$ or $x \in B \backslash C$

$\Rightarrow$ $x \in (A \backslash C) \cup (B \backslash C)$.

"$\supset$" Reverse all arrows in the above argument.

b) To <u>dis</u>prove, let $A = \{1, 2\}$, $B = \{1\}$, $C = \{2\}$

$A \cup B = \{1, 2\} = A \cup C$, but $B \neq C$.

2. (a) (5 points) Let $x, y \in \mathbb{Z}$. Let $x \sim y$ if and only if $7 | 3x - 10y$. Prove that $\sim$ is an equivalence relation on $\mathbb{Z}$.

(b) (5 points) Let $A = \mathbb{Q} \backslash \{0\}$. Let $(a, b), (c, d) \in A \times A$. Let $(a, b) \sim (c, d)$ if and only if $ad = bc$. Prove that $\sim$ is an equivalence relation on $A \times A$.

a) reflexive: $x \sim x \Leftrightarrow 7 | -7x$ ✓

symm.: $x \sim y \Leftrightarrow 7 | 3x - 10y$

$\Leftrightarrow 7 | (3x - 10y + 7(x+y))$

$\Leftrightarrow 7 | 10x - 3y$

$\Leftrightarrow 7 | 3y - 10x$

$\Leftrightarrow y \sim x$ ✓

transitive: $x \sim y$ and $y \sim z \Rightarrow 7 | 3x - 10y$ and $7 | 3y - 10z$

$\Rightarrow 7 | 3x - 10y + 3y - 10z$

$\Rightarrow 7 | 3x - 7y - 10z \Rightarrow 7 | 3x - 10z \Rightarrow x \sim z$ QED

b) refl. $(a, b) \sim (a, b) \Leftrightarrow ab = ba$ ✓

symm. $(a, b) \sim (c, d) \Leftrightarrow ad = bc \Leftrightarrow bc = ad$

$\Leftrightarrow (c, d) \sim (a, b)$ ✓

trans. $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$

$\Rightarrow ad = bc$ and $cf = de$

$\Rightarrow adf = bcf$ and $bcf = bde$

$\Rightarrow adf = bde \underset{d \neq 0}{\Rightarrow} af = be \Rightarrow (a, b) \sim (e, f)$

QED

3. (10 points) Prove by induction that for $n \in \mathbb{N}$ the following holds.

$$1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \ldots + n \cdot 2^n = (n-1)2^{n+1} + 2.$$

$n=1$: $\quad 1 \cdot 2 = (1-1)2^{1+1} + 2$ ✓

$n \rightsquigarrow n+1$: $\quad 1 \cdot 2 + 2 \cdot 2^2 + \ldots + n \cdot 2^n + (n+1)2^{n+1} \underset{\substack{\uparrow \\ \text{Induction} \\ \text{hypothesis}}}{=}$

$$= (n-1)2^{n+1} + 2 + (n+1)2^{n+1}$$

$$= 2^{n+1}(n-1+n+1) + 2$$

$$= 2^{n+1} \cdot 2 \cdot n + 2$$

$$= n \cdot 2^{n+2} + 2 \quad \text{QED}$$

4. (a) (5 points) Find the value of $d = \gcd(200, 56)$. Find m, n such that $d = m \cdot 200 + n \cdot 56$.

(b) (5 points) Find a solution $x \in \mathbb{N}$ with $0 \le x < 42$ such that

$$55x \equiv 50 \mod 42.$$

a) $d = \gcd(200, 56) = \gcd(200 - 3 \cdot 56 = 32, 56)$

$= \gcd(32, 56 - 32 = 24) = 8$

To find m, n:

$8 = 32 - 24 = 32 - (56 - 32) = 2 \cdot 32 - 56 =$

$= 2(200 - 3 \cdot 56) - 56 = \underbrace{2}_{m} \cdot 200 + \underbrace{(-7)}_{n} \cdot 56$

b) $55x \equiv 50 \mod 42$

$\Leftrightarrow 55x \equiv 8 \mod 42$

Note: $\gcd(55, 42) = 1$

As above, we find $1 = 13 \cdot 55 + (-17) \cdot 42$

Multiply through by 8:

$8 = \underbrace{(8 \cdot 13)}_{= 104} \cdot 55 + (-17) \cdot 8 \cdot 42$

$\Rightarrow \underline{\underline{x}} = 104 - 2 \cdot 42 = \underline{\underline{20}}$

5. (a) (5 points) Let a be an element of a group G. Prove that a and a^{-1} have the same order.

(b) (5 points) Let $f : G \to H$ be an epimorphism of groups. Let G be abelian. Prove that H is abelian.

a) By symmetry, it suffices to show
$\operatorname{ord}(a) \geq \operatorname{ord}(a^{-1})$.
Let $k := \operatorname{ord}(a)$. Then
$(a^{-1})^k = (a^k)^{-1} = e^{-1} = e$
$\Rightarrow \operatorname{ord}(a^{-1}) \leq k$ QED

b) Let $a, b \in H$. Since f onto $\Rightarrow$
$\exists c, d \in G$: $f(c) = a$, $f(d) = b$.

$\Rightarrow a \cdot b = f(c) \cdot f(d) \underset{\text{homom.}}{=} f(cd) \overset{G \text{ abelian}}{=} f(dc) =$

$\underset{\text{homom.}}{=} f(d) \cdot f(c) = b \cdot a$ QED

6. (a) (5 points) Let f be the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 5 & 1 & 2 \end{pmatrix}$. Write f as a product of disjoint cycles.

(b) (4 points) Write f as a product of transpositions.

(c) (1 point) Is f even or odd?

a) $f = (1\ 3\ 4\ 5)(2\ 6)$

b) $(1\ 3\ 4\ 5) = (5\ 1)(4\ 1)(3\ 1)$

$\Rightarrow f = (5\ 1)(4\ 1)(3\ 1)(2\ 6)$

c) 4 transpositions $\Rightarrow$ f even

7. (10 points) Prove that if H and K are normal subgroups of a group G such that $H \cap K = \{e\}$, then $hk = kh$ for all $h \in H$, $k \in K$.

First, note that $hk = kh$

$$\Leftrightarrow hkh^{-1}k^{-1} = e$$

Next, observe that

$$\underbrace{hkh^{-1}}_{\substack{\in K \\ \text{due to normality}}}\underbrace{k^{-1}}_{\in K} \in K.$$

Also

$$\underbrace{h}_{\in H}\underbrace{kh^{-1}k^{-1}}_{\substack{\in H \\ \text{due to normality}}} \in H.$$

$$\underset{\substack{\uparrow \\ H \cap K = \{e\}}}{\Rightarrow} hkh^{-1}k^{-1} = e \quad \text{QED}$$

8. (10 points) Prove that for a fixed element a in a ring R, the set $S = \{x \in R \mid ax = 0\}$ is a subring of R.

a) $0 \in S$ ✓ b/c $a \cdot 0 = 0$

b) let $x, y \in S$.

$\Rightarrow a(xy) = (ax)y = 0 \cdot y = 0$

and $a(x+y) = ax + ay = 0 + 0 = 0$

c) $a(-x) = -(ax) = -0 = 0$

$\Rightarrow$ subring ✓.

9. (10 points) Let R be a ring in which all elements x satisfy $x^2 = x$. Prove that are every element of R is equal to its own additive inverse. (Hint: Consider $(x + x)^2$.)

assumption

$$4 \cdot x^2 = (x + x)^2 \overset{\text{assumption}}{=} x + x = 2 \cdot x$$

$\parallel \leftarrow$ assumption

$4x$

$\Rightarrow \quad 2 \cdot x = 0$

$\Rightarrow \quad x + x = 0$

$\Rightarrow \quad x = -x$ QED

10. (a) (5 points) Give an example of a ring R and elements $a, b \in R$ such that both a and b are not zero divisors, but the sum $a+b$ is a zero divisor.

(b) (5 points) Give an example where a and b are zero-divisors in a ring R with $a+b \neq 0$, and $a+b$ is not a zero divisor.

a) $R = Mat_{2\times 2}(\mathbb{R})$

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Both a, b are invertible, so they are not zero divisors.

But $a+b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is a zero-divisor

b/c $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

b) $R = Mat_{2\times 2}(\mathbb{R})$.

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Both a, b are zero divisors:

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$; $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

But

$a+b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which is not a zero divisor.