UH - Math 6303 (Modern Algebra II) - Dr. Heier - Spring 2014

Midterm Exam

Thursday, March 20, 2014

Print your NAME:

Solve all of the 7 problems. Please show all work to support your solutions. Our policy is that if you show no supporting work, you will receive no credit. This is a closed book test. Please close your textbook, notebook, cell phone. No calculators are allowed in this test. Please do not start working before you are told to do so. The time allowed will be announced by the proctor.

Problem 1 ____________________ /10 points

Problem 2 ____________________ /20 points

Problem 3 ____________________ /10 points

Problem 4 ____________________ /15 points

Problem 5 ____________________ /10 points

Problem 6 ____________________ /20 points

Problem 7 ____________________ /15 points

Total ____________________ /100 points

1. (10 points) Let the polynomials $f_1, f_2 \in \mathbb{Q}[x, y, z]$ be given by

$$f_1(x, y, z) = x + 2y + 5z, \quad f_2(x, y, z) = x + y + 4z.$$

Consider the lexicographical ordering given by $x > y > z$. Let $I = (f_1, f_2)$. Is f_1, f_2 a Gröbner basis for I? If not, find a Gröbner basis. Explain your answer as best you can, but you don't need to provide complete proofs.

2. Consider the irreducible polynomial $p(x) = x^3 + 6x + 2$ in $\mathbb{Q}[x]$ (you don't have to prove it is irreducible). Let θ be a root of $p(x)$.

(a) (10 points) Find the inverse of θ in $\mathbb{Q}(\theta)$.

(b) (10 points) Compute $\frac{\theta}{1+\theta}$ in $\mathbb{Q}(\theta)$.

3. (10 points) Let $F = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ with $\alpha_i^2 \in \mathbb{Q}$ for $i = 1, \ldots, n$. Prove that $\sqrt[3]{2} \notin F$.

4. (15 points) Determine the splitting field and degree over $\mathbb{Q}$ of $x^6 - 4$.

5. (10 points) Are the fields $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(i\sqrt{2})$ isomorphic? Prove your answer.

6. (a) (10 points) Let $K/\mathbb{Q}$ be a field extension of degree 2. Prove that it is Galois.

(b) (10 points) Let $L/\mathbb{Q}$ be a field extension of degree 3. Is it necessarily Galois? Prove your answer.

7. In the following problem, you may freely use the Fundamental Theorem of Galois Theory. Let K/F be a Galois extension with $\mathrm{Gal}(K/F) = \mathbb{Z}_6$.

(a) (5 points) How many field extensions E/F with $F \subseteq E \subseteq K$ are there? (Do NOT include the trivial cases F/F and K/F in your count.) Justify your answer.

(b) (5 points) How many of the above non-trivial extensions E/F are Galois? Justify your answer.

(c) (5 points) Among the above non-trivial extensions E/F, are there extensions E_1/F and E_2/F with

$$F \subsetneqq E_1 \subsetneqq E_2 \subsetneqq K?$$

Justify your answer.