UH - Math 3330 - Dr. Heier - Spring 2014
HW 9 - Solutions to *Selected* Homework Problems
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2. (Section 4.2, Problem 1) Write out the elements of a group of permutations that is isomorphic to G and exhibit and isomorphism from G to this group: Let G be the additive group $\mathbb{Z}_3$.

Solution. The elements of $\mathbb{Z}_3$ are $\mathbb{Z}_3 = \{[0],[1],[2]\}$. For $[a] \in \mathbb{Z}_3$, define a map $f_{[a]} : \mathbb{Z}_3 \to \mathbb{Z}_3$ by $f_{[a]}([b]) = [a] + [b]$. Thus, we have,

$$\begin{array}{lll} f_{[0]}([0]) = [0] & f_{[1]}([0]) = [1] & f_{[2]}([0]) = [2] \\ f_{[0]}([1]) = [1] & f_{[1]}([1]) = [2] & f_{[2]}([1]) = [0] \\ f_{[0]}([2]) = [2] & f_{[1]}([2]) = [0] & f_{[2]}([2]) = [1] \end{array}$$

Thus, our group of permutations can be defined by $H = \{f_{[0]}, f_{[1]}.f_{[2]}\}$. Define a map, $\varphi : \mathbb{Z}_3 \to H$ by $\varphi([a]) = f_{[a]}$. This is the desired isomorphism.

5. (Section 4.2, Problem 8) For each a in the group G, define a mapping $h_a : G \to G$ by $h_a(x) = xa$ for all $x \in G$.

- **(a)** Prove that h_a is a permutation on the set of elements in G.
- **(b)** Prove that $H = \{h_a \mid a \in G\}$ is a group with respect to mapping composition.
- **(c)** Define $\phi : G \to H$ by $\phi(a) = h_a$ for each $a \in G$. Determine whether ϕ is always an isomorphism.

Solution.
(a) The map h_a is a permutation on G.

Proof. Assume that for $x, y \in G$, $h_a(x) = h_a(y)$. Then $xa = ya$. Multiplication by a^{-1} yields $x = y$. So h_a is one-to-one. Now, let $y \in G$. The pre-image of y under h_a is $ya^{-1} \in G$. We check this: $h_a(ya^{-1}) = yaa^{-1} = y$. So, h_a is onto and thus is a permutation on G. □

(b) $H = \{h_a \mid a \in G\}$ is a group with respect to mapping composition.

Proof. Let h_a and h_b be in H. Then for $x \in G$, we have

$$(h_a h_b)(x) = h_a\,(h_b(x)) = h_a(xb) = xba = x(ba) = h_{ba}(x)$$

Thus, $(h_a h_b) = h_{ba} \in H$. So H is closed. H is also associative under mapping composition. The identity element of H is h_e, where e is the identity element of G. Note that $h_a h_e = h_e h_a = h_a$, as desired. Lastly, for each $h_a \in H$, the inverse element is $(h_a)^{-1} = h_{a^{-1}}$. We check this: $h_a h_{a^{-1}} = h_e = h_{a^{-1}} h_a$. Thus, H is a group. □

(c) The map $\phi : G \to H, a \mapsto h_a$, is **not always** an isomorphism, because ϕ is not always a homomorphism. Let $a, b \in G$. Then

$$\phi(ab) = h_{ab} = (h_b)(h_a) = \phi(b)\phi(a)$$

Hence, $\phi(ab) \neq \phi(a)\phi(b)$. Note that ϕ **is** a homomorphism if G is abelian.

7. (Section 4.4, Problem 4a) Let $H = \{(1), (2,3)\}$ of S_3. Find the distinct left cosets of H in S_3, write out their elements, and partition S_3 into left cosets of H.

Solution. The distinct left cosets of H are

$$H \text{ itself}, \quad (1,3)H = \{(1,3),(1,3,2)\}, \text{ and } (1,2)H = \{(1,2),(1,2,3)\}$$

Thus, S_3 can be partitioned as $S_3 = H \cup (1,3)H \cup (1,2)H$.

8. (Section 4.4, Problem 8) Let H be a subgroup of a group G.

- **(a)** Prove that gHg^{-1} is a subgroup of G for any $g \in G$. We say that gHg^{-1} is a **conjugate** of H and that H and gHg^{-1} are **conjugate subgroups**.
- **(b)** Prove that if H is abelian, then gHg^{-1} is abelian.
- **(c)** Prove that if H is cyclic, then gHg^{-1} is cyclic.
- **(d)** Prove that H and gHg^{-1} are isomorphic.

Solution.
(a) For any $g \in G$, gHg^{-1} is a subgroup of G.

Proof. Let e be the identity element in G. Then, the identity element in gHg^{-1} is geg^{-1} and gHg^{-1} is nonempty. Now let $x, y \in gHg^{-1}$. This means that there exist $h, h' \in H$ such that $x = ghg^{-1}$ and $y = gh'g^{-1}$. Note that $y^{-1} = gh'^{-1}g^{-1}$. So

$$xy^{-1} = (ghg^{-1})(gh'^{-1}g^{-1}) = ghh'^{-1}g^{-1} = g\hat{h}g^{-1} \in H$$

because $hh' = \hat{h} \in H$ due to H being a subgroup. Thus, by Theorem 3.10, gHg^{-1} is a subgroup of G. □

(b) If H is abelian, then gHg^{-1} is abelian.

Proof. Assume H is abelian and let $x, y \in H$. This means that there exist $h, h' \in H$ such that $x = ghg^{-1}$ and $y = gh'g^{-1}$. Then

$$xy = (ghg^{-1})(gh'g^{-1}) = g(hh)'g^{-1} = g(h'h)g^{-1} = gh'ehg^{-1} = (gh'g^{-1})(ghg^{-1})) = yx$$

So, gHg^{-1} is abelian. □

(c) If H is cyclic, then gHg^{-1} is cyclic.

Proof. Assume H is cyclic—that is, $\exists\ h \in H$ such that $H = \langle h \rangle$. Claim that $gHg^{-1} = \langle ghg^{-1} \rangle$. Let $x \in gHg^{-1}$ be arbitrary. This means that there exists $h' \in H$ such that $x = gh'g^{-1}$. Because $h' \in H$ and H is cyclic, there exists $k \in \mathbb{Z}$ such that $h' = h^k$. Thus, $x = gh'g^{-1} = g(h^k)g^{-1} = \left(ghg^{-1}\right)^k$. So gHg^{-1} is cylcic. □

(d) H and gHg^{-1} are isomorphic.

Proof. Define a map $\varphi : H \to gHg^{-1}$ by $\varphi(h) = ghg^{-1}$. We must show that this map is indeed an isomorphism.

Let $h, h' \in H$ and let e be the identity element in H. Then

$$\varphi(hh') = g(hh')g^{-1} = gheh'g^{-1} = (ghg^{-1})(gh'g^{-1}) = \varphi(h)\varphi(h')$$

So, φ is a homomorphism. Now assume that $\varphi(h) = \varphi(h')$. This means that $ghg^{-1} = gh'g^{-1}$, and that $h = h'$. So φ is one-to-one. Also, let $ghg^{-1} \in gHg^{-1}$ has a pre-image of $h \in H$—that is, $\varphi(h) = gHg^{-1}$. Thus, φ is onto. Hence, φ is an isomorphism and $H \cong gHg^{-1}$. □

9. (Section 4.4, Problem 19) If H and K are arbitrary subgroups of G, prove that $HK = KH$ if and only if HK is subgroup of G.

Proof. $\boxed{\Rightarrow}$ Assume $HK = KH$. Because H and K are subgroups, they both contain the identity element, e. Thus, $e = e(e) \in HK$. So $HK \neq \emptyset$. Now let $x, y \in HK$. This means there exists $h, h' \in H$ and $k, k' \in K$ such that $x = hk$ and $y = h'k'$. Then, using the fact that $HK = K'$, we have

$$xy^{-1} = (hk)(h'k')^{-1} = hkk'^{-1}h'^{-1} = hh'^{-1}kk'^{-1} \in HK$$

Thus, by Theorem 3.10, HK is a subgroup of G.

$\boxed{\Leftarrow}$ Now assume that HK is a subgroup of G. Let $x \in HK$. Because HK is a subgroup, x^{-1} exists and is given by $x^{-1} = hk$, for some $h \in H$ and $k \in K$. Then $x = (x^{-1})^{-1} = k^{-1}h^{-1} \in KH$. Hence, $HK \subset KH$. Similarly, we have that $KH \subseteq HK$. Thus, $HK = KH$. $\square$