4/3/2014 Abstract Algebra

* Review all past course material and recent HW problems.

Going Quiz #8

1) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 8 & 5 & 2 & 3 & 4 & 10 & 6 & 9 & 1 \end{bmatrix}$

$\Longrightarrow (1\ 7\ 10)\circ(2\ 8\ 6\ 4)\circ(3\ 5)$ "In cycles" form

$= (1\ 10)(1\ 7)(2\ 4)(2\ 6)(2\ 8)(3\ 5)$

a product of transpositions (placement of cycles is very important)

2) Prove the square of a cycle is not necessarily a cycle. (Hint look in S_4)

Answer (Possibility #1):

so, ~~$(1\ 2)^2 = id$ "a cycle squared"~~

$(1\ 2)^2 = id$ is a cycle

then, $(1\ 2\ 3)^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$

$= (1\ 2\ 3)$ is a cycle

then, $(1\ 2\ 3\ 4)^2 = \begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 3 & 4 & 1 & 2 & 5\end{pmatrix}$

$= (1\ \ 3)\circ(2\ \ 4)$ is not case where the square of a cycle is a cycle (we mean the same cycle).

Note: Being a cycle is a well-defined property.

Continuitation of Lecture on Even/Odd Permutations

Defn: A permutation which can be expressed as a product of an even # of transpositions is called an even permutation.

then, odd # of transpositions $\Longleftrightarrow$ odd permutation

Ex: Is $\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 3 & 4 & 1 & 5 & 2\end{pmatrix}$ even or odd?

Answer: $(1\ \ 3)(2\ 4\ 5) = \circledast$ — indicating the decomp. of the given permutation

$\Longrightarrow (1,3)(2,5)(2,4)$

$\Longrightarrow$

Remark:

Defn/

Proof:

let

let

tran

So,

Chap

$\Rightarrow$ The given permutation is odd $\boxed{\text{QED}}$

Remark: The composition of even permutations is even.
The converse is also true.

Defn/Proposition: The set of even permutations in S_n (for any $n \geq 2$) is a subgroup of S_n, called the alternating group, A_n.

Proof: $id = (1\,2)(1\,2)$ ✓

let $f, g \in A_n \Rightarrow f \circ g$ is a product of the transpositions in f and g. The total number is even again (because even + even = even)

let $f \in A_n$, $f = \tau_1 \circ \ldots \circ \tau_k$ as a product of transpositions.

so, $f^{-1} = \tau_k^{-1} \circ \ldots \circ \tau_1^{-1}$
$= \tau_k \circ \ldots \circ \tau_1$

$\boxed{\text{QED}}$

Chapter 4 Section 2
Cayley's Theorem

Theorem (itself) the proof of Cayley's:

Every finite group G is a subgroup of $S_{\#G}$

<u>Proof:</u> We will write down a monomorphism $G \to S_{\#G}$ as follows:

$\Rightarrow$ For an arbitrary $g \in G$, $\exists$ bijection $\ell_g : G \to G$

Namely: $\ell_g(h) = gh$

Proof that this is a bijection,

$\ell_g(h_1) = \ell_g(h_2)$

then, $g^{-1} \cdot 1 \iff gh_1 = gh_2$

$\Rightarrow h_1 = h_2$

so, the monomorphism is $\phi : G \to S_{\#G}$
$g \mapsto \ell_g$

Proof that ϕ is a monomorphism:

1) ϕ is a homomorphism

then, $\phi(g_1) \circ \phi(g_2) = \phi(g_1 g_2)$

a. Now we apply $\phi(g_1) \circ \phi(g_2)$ to h:

then, $(\phi(g_1) \circ \phi(g_2))(h)$

$= \phi(g_1)(\ell_g(h))$

$= \phi(g_1)(g_2 h) = \ell_{g_1}(g_2 h) = \boxed{g_1 g_2 h}$ ✓

then, apply $\phi(g_1 g_2)(h)$,

b. $\phi(g_1 g_2)(h) = \ell_{g_1 g_2}(h) = \boxed{g_1 g_2 h}$ ✓

2) ϕ is injective

then, let $\phi(g_1) = \phi(g_2)$

then, in particular, $\underbrace{\phi(g_1)(e)}_{\ell_{g_1}(e) = g_1} = \underbrace{\phi(g_2)(e)}_{\ell_{g_2}(e) = g_2}$

$\boxed{\text{QED}}$

4/8/2014 Abstract Algebra

Chapter 4 Section 4 – Cosets of a subgroup

Defn: Let $H \subset G$ be a subgroup.
For any $a \in G$,
$aH = \{x \in G \mid x = ah \text{ for some } h \in H\}$
is the left coset of H with respect to the group element "a".

* Analogously, $Ha \Rightarrow$ the right coset of H with respect to the group element "a".

Lemma 4.11 – "Left Coset Partition Lemma"

Let aH and bH be 2 cosets. Then either $aH = bH$ or $aH \cap bH = \phi$

implies only 1 of these statements is true

Proof of Lemma 4.11:

Lets assume that $aH \cap bH = \phi$ is false.
$\Rightarrow \exists z \in G: z \in aH \cap bH$

Show: $aH \subseteq bH$ ($\supseteq$ by symmetry)

Proof of: Let $z = ah_1 = bh_2$
Show

$h_1 \in H \quad h_2 \in H$

then, solve for a:

$\iff a = b h_2 h_1^{-1}$

From this we can conclude, $\underbrace{}_{}$ this product becomes another element in H.

$\forall h \in H : ah = b h_2 h_1^{-1} h$, where $h_2 h_1^{-1} h \in H$

$\Rightarrow b h_2 h_1^{-1} h \in bH$ $\boxed{\text{QED}}$

"break up"

<u>Corollary:</u> The left cosets <u>partition</u> G into mutually disjoint subsets.

<u>Defn:</u> Let H be a subgroup of G.
Define "index of H in G" :=

Note:

$[G:H]$ = # of disjoint left cosets of H.

Note:

$eH = H$

~~Note:~~

Theorem 4.13 - Lagrange's Theorem

Let G be a finite group.
Let H be a subgroup of G.
then,

$$\text{ord}\, G = (\text{ord}\, H) \cdot [G:H]$$

this works when you have the same # of elements in the

Recheck this later. $\Longrightarrow$

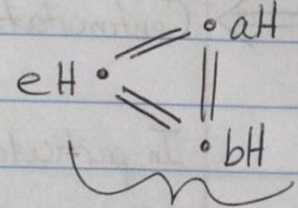
<u>Proof:</u> Our strategy will be to show that <u>all</u>

left cosets have the same cardinality.

It clearly suffices to show that,

$\forall a \in H$: the left coset H and aH have the same cardinality.

Note:



indicating transitivity

To prove this, we prove $f: H \to aH$

Note $\Rightarrow$ "aH is not a group unless $aH = H$."

so, $f: H \longrightarrow aH$

$h \longmapsto ah$ is bijective (we need to prove that the bijectivity holds)

so, Injectivity $\Rightarrow f(h_1) = f(h_2)$

$\Leftrightarrow ah_1 = \cancel{b}\,^{a}h_2$

then, $a^{-1} \cdot 1 \Leftrightarrow h_1 = h_2$ ✓

and Surjectivity $\Rightarrow$ let $z \in aH$ be "$z = ah_0$"

then, $f(h_0) = ah_0 = z$ $\boxed{QED}$

Corollary: The order of H (aka ord H) divides the order of G (aka ord G)

$\Longleftrightarrow$ ord H / ord G

Ex: If ord $G = 13$, then ord H = the trivial answer (aka 1)

$\Rightarrow$ Continuitation of previous Corollary:

In particular, $\forall g \in G$: $\text{ord}(g) = \text{ord}(\langle g \rangle) \mid \text{ord}\, G$

the order of "g" divides the order of "G".

Ex: Find all subgroups of S_3

$\Rightarrow$ Because of the previous corollary, we know that the order of any subgroup is 1, 2, 3, or 6.

then, ord $H = 1$: $H = \{e\}$

then, ord $H = 2$: $H = \{e, (1,2)\} = \boxed{\langle (1,2) \rangle}$

and $H = \{e, (1,3)\} = \boxed{\langle (1,3) \rangle}$

and $H = \{e, (2\,3)\} = \boxed{\langle (2,3) \rangle}$

Note!

$S_3 = \{e, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}$

and ord $H = 3$: $H = \{e, (1\,2\,3), (1\,3\,2)\}$

Ex: Find all subgroups of $(\mathbb{Z}_{15}, +)$

then, $15 = 3 \cdot 5 \Rightarrow$

Note:

ord $H = 3$:
$H = \{[0], [5], [10]\}$

ord $H = 5$:
$H = \{[0], [3], [6], [9], [12]\}$

Proposition Any group of prime order is cyclic.

Proof: Let $e \neq g \in G$ group (we claim that g is arbitrary)

Proof of Claim: $\langle g \rangle = G$ (This is given claim)

then, $1 < \text{ord}(\langle g \rangle)$ and by Lagrange's Theorem, $\text{ord}(g) \mid \text{ord}(G) = \text{prime}$

$\Rightarrow \text{ord}(\langle g \rangle) = \text{ord}\, G$

$\Rightarrow \langle g \rangle = G$ $\boxed{\text{QED}}$

Proposition Let $\text{ord}(G) = p \cdot q$, where "$p$" and "$q$" are prime numbers.

then, any proper subgroup is cyclic.

Proof: For any proper subgroup H, one of the following holds:

1) $\text{ord} H = 1$
2) $\text{ord} H = p$ $\Longrightarrow$ Now, we can apply the previous proposition
3) $\text{ord} H = q$

$\boxed{QED}$

** Review course material from Tuesday, April 8th

4/8/2014 Vector Analysis

$\Rightarrow$ Given Set $R \in \mathbb{R}$ is open if for any $x \in R$, there is $B_\varepsilon(x) \subset R$

a ball centered at x of a small radius ε.

so, $B_\varepsilon(x)$ - a ball centered at x of radius ε.

$\mathbb{R}^3 \Rightarrow$ Ball
$\mathbb{R}^2 \Rightarrow$ Disc
$\mathbb{R}^1 \Rightarrow$ Interval

Ex: $R = \mathbb{R}^2 \setminus \{(x,y) : y=0,\ 0<x<1\}$

☆☆ Review class notes on Discrete Fourier transform and Fast Fourier Transform ☆☆

4/10/2014

Abstract Algebra

Continuitation of 4.4- Cosets and Starting 4.5- Normal Subgroups

Chapter 4 Section 5 - Normal Subgroups

Recall: Let $H \subset G$ be a subgroup. Then for $a \in G$,

$\Rightarrow$ $aH = \{x \in G \mid x = ah, \text{ for some } h \in H\}$ is the LEFT COSET OF H in G.

$\Rightarrow$ $Ha = \{x \in G \mid x = ha, \text{ for some } h \in H\}$ is the RIGHT COSET OF H in G.

Note: Special subgroup $\Rightarrow$ when LEFT COSET = RIGHT COSET

Defn: Let $H \subset G$ be a subgroup. Then H is NORMAL if $\forall x \in G$, $xH = Hx$. (In other words, left coset = right coset).

WARNING: $\underbrace{xH}$ $= Hx$ ← "Equality of Sets"

this does not mean $xh = hx : \forall x \in G, h \in H$.

Ex: Let $G = S_3 = \{(1),(1,2),(1,3),(2,3),(1,2,3),(1,3,2)\}$

and $H = \{(1),(1,2,3),(1,3,2)\}$

Let $x = (1,2) \Rightarrow xH = (1,2)H$

$$= \{(1,2)(1),(1,2)(1,2,3),(1,2)(1,3,2)\}$$

$$= \{(1,2),(2,3),(1,3)\}$$

and $Hx = \{(1)(1,2),(1,2,3)(1,2),(1,3,2)(1,2)\}$

$$= \{(1,2),(1,3),(2,3)\}$$

so, $(1,2)H = H(1,2)$ ✓

What you can also check later:

$(1)H = H(1)$

$(1,2,3)H = H(1,2,3)$

$(1,3,2)H = H(1,3,2)$

$(1,3)H = H(1,3)$

$(2,3)H = H(2,3)$

Theorem 4.16

If H is any subgroup of G, then

$xH = H = Hx \iff x \in H$

Note:

$(1,2)(1,2,3)$

$1 \mapsto 2 \mapsto 1$

$2 \mapsto 3 \mapsto 3$

$3 \mapsto 1 \mapsto 2$

$(1,3,2)(1,2)$

$1 \to 2 \to 1$

$2 \to 1 \to 3$

$3 \to 3 \to 2$

☆ Review Permutation Multiplication

Theorem 4.18 - Conjugates and Normality

Let $H \subset G$ be a subgroup. Then H is normal

$\Longleftrightarrow \forall x \in G, \forall h \in H, \underbrace{xhx^{-1}}_{} \in H$

same notation used in the April 8th lecture

Note: xhx^{-1} is called the conjugate of h.

$\Rightarrow H$ is normal $\Leftrightarrow xHx^{-1} \in H$ ✓

Proof of Theorem 4.18

"$\Longrightarrow$" Assume H is normal

then, $\forall x \in G$, $xH = Hx$ [By definition]

$\Rightarrow \forall h \in H, \forall x \in G, \exists h' \in H$ such that $xh = h'x$

$\Rightarrow xhx^{-1} = h'$ and $h' \in H$ (multiplication by x^{-1})

$\Rightarrow xhx^{-1} \in H$

"$\Longleftarrow$" Reverse steps used in "$\Rightarrow$"

so, assume $xhx^{-1} \in H \Rightarrow xhx^{-1} = h'$, $h' \in H$

$\Rightarrow \forall h \in H, \forall x \in G, \exists h' \in H$ s.t. $xh = h'x$

$\Rightarrow \forall x \in G$, $xH = Hx$ [By defn.]

$\Rightarrow H$ is normal

QED

Recall: If $H \subset G$ is a subgroup, then the INDEX of H in G is $[G:H] = \#$ of left cosets of H (definition)

$$= \frac{\text{order } G}{\text{order } H} \quad \text{[by Lagrange's Theorem]}$$

Theorem - Every subgroup H of G of index 2 is normal. (So is this true? Lets prove it)

Proof: Assume $H \subset G$ has index of 2 (in other words, $[G:H] = 2$)

Let $x \in G$ be arbitrary

Using Theorem 4.16

Case I: $x \in H \Longrightarrow xH = H = Hx$, so H is normal $\boxed{\text{QED}}$

Case II: $x \notin H$

or $x \in G$, but $x \notin H$

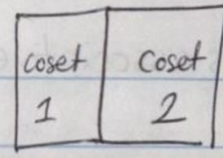
$[G:H] = 2 \Longrightarrow 2$ left cosets of H

so, $eH = H$ (we know that least 1 left coset exists and that coset is H itself)

then, xH will be the other left coset.

Recall: Left cosets partition G into disjoint sets

$\Rightarrow$ pictorally,



then, $G = H \,\dot\cup\, xH$ [$\dot\cup$ is used to represent "disjoint"]

Similarly, $G = H \,\dot\cup\, Hx$ [for the right cosets]

so, $G = H \,\dot\cup\, xH = H \,\dot\cup\, Hx \Longleftrightarrow xH = Hx$, so H is normal

$\boxed{\text{QED}}$

Chapter 4 Section 6 - Quotient Groups

Theorem - Groups of Cosets

Let H be a normal subgroup. Then the set of all cosets of H in G form a group with respect to the following binary operation:

written as G/H

> Side Note:
> whenever H is normal, left coset = right coset
> $\Rightarrow$ for brevity, we will then just say "coset", when left = right

if $a, b \in G$, $(aH)(bH) := (ab)H$

Proof:

Closed: Let $aH, bH \in G/H$

then, $(aH)(bH) = \underbrace{a(Hb)H}_{\text{because } G \text{ associative}}$

> Side Note:
> 4 Conditions for defining a group.
> 1) closed
> 2) associative
> 3) identity elements
> 4) inverses

$= a(bH)H$ } because H is normal

$= (ab)HH$ } G is associative

$= (ab)H$ } because $HH = H$

$\Rightarrow G/H$ is closed ✓ $\boxed{\text{QED}}$

Associativity: → Associativity is inherited from G itself $\boxed{\text{QED}}$

Identity elements: → The identity is $eH = H$. Now we check this to be true.

so, $(aH)(eH) = (ae)H$
$= aH$ ✓

* This is the same for $(eH)(aH) = aH$ ✓ $\boxed{\text{QED}}$

Inverses: Let $aH \in G/H$
so, $(aH)(a^{-1}H) = eH$

Check: $(aH)(a^{-1}H) = (aa^{-1}H) = eH$ ✓

This is the same for $(a^{-1}H)(aH) = eH$ ✓

$\Rightarrow G/H$ is a group

$\boxed{\text{Final QED}}$ → So the set of all cosets of H

in G is a group.

Defn: If $H \subset G$ is normal, the group of cosets, G/H is called the quotient group of G by H.

Remark: If G is abelian (aka commutative), then any subgroup of G is normal. Also G abelian implies the quotient group, G/H, is also abelian.

Note:

DO NOT MIX UP,

G/H "quotient" group

$G \setminus H$ "G excluding H"

Note: G/H only makes sense only if H is normal.

so, $(aH)(bH) = (ab)H = (ba)H = (bH)(aH)$

$\Rightarrow$ then associativity holds to be true.

4/10/2014 Vector Analysis

Section 4.4 Problem 7b

since $F = [(1+x)e^{x+y}]\hat{i} + [xe^{x+y} + 2y]\hat{j} - 2z\hat{k}$

then, $G = [(1+x)e^{x+y}]\hat{i} + [xe^{x+y} + 2z]\hat{j} - 2y\hat{k}$