# UH Math 3330-01 Dr.Heier-Spring 2017
# HW12 Key

Yan He

(1)Note that a finite integral domain is a field. Then it's sufficient to show that $\mathbb{Z}_n$ has no zero-divisors iff $n$ is prime.

$\mathbb{Z}_n$ has no zero-divisors if and only if for $[a],[b] \in \mathbb{Z}_n$ we have $[ab]=[0] \iff [a]=[0]$ or $[b]=[0]$. Then we have $n|ab \iff n|a$ or $n|b$ for $a,b \in \mathbb{Z}$. This holds only when $n$ is prime.

(2)From assumption we have $x(xy-yx)y=0$. The problem becomes very easy if $R$ has no zero divisor. However, if $R$ might has zero divisor we may not naively cancel $x$ and $y$. We need some clever way to do this.

Use the assumption on $(1-x)y$ we have

$$(1+x)y(1+x)y=(1-x)^2y^2$$

Simplify we get $yxy=xy^2$, or $(xy-yx)y=0$. Then substitute $y$ with $(y+1)$:

$$(y+1)x(y+1)=x(y+1)^2$$

Simplify we get $yx=xy$.

(3) $\Rightarrow$ If $a$ is a unit then every $r \in R$ we have $r=r1=raa^{-1}=a(ra^{-1}) \in aR$.
$\Leftarrow$ If $aR=R$ then $1 \in aR$. Then there exists $b \in R$ s.t. $1=ab$.

(4)(a)$\frac{a_1}{b_1} \in S$, $\frac{a_2}{b_2} \in S$, then $\frac{a_1}{b_1}+\frac{a_2}{b_2}=\frac{a_1b_2+a_2b_1}{b_1b_2}$ where $b_1,b_2$ are odds so $b_1b_2$ is odd, then $\frac{a_1}{b_1}+\frac{a_2}{b_2} \in S$.

Claim:$S_0=\{\frac{a}{b} \in S : a \text{ is even}\}$ is the unique maximal ideal. Actually, every element in $S-S_0$ is unit. For $\frac{a}{b} \in S-S_0$, $a$ is odd, so its inverse $\frac{b}{a} \in S$, then it is unit.

(5)Note that $\{0\}$ is the kernel of the identity map $id : R \to R$. The rest of proof follows from the fact that $R/I$ is integral domain iff $I$ is prime and is a field iff $I$ is maximal.