UH - Math 6353 - Dr. Heier - Take Home Midterm Exam - Spring 2017
Due: Monday, April 3, in class
(corrected version as of March 31, 12:05pm)

Note: Each Problem is worth the amount of points indicated.

1. (10 points) Let X be a projective manifold. Let D be an $\mathbb{R}$-divisor on X. Prove that if D is $\mathbb{R}$-ample, then $\overline{\mathrm{NE}}(X) \setminus \{\vec{0}\} \subseteq D_{>0}$.

2. (20 points) Prove that $\dim N^1(\mathbb{P}^1 \times \mathbb{P}^1)_{\mathbb{R}} = 2$. Describe a basis.

3. (20 points) Let X be a projective manifold. Let D be a strictly nef integral divisor on X. (Recall that *strictly nef* means that D intersects every effective curve positively and not just non-negatively.) Prove that $K_X + tD$ is a nef $\mathbb{R}$-divisor for all reals $t \geq \dim X + 1$. Hint: Use the Cone Theorem.

4. (20 points) Let a, b, be positive reals. Let φ be the plurisubharmonic function $\varphi(z, w) = \log(|z|^{2a} + |w|^{2b})$ on the bidisk. Determine those pairs (a, b) such that $\mathcal{I}(\varphi)_{(0,0)} \neq \mathcal{O}_{(0,0)}$, i.e., such that the multiplier ideal sheaf of φ is not trivial at the origin.

5. (20 points) For $\mathbb{P}^2$ with the Fubini-Study metric g, compute the Ricci curvature form Ric. Compare Ric and the associated $(1,1)$-form of g. Also, using the trace formula discussed in class (and appearing as an exercise below), compute the scalar curvature.

Hint: Choose affine coordinates z_1, z_2 and work in terms of the frame $\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}$. Recall that the associated $(1,1)$-form of g is given by

$$\omega = \frac{\sqrt{-1}}{2} \partial\bar{\partial} \log(1 + z_1\bar{z}_1 + z_2\bar{z}_2).$$

Moreover,

$$r_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log\det(g_{k\bar{l}}).$$

and

$$Ric = \sqrt{-1} \sum_{i,j=1}^{2} r_{i\bar{j}} dz_i \wedge d\bar{z}_j.$$

Finally, note that the above definitions (which are the ones I gave in class) may differ slightly from the ones you may be used to. Thus, the numerical values, e.g., of the scalar curvature, may also be slightly different from the ones you may be used to.

6. (10 points) Let (M, h) be a Kähler manifold such that at a point $p \in M$, there exist coordinates $z_1, \ldots, z_n$ such that

$$\omega_h = \frac{\sqrt{-1}}{2} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$$

and

$$Ric = \sqrt{-1} \sum_{i=1}^{n} r_{i\bar{i}} dz_i \wedge d\bar{z}_i.$$

Prove that at the point p,

$$Ric \wedge \omega_h^{n-1} = \frac{2}{n} s \omega_h^n.$$