Asymptotic Analysis -1 (2020) 1–8
DOI 10.3233/ASY-201623
IOS Press

On the approximate controllability of parabolic problems with non-smooth coefficients

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Abstract. In this paper we prove the existence of approximate controls for certain classes of parabolic problems with non-smooth coefficients and discuss as examples the problem of approximate controllability for the heat flow in heterogeneous media such as, periodic composites, perforated domains or periodic microstructures separated by rough interfaces.

Keywords: Parabolic equations, approximate control, non-smooth coefficients, homogenization

1. Introduction

Interior approximate controllability of heat flows through various materials within complex given geometrical settings is of paramount practical importance [7–9,11,15]. As an example, let us introduce $\Omega \subset \mathbb{R}^N$ bounded open set with smooth boundary, $\omega \subset \Omega$ a non empty open subset and consider A a symmetric $N \times N$-matrix field in $\mathcal{M}(\alpha, \beta, \Omega)$, that is,

$$\begin{cases} (i) \quad A \in (L^\infty(\Omega))^{N^2} \quad \text{and} \quad a_{ij} = a_{ji}, 1 \leqslant i, j \leqslant N, \\ (ii) \quad (A(x)\lambda, \lambda) \geqslant \alpha|\lambda|^2, \quad |A(x)\lambda| \leqslant \beta|\lambda|, \end{cases} \tag{1.1}$$

for every $\lambda \in \mathbb{R}^N$ and a.e. in Ω where $\alpha, \beta \in \mathbb{R}$ with $0 < \alpha < \beta$.

In the sequel we denote by $'$ the first time-derivative. The classical linear parabolic interior control problem reads:

Let δ be a real number with $0 < \delta < 1$ and $z \in L^2(\Omega)$ a given target function. Given $f \in L^2((0, T) \times \Omega)$ and $u^0 \in L^2(\Omega)$, determine $v \in L^2((0, T) \times \omega)$ such that the solution u in $C([0, T], L^2(\Omega))$ of the

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following problem

$$\begin{cases} u' - \operatorname{div}(A\nabla u) = f + v\chi_\omega & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u^0 & \text{in } \Omega, \end{cases}$$

satisfies

$$\|u(T) - z\|_{L^2(\Omega)} \leqslant \delta.$$

The question of interior approximate controllability for the classical linear heat equation has been thoroughly studied in the literature ([18,26] and references therein). It is well known that, in the case of time-independent coefficients A, this question is equivalent to the question of unique continuation for elliptic problems. The latter question has been answered in the affirmative in two dimensions in [3] for the classical elliptic divergence operator with measurable coefficients) (see also [1] for more general linear elliptic operators and bounded coefficients). In three or higher dimensions it is known to hold only for Lipschitz coefficients [14] (see also [24] with counterexamples (for less smooth coefficients) provided for instance in [19,20,23].

In the control literature, there exist several papers studying the control of parabolic problems with nonsmooth coefficients. In this context, null controllability for non smooth coefficients was addressed for the 1-D case in papers [13,16,27] while null and exact controllability for higher dimensional case with piece-wise smooth coefficients was discussed in [12,17].

In this paper we will provide a general strategy for obtaining approximate controls for parabolic problems via periodic approximations. In this regard, we will first prove our strategy for the approximate control of the classical linear parabolic problem with non-smooth coefficients in Theorem 2.1 (in particular Corollary 2.2 presents the proof of approximate control of periodic microstructures) and then state the general control scheme and its application for the approximate controllability of multi-scale parabolic problems considered in [7–11,15] in the general case of non-smooth coefficients.

2. The classical problem

In this section, we will present our ideas in the context of the classical diffusion problem. Thus, the next result describes a general scheme for computing approximate controls in the context of general linear parabolic problems with non-smooth coefficients. We have,

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with smooth boundary, $\omega \subset \Omega$ a non-empty open subset, and $Y \subset \mathbb{R}^N$ a cell with the paving property in $\mathbb{R}^N$. Assume that $\{\varepsilon\}$ is a parameter taking its values in a sequence of positive real numbers that converges to zero.*

Let δ be a real number with $0 < \delta < 1$ and $z \in L^2(\Omega)$ a given target function. For $f \in L^2((0,T)\times\Omega)$, $u^0 \in L^2(\Omega)$, and $A \in \mathcal{M}(\alpha,\beta,\Omega)$ a given symmetric $N \times N$-matrix field as defined in (1.1), consider the following problem

$$\begin{cases} u' - \operatorname{div}(A\nabla u) = f & \textit{in } \Omega \times (0,T), \\ u = 0 & \textit{on } \partial\Omega \times (0,T), \\ u(x,0) = u^0 & \textit{in } \Omega. \end{cases}$$

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Suppose that, for some $\alpha_B, \beta_B \in \mathbb{R}$ with $0 < \alpha_B < \beta_B$, there exists $v_0 \in L^2(\Omega)$ and a Y-periodic matrix field $B \in \mathcal{M}(\alpha_B, \beta_B, Y)$, such that the sequence of matrices $B^\epsilon(x) = B(\frac{x}{\epsilon})$ has the property that for any $f \in L^2((0,T) \times \Omega)$, there exists $\{u_\epsilon^0\}_\epsilon \subset L^2(\Omega)$ (depending on f, B and v_0) with $u_\epsilon^0 \to v_0$ strongly in $L^2(\Omega)$, such that the solution u_ϵ of

$$\begin{cases} u_\epsilon' - \operatorname{div}(B^\epsilon \nabla u_\epsilon) = f & \textit{in } \Omega \times (0,T), \\ u_\epsilon = 0 & \textit{on } \partial\Omega \times (0,T), \\ u_\epsilon(x,0) = u_\epsilon^0 & \textit{in } \Omega, \end{cases}$$

verifies, for some $\epsilon_0 > 0$,

$$\left\| u_\epsilon(\cdot,T) - u(\cdot,T) \right\|_{L^2(\Omega)} \leqslant \frac{\delta}{3} \quad \textit{for all } \epsilon < \epsilon_0. \tag{2.1}$$

Then there exists a control $v \in L^2((0,T) \times \omega)$ such that solution of the following problem

$$\begin{cases} p' - \operatorname{div}(A \nabla p) = f + \chi_\omega v & \textit{in } \Omega \times (0,T), \\ p = 0 & \textit{on } \partial\Omega \times (0,T), \\ p(x,0) = u^0 & \textit{in } \Omega, \end{cases}$$

verifies

$$\left\| p(\cdot,T) - z \right\|_{L^2(\Omega)} \leqslant \delta.$$

Proof. For every function $\psi \in L^2(Y)$ define its average $M_Y(\psi) = \frac{1}{|Y|}\int_Y \psi\, dy$ where $|Y|$ denotes the measure of the set Y. Next consider the space of periodic functions with mean zero defined by $W_{\text{per}}(Y) = \{\psi \in H^1_{\text{per}}(Y), M_Y(\psi) = 0\}$. Let B_0 be the homogenized matrix corresponding to B_ϵ (see [2,6]), i.e.,

$$B_0 \lambda = M_Y(B \nabla q_\lambda), \quad \text{for every } \lambda \in \mathbb{R}^N, \tag{2.2}$$

where w_λ solves

$$\begin{cases} \int_Y B \nabla q_\lambda \nabla \psi = 0, & \text{for all } \psi \in W_{\text{per}}(Y), \\ (q_\lambda - \lambda \cdot y) \in W_{\text{per}}(Y). \end{cases} \tag{2.3}$$

Classical controllability results (see [25,26]) imply there exists $\varphi_0 \in L^2(\Omega)$ such that the solution of

$$\begin{cases} w_0' - \operatorname{div}(B_0 \nabla w_0) = f + \chi_\omega \varphi_0 & \text{in } \Omega \times (0,T), \\ w_0 = 0 & \text{on } \partial\Omega \times (0,T), \\ w_0(x,0) = v_0 & \text{in } \Omega, \end{cases} \tag{2.4}$$

verifies

$$\left\| w_0(\cdot,T) - z \right\|_{L^2(\Omega)} \leqslant \frac{\delta}{3}. \tag{2.5}$$

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Consider now the problem

$$\begin{cases} p' - \operatorname{div}(A\nabla p) = f + \chi_\omega \varphi_0 & \text{in } \Omega \times (0,T), \\ p = 0 & \text{on } \partial\Omega \times (0,T), \\ p(x,0) = u^0 & \text{in } \Omega. \end{cases}$$

The hypothesis implies that there exists $\{\xi_0^\epsilon\}_\epsilon \subset L^2(\Omega)$ with $\xi_0^\epsilon \to v_0$ strongly in $L^2(\Omega)$ such that the solutions of

$$\begin{cases} w_\epsilon' - \operatorname{div}(B^\epsilon \nabla w^\epsilon) = f + \chi_\omega \varphi_0 & \text{in } \Omega \times (0,T), \\ w_\epsilon = 0 & \text{on } \partial\Omega \times (0,T), \\ w_\epsilon(x,0) = \xi_0^\epsilon & \text{in } \Omega, \end{cases} \tag{2.6}$$

satisfy, for some $\epsilon_1 > 0$, the inequality

$$\left\| w_\epsilon(\cdot,T) - p(\cdot,T) \right\|_{L^2(\Omega)} \leqslant \frac{\delta}{3} \quad \text{for all } \epsilon < \epsilon_1. \tag{2.7}$$

Note that classical homogenization theory and correctors ([6]) implies that problem (2.4) is the homogenized limit of the sequence of problems (2.6) and one has the following corrector results

$$\left\| w_\epsilon(\cdot,T) - w_0(\cdot,T) \right\|_{C([0,T];L^2(\Omega))} \to 0.$$

This implies that there exists $\epsilon_2 > 0$ such that

$$\left\| w_\epsilon(\cdot,T) - w_0(\cdot,T) \right\|_{L^2(\Omega)} \leqslant \frac{\delta}{3} \quad \text{for all } \epsilon < \epsilon_2. \tag{2.8}$$

From (2.5), (2.7), (2.8) we conclude that

$$\left\| p(\cdot,T) - z \right\|_{L^2(\Omega)} \leqslant \delta,$$

and so $v = \varphi_0$ satisfies the conclusion of the theorem. $\square$

The next corollary shows that one can prove and compute approximate controls for parabolic problems associated with given micro-structures as long as their characteristic length is small enough, depending on the desired level of control accuracy. Indeed we have,

Corollary 2.2. *Let δ be a real number with $0 < \delta < 1$, $z \in L^2(\Omega)$ a given target function and $f \in L^2((0,T) \times \Omega)$. Suppose that $\{\eta\}$ is a parameter taking its values in a sequence of positive real numbers that converges to zero, and $\{y_\eta^0\}_\eta \subset L^2(\Omega)$ a sequence such that $y_\eta^0 \to y_0$ strongly in $L^2(\Omega)$ for some $y_0 \in L^2(\Omega)$.*

Also consider a sequence of given matrices $D^\eta(x) = D(\frac{x}{\eta})$, with D being Y-periodic, $D \in \mathcal{M}(\alpha_D, \beta_D, Y)$ for some $\alpha_D, \beta_D \in \mathbb{R}$ with $0 < \alpha_D < \beta_D$.

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Then there exists a number $0 < \eta_0 < 1$ and a function φ_0 independent of $\eta < \eta_0$ such that for any $\eta < \eta_0$ the solution of

$$\begin{cases} y'_\eta - \operatorname{div}(D^\eta \nabla y_\eta) = f + \chi_\omega \varphi_0 & \textit{in } \Omega \times (0,T), \\ y_\eta = 0 & \textit{on } \partial\Omega \times (0,T), \\ y_\eta(x,0) = y^0_\eta & \textit{in } \Omega, \end{cases} \tag{2.9}$$

satisfies

$$\left\| y_\eta(\cdot,T) - z \right\|_{L^2(\Omega)} \leqslant \delta \quad \textit{for all } \eta < \eta_0. \tag{2.10}$$

Proof. The proof of the corollary follows as a consequence of Theorem 2.1, written for $B = D$ and $u = y_\eta$, but here we present a shorter more direct solution. Let D_0 be the homogenized matrix defined by (2.2) and (2.3) with matrix B replaced by matrix D. Classical control results imply that there exists a control function $\varphi_0 \in L^2(\Omega)$ such that the solution of

$$\begin{cases} y' - \operatorname{div}(D_0 \nabla y) = f + \chi_\omega \varphi_0 & \text{in } \Omega \times (0,T), \\ y = 0 & \text{on } \partial\Omega \times (0,T), \\ y(x,0) = y_0 & \text{in } \Omega, \end{cases}$$

satisfies

$$\left\| y(\cdot,T) - z \right\|_{L^2(\Omega)} \leqslant \frac{\delta}{2}. \tag{2.11}$$

On the other hand, the classical corrector results applied to the sequence of problems (2.9) imply that there exists $0 < \eta_0 < 1$ such that

$$\left\| y_\eta(\cdot,T) - y(\cdot,T) \right\|_{L^2(\Omega)} \leqslant \frac{\delta}{2} \quad \text{for all } \eta < \eta_0. \tag{2.12}$$

Finally, from (2.11) and (2.12), we obtain (2.10). □

Next we make the observation that our results presented above can be easily adapted to remain true (under certain assumptions to be described below) for the case when the periodicity assumption is replaced by the more general concept of H-convergence.

Remark 2.3. We recall (see [21,22]) that a sequence $\{A^\epsilon\}_\epsilon \subset \mathcal{M}(\alpha,\beta,\Omega)$ H-converges to $A_0 \in M(\alpha',\beta',\Omega)$ (for some α', β' with $0 < \alpha' < \beta'$) iff for every function $h \in H^{-1}(\Omega)$ the solution ξ_ϵ of

$$\begin{cases} -\operatorname{div}(A^\epsilon \nabla \xi_\epsilon) = h & \text{in } \Omega, \\ \xi_\epsilon = 0 & \text{on } \partial\Omega \end{cases}$$

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is such that

$$\begin{cases} i) \quad \xi_\epsilon \rightharpoonup \xi_0 & \text{weakly in } H_0^1(\Omega) \\ ii) \quad A^\epsilon \nabla \xi_\epsilon \rightharpoonup A_0 \nabla \xi_0 & \text{weakly in } (L_2(\Omega))^N, \end{cases}$$

where ξ_0 is the unique solution of the problem

$$\begin{cases} -\operatorname{div}(A_0 \nabla \xi_0) = h & \text{in } \Omega, \\ \xi_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

By using the corrector results for the classical parabolic equations associated to an H-convergent matrix of coefficients (see [4]) and the unique continuation property for parabolic problems with C^1 coefficients (see [24]), the results of Theorem 2.1 and Corollary 2.2 remain true (with identical proofs) if, while still requiring that the approximation property (2.1) holds true, instead of periodicity of the sequences $\{B_\epsilon\}_\epsilon$ and $\{D_\eta\}_\eta$ one assumes that they are H-convergent with smooth (i.e., C^1) limits.

3. General strategy

In this section, we will state our strategy for the approximate control of general linear parabolic problems in heterogeneous media occupying possibly complex geometries (e.g. perforated domains, domains with inclusions or materials separated by interfaces). Thus consider parameter δ, Ω, ω and target function $z \in L^2(\Omega)$ as above and denote by $\Gamma \subset \mathbb{R}^{N-1}$ a possible connected interface separating Ω in two components or a disconnected set describing the boundary of perforations. In what follows, we will consider a generic parabolic flow $\{\mathcal{P}\}$ associated to a general source $f \in L^2((0,T)\times\Omega)$ and initial condition u^0. For this problem we assume a general heterogeneous media with given boundary conditions $\{\mathcal{BC}\}$ on $\partial\Omega$, for instance Dirichlet conditions and, if an interface Γ is considered as part of the geometry (i.e., as described above), with possibly interface conditions $\{\mathcal{IC}\}$ prescribed on Γ, e.g., flux-temperature proportionality conditions, (see [7–9,11,15]). We can also treat the case of a perforated domain, where one can assume for instance homogeneous or nonhomogeneus Robin conditions (in particular Neumann conditions, see [11]) on the boundaries of the holes. In this case, in the statement below, the fixed domain Ω has to be replaced by a varying one, with the obvious modifications, and for a nonperiodic setting one can use the H-convergence extension to perforated domains (see [5]).

We have:

Theorem 3.1. *Consider problem $\{\mathcal{P}\}$, $\{\mathcal{BC}\}$ or, if an interface is part of the geometry, problem $\{\mathcal{P}\}$, $\{\mathcal{BC}\}$, $\{\mathcal{IC}\}$ and assume that each of these two problems admits a unique solution, denoted generically by u. Assume that there exist a function v_0 and a sequence of well posed periodic parabolic flow problems $\{\mathcal{P}_\epsilon\}$, $\{\mathcal{BC}_\epsilon\}$ or $\{\mathcal{P}_\epsilon\}$, $\{\mathcal{BC}_\epsilon\}$, $\{\mathcal{IC}_\epsilon\}$ (with solution generically denoted by u_ϵ), with the property that for any associated source $f \in L^2((0,T)\times\Omega)$ there exists initial condition $u_\epsilon^0 \to v_0$ (where the convergence holds in appropriate strong topologies depending on the problem considered) such that u_ϵ satisfies:*

There exists ϵ_0, ϵ_1 positive parameters such that

$$\begin{cases} \text{1. } \underline{\text{Approximation property:}} \quad \|u_\epsilon(\cdot,T) - u(\cdot,T)\|_{L^2(\Omega)} \leqslant \frac{\delta}{3} \text{ for all } \epsilon < \epsilon_0, \\ \text{2. } \underline{\text{Corrector property:}} \quad \|u_\epsilon(\cdot,T) - u_h(\cdot,T)\|_{L^2(\Omega)} \leqslant \frac{\delta}{3} \text{ for all } \epsilon < \epsilon_1, \end{cases}$$

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where u_h solves the limit homogenized problem $\{\mathcal{P}_h\}$, $\{\mathcal{BC}_h\}$ or $\{\mathcal{P}_h\}$, $\{\mathcal{BC}_h\}$, $\{\mathcal{IC}_h\}$ associated to the same source $f \in L^2((0,T)\times\Omega)$ and initial condition v_0. If the limit homogenized problem $\{\mathcal{P}_h\}$, $\{\mathcal{BC}_h\}$ or $\{\mathcal{P}_h\}$, $\{\mathcal{BC}_h\}$, $\{\mathcal{IC}_h\}$ admits an approximate control $\varphi_h \in L^2(\omega)$ then the initial problem $\{\mathcal{P}\}$, $\{\mathcal{BC}\}$ or $\{\mathcal{P}\}$, $\{\mathcal{BC}\}$, $\{\mathcal{IC}\}$ can be approximately controlled by $\varphi_h \in L^2(\omega)$.

Proof. The proof follows the identical steps as in Theorem 2.1. $\square$

Remark 3.2. Theorem 3.1 implies the possibility to extend the approximate control results for the parabolic problems considered in [7,8,11,15] to the general case of non-smooth coefficients (i.e., general heterogeneous materials). We also mention that, for the multiscale parabolic flow through a connected interface considered in [9], assuming approximate controllability of the proposed limit problem (which can be proved by adaptation of standard control techniques (i.e., adaptation of the HUM method proposed in [18])), the limit analysis and corrector results obtained in [9] together with Theorem 3.1 imply the possibility to characterize approximate controls for the initial multiscale parabolic flow.

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Acknowledgements

This research was supported in part by the International Centre for Theoretical Sciences (ICTS) during a visit for participating in the program – Multi-scale Analysis and Theory of Homogenization (Code: ICTS/math2019/08).

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