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THE UNFOLDING OPERATOR NEAR A HYPERPLANE AND ITS APPLICATIONS TO THE NEUMANN SIEVE MODEL

Daniel Onofrei
Mathematical Sciences Department, Worcester Polytechnic Institute
100 Institute Road, Worcester, MA, 01609, USA.
(onofrei@wpi.edu)

Abstract. Following the ideas in [4] (see also [6]) we introduce the bl-Unfolding Operator, and describe some of its most important properties. Using the bl-Unfolding Operator we present a new proof of the classical Neuman sieve model and provide the limit analysis of the thick Neumann sieve model.

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1 Introduction

The Periodic Unfolding Method, was first introduced in [4], (see also [6]), for the homogenization of problems in fixed as well as in variable domains, whenever one can assume the scale separation.

The Method is operatorial in essence, and based on the properties of the Unfolding Operator. The Unfolding Operator, depends on the structure of the problem to be analyzed, and is defined as

$$\mathcal{T}_\epsilon : L^2(\tilde{\Omega}_\epsilon) \to L^2(\tilde{\Omega}_\epsilon \times Y)$$

where $\tilde{\Omega}_\epsilon = \bigcup_{\xi \in \Xi_\epsilon} (\epsilon\xi + \epsilon Y)$ and $\Xi_\epsilon = \{\xi \in \mathbb{Z}^N; (\epsilon\xi + \epsilon Y) \cap \Omega \neq \emptyset\}$. One of the main properties of the Unfolding Operator is that it replace, integrals on Ω, with integrals on the product space $\Omega \times Y$ and weak convergence by strong convergence.

In this paper we present the limit analysis for the classic Neumann Sieve model and the thick Neumann Sieve model. The geometry of the model is composed of a domain Ω cut in two parts by a hyperplane Σ which, for the simplicity of the exposition is assumed to be a subset of the plane $\Pi = \{x_N = 0\}$. A periodical 2-dimensional network of size ϵ is considered on Σ, and an open set (hole in the sieve) is brought by scaling of ratio $\delta\epsilon$ in each cell of the network, where $\delta \doteq \delta(\epsilon)$. Let the reunion of all the barriers be denoted by T_ϵ. Then, the barriers T_ϵ are considered part of the domain and Neumann homogenous boundary condition are imposed on $\Sigma \setminus T_\epsilon$. When the Sieve has a certain thickness $h(\epsilon) > 0$ we have the thick Neumann Sieve model. We will only consider in this paper the case when $h(\epsilon) \leq \epsilon$ the other situations being trivial. Depending on the limit behavior of the ratio $\dfrac{\delta^{N-2}}{\epsilon}$ we obtain different limit equations. In order to obtain the limit problems for these models we define the bl-Unfolding Operator, which characterizes the geometry of the models, and acts only on a thin layer of size ϵ around the hyperplane Σ. We present a few of its most important properties and apply the Periodic Unfolding Method to complete our limit analysis. The homogenization of the classical Neumann Sieve was discussed by many authors, see ([5], [11], [1], [18],[12]). The ϵ-problem can be expressed in a variational form as,

$$\int_{\Omega_\epsilon} \nabla u_\epsilon \nabla \psi dx = \int_{\Omega_\epsilon} f\psi \quad \text{for all} \quad \psi \in V_\epsilon$$

where $\Omega_\epsilon = \Omega \setminus \{\Sigma \setminus T_\epsilon\}$. Let $\mathbb{R}^N_+ = \{(x', x_N)\ , x_N > 0\}$ and similarly define $\mathbb{R}^N_-$. If we set $\Omega_+ = \Omega \cap \mathbb{R}^N_+$ and similarly for Ω_-, then we can write $V_\epsilon = \{v \in H^1(\Omega_+ \cup \Omega_-)\ ; v = 0 \text{ on } \partial\Omega\ ; [v] = 0 \text{ on } T_\epsilon\}$, where $[v] \doteq v^+|_\Sigma - v^-|_\Sigma$ for every function $v \in H^1(\Omega_+ \cup \Omega_-)$. The solution u_ϵ converges weakly in $H^1(\Omega_+ \cup \Omega_-)$ to a limit function u_0. It is well-known that u_0 satisfies the following problem,

$$\begin{cases} -\Delta u_0 = f & \text{on } \Omega_+ \cup \Omega_- \\ \dfrac{\partial u_0^+}{\partial n^+} = -\dfrac{\partial u_0^-}{\partial n^-} = \dfrac{k^2}{4|Y|\theta(T)}[u_0] & \text{on } \Sigma \end{cases} \tag{1}$$

where n^+ and n^- are the normals to Σ towards Ω_- and Ω_+ respectively and $\frac{1}{\theta(T)}$ equals the usual capacity of the set T. The nonlinear Neumann Sieve was recently discussed

([2]) where Γ-convergence techniques are used.
In Section 3 we obtain the unfolded limit problem for this model and as a simple consequence of this result we recover problem (1) for u_0. Although the limit problem for u_0 is well-known in the mathematical community, our proof provides the unfolded formulation for the limit problem which contain in it the problem for u_0 together with useful corrector information. The idea of the proof is new and highlights the elegance of the Periodic Unfolding Method, when one uses the Unfolding Operator which is well adapted to the model.
The homogenization of the thick Neumann Sieve was studied in [10], for the case of an uniform Sieve (i.e, it has an uniform thickness), symmetric with respect to Σ. Our model is different then the one proposed in [10] in the sense that the sieve is nonuniform, but we suppose that domain has the extension property as defined in [7](this is not the case in [10]). To analyze the case of symmetric thick sieve we use similar arguments as in Section 3.
The paper is organized as follows: In Section 2 we present the basic properties of the Unfolding Operator. In Section 3 we introduce the functional setting for our models and present the bl-Unfolding Operator with its most important properties. In Section 4 we study the homogenization of the Neumann Sieve Model. The thick Neumann Sieve model is presented in Section 5. As an application we will show how the bl-Unfolding Operator can be used to pass to the limit in both models, the classical Neumann sieve model and the thick Neumann sieve model. The case of non-symmetric sieve with variable coefficients, will be discussed in a forthcoming paper.

2 General notations and properties for the Unfolding Operator

In the beginning of this section we fix a few notations which will be used frequently in the rest of the paper. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set, Y be an open cube centered in the origin of $\mathbb{R}^N$, $Y_N = \{y \in \bar{Y}; y_N = 0\}$, $T \subset\subset Y_N$ and $B \subset\subset Y$. Let $\epsilon > 0$, $\delta \doteq \delta(\epsilon) > 0$, be two small parameters, and define

$$k = \lim_{\epsilon \to 0} \frac{\delta^{\frac{N-2}{2}}}{\epsilon^{\frac{1}{2}}}. \tag{2}$$

Consider $\Xi_\epsilon = \{\xi \in \mathbb{Z}^N; (\epsilon\xi + \epsilon Y) \cap \Omega \neq \emptyset\}$ and define $\tilde{\Omega}_\epsilon = \bigcup_{\xi \in \Xi_\epsilon} (\epsilon\xi + \epsilon Y)$.

Let $M \subset \mathbb{R}^N$ and $A \subset \mathbb{R}^N$. We define the capacity space corresponding to the bounded set $M \subset \mathbb{R}^N \setminus A$ with respect to $\mathbb{R}^N \setminus A$ to be,

$$H^{1,*}_M(\mathbb{R}^N \setminus A) = \{\Phi \in L^{2^*}(\mathbb{R}^N \setminus A)\,;\, \nabla_z \Phi \in [L^2(\mathbb{R}^N \setminus A)]^N \text{ and } \Phi(z) \text{ constant on } M\}. \tag{3}$$

Define also the space $K^2(\mathbb{R}^N)$ as:

$$K^2(\mathbb{R}^N) = \{\Phi \in L^{2^*}(\mathbb{R}^N \setminus A)\,;\, \nabla_z \Phi \in [L^2(\mathbb{R}^N \setminus A)]^N\}.$$

One can consult [14] for more properties of these spaces. Now, if we have a periodic net on $\mathbb{R}^N$ with period Y, by analogy with the one-dimensional case, to each $x \in \mathbb{R}^N$ we can associate its integer part, $[x]_Y$, such that $x - [x]_Y \in Y$ and its fractional part respectively, i.e, $\{x\}_Y = x - [x]_Y$. Therefore we have:

$$x = \epsilon \left\{ \frac{x}{\epsilon} \right\}_Y + \epsilon \left[\frac{x}{\epsilon} \right]_Y \quad \text{for any } x \in \mathbb{R}^N.$$

Next, consider the hyperplane $\Pi = \{x_N = 0\}$ and define $\Sigma = \Pi \cap \Omega$. For any given set $A \subset \mathbb{R}^N$, we will systematically use the notation A_+ for $\mathbb{R}^N_+ \cap A$ and A_- for $\mathbb{R}^N_- \cap A$ manner. For the simplicity of the exposition we will make the convention that all the results stated for A_+, are true also for A_- unless specified otherwise. We denote by n^+ the normal to Σ oriented toward Ω_-.
For a function u, by u^+ we denote it's restriction to the domain Ω_+, i.e., $u^+ \equiv u|_{\Omega_+}$.Let $[.]$ denote the jump over Σ, i.e, $[u] = u^+_\Sigma - u^-_\Sigma$.
The letter c will denote a positive constant independent of any small parameter, otherwise specified. Let $M \subset \mathbb{R}^N$ and $S \subset M$. By $cap_2(S, M)$ we denote the classical 2-capacity of the set S with respect to M, see [14].
Next we will recall the definition of the unfolding operator, as it have been introduced in ([4]) and review a few of its principal properties. Let the unfolding operator be defined as $\mathcal{T}_\epsilon : L^2(\tilde{\Omega}_\epsilon) \to L^2(\tilde{\Omega}_\epsilon \times Y)$ with

$$\mathcal{T}_\epsilon(\phi)(x,y) = \phi(\epsilon \left[\frac{x}{\epsilon} \right]_Y + \epsilon y) \quad \text{for all} \quad \phi \in L^2(\tilde{\Omega}_\epsilon).$$

We have (see [4]):

Theorem 2.1. *For any $v, w \in L^2(\Omega)$ we have*

1.

$$\mathcal{T}_\epsilon(vw) = \mathcal{T}_\epsilon(v)\mathcal{T}_\epsilon(w)$$

2.

$$\nabla_y \left(\mathcal{T}_\epsilon(u)\right) = \epsilon \mathcal{T}_\epsilon(\nabla_x u) \quad \textit{where} \quad u \in H^1(\Omega)$$

3.

$$\int_\Omega v dx = \frac{1}{|Y|} \int_{\tilde{\Omega}_\epsilon \times Y} \mathcal{T}_\epsilon(v) dx dy$$

4.

$$\left| \int_\Omega v dx - \int_{\Omega \times Y} \mathcal{T}_\epsilon(v) dx dy \right| < |v|_{L^1(\{x \in \tilde{\Omega}_\epsilon;\, dist(x, \partial\Omega) < \sqrt{(n)}\epsilon\})}$$

5.

$$\mathcal{T}_\epsilon(w) \to w \quad \textit{strongly in} \quad L^2(\Omega \times Y)$$

6. Let $\{w_\epsilon\} \subset L^2(\Omega)$ such that $w_\epsilon \to w$ in $L^2(\Omega)$. Then

$$\mathcal{T}_\epsilon(w_\epsilon) \to w \textit{ in } L^2(\Omega \times Y)$$

7. Let $w_\epsilon \rightharpoonup w$ in $H^1(\Omega)$. Then there exists a subsequence still denoted by ϵ and $\hat{w} \in L^2\left(\Omega; H^1_{per}(Y)\right)$ such that:

$$a)\ \mathcal{T}_\epsilon(w_\epsilon) \rightharpoonup w \text{ in } L^2(\Omega; H^1(Y))$$

$$b)\ \mathcal{T}_\epsilon(\nabla w_\epsilon) \rightharpoonup \nabla_x w + \nabla_y \hat{w} \text{ in } L^2(\Omega \times Y).$$

See [4], [6] for the proofs and other properties of the Unfolding operator $\mathcal{T}_\epsilon$.

3 The bl-Unfolding operator

3.1 The functional setting

Let $\Omega \subset \mathbb{R}^N$, $\Omega_{+,-}$, Σ be defined as in Section 2. Define the set of the holes in the sieve T_ϵ to be :

$$T_\epsilon = \left[\bigcup_{\xi \in \mathbf{Z}^{\mathbf{N-1}} \times \{0\}} \{\xi\epsilon + \epsilon\delta T\}\right] \cap \Omega.$$

Define $\Omega_\epsilon = \Omega_+ \cup \Omega_- \cup T_\epsilon$ and $\Sigma^\epsilon = \Omega_\epsilon \cap \{x; |x_N| < \frac{\epsilon}{2}\}$.

In this section, following the idea presented in [4] we will introduce the bl-Unfolding Operator which is designed to capture the contribution of the barriers in the limit process.

We need some more notations. Let $Y = \left]-\frac{1}{2}, \frac{1}{2}\right[$ and let $Y^\delta = Y_+ \cup Y_- \cup \delta T$.
Define $A_\epsilon = \{\xi \in \mathbf{Z}^{N-1} \times \{0\}; (\epsilon\xi + \epsilon Y^\delta) \cap \Omega \neq \emptyset\} = \Xi_\epsilon \cap \Pi$ and $\tilde{\Sigma}^\epsilon = \bigcup_{\xi \in A_\epsilon} \{\epsilon\xi + \epsilon Y^\delta\}$.
Consider also $B_\epsilon = \{\xi \in \mathbf{Z}^N; \xi_N \neq 0 \text{ and } (\epsilon\xi + \epsilon Y) \cap \Omega \neq \emptyset\}$ and let $\tilde{\Omega}'_\epsilon = \bigcup_{\xi \in B_\epsilon} \{\epsilon\xi + \epsilon Y\}$.
Let $\tilde{\Omega}^*_\epsilon = \tilde{\Sigma}^\epsilon \cup \tilde{\Omega}'_\epsilon$ and $\tilde{\Omega}_+ = \tilde{\Omega}^*_\epsilon \cap \mathbb{R}^N{}_+$.

The bl-Unfolding Operator, corresponding to the Sieve model, (see Figure 1), is defined as, $\mathcal{T}_{\epsilon,\delta} : L^2(\tilde{\Omega}^*_\epsilon) \to L^2(\Omega \times \mathbb{R}^N)$ with

$$\mathcal{T}_{\epsilon,\delta}(\phi)(x,z) = \begin{cases} \phi(\epsilon\left[\frac{x'}{\epsilon}\right]_Y + \epsilon\delta z) & \text{if } \delta z \in Y^\delta \text{ and } |x_N| < \frac{\epsilon}{2} \\ 0 & \text{if } \delta z \in \mathbb{R}^N \setminus Y^\delta \text{ or } |x_N| \geq \frac{\epsilon}{2}. \end{cases} \tag{4}$$

Following the proof of Theorem 2.1 (see [4], [6]) one can show that properties similar to 1, 2, 3, and 4, hold for the bl-Unfolding Operator, $\mathcal{T}_{\epsilon,\delta}$ (see Theorem 3.1 bellow).
Next we define the spaces

$$V = \{v \in H^1(\Omega_+ \cup \Omega_-);\ v = 0 \text{ on } \partial\Omega\}$$

$$V^\epsilon = \{v \in V, [v] = 0 \text{ on } T_\epsilon\}.$$

V is a Hilbert space with the scalar product defined by

$$< u, v >_V = \int_{\Omega_+ \cup \Omega_-} \nabla u \cdot \nabla v \quad \text{for all} \quad u, v \in V.$$

From now on we will only consider the case where k as defined in (2) satisfies:

$$0 \leq k < \infty. \tag{5}$$

3.2 Estimates for the bl-Unfolding Operator

In order to obtain the estimates we will use the following identity:

Theorem 3.1. *For all* $\phi \in L^2(\Omega_+)$ *we have that,*

$$\frac{\delta^N}{|Y|}\int_{\bar{\Omega}_+\times\mathbb{R}^N_+} \mathcal{T}_{\epsilon,\delta}(\phi)(x,z)\,dxdz = \frac{1}{2}\int_{\Sigma^\epsilon_+} \phi\,dx.$$

Proof. We successively have

$$\frac{\delta^N}{|Y|}\int_{\bar{\Omega}_+\times\mathbb{R}^N_+} \mathcal{T}_{\epsilon,\delta}(\phi)(x,z)dxdz = \frac{\delta^N}{|Y|}\int_{\hat{\Sigma}^\epsilon_+\times\mathbb{R}^N_+} \mathcal{T}_{\epsilon,\delta}(\phi)(x,z)dxdz =$$

$$= \tfrac{\delta^N}{|Y|}\textstyle\int_{\hat{\Sigma}^\epsilon_+\times(\frac{1}{\delta})Y_+} \phi(\epsilon\left[\frac{x'}{\epsilon}\right]_Y + \epsilon\delta z)dxdz = \frac{1}{|Y|}\int_{\hat{\Sigma}^\epsilon_+\times Y_+} \phi(\epsilon\left[\frac{x'}{\epsilon}\right]_Y + \epsilon y)dxdy =$$

$$= \frac{1}{|Y|}\sum_{\xi\in A_\epsilon}\int_{Y_+} dy \int_{\epsilon\xi+\epsilon Y_+} \phi(\epsilon\left[\frac{x'}{\epsilon}\right]_Y + \epsilon y)dx = \frac{1}{|Y|}|\epsilon\xi + \epsilon Y_+|\sum_{\xi\in A_\epsilon}\int_{Y_+}\phi(\epsilon\xi + \epsilon y)dy =$$

$$= \frac{1}{2}\epsilon^N \sum_{\xi\in A_\epsilon}\int_{Y_+}\phi(\epsilon\xi+\epsilon y)dy = \frac{1}{2}\int_{\Sigma^\epsilon_+}\phi(x)dx,$$ where , in the last two equalities above, we have used that $\phi(\epsilon\left[\frac{x'}{\epsilon}\right]_Y + \epsilon y) = \phi(\epsilon\xi + \epsilon y)$ for all $x \in \epsilon\xi + \epsilon Y_+$, and an obvious change of variables. □

Corollary 3.2. *For all* $\phi \in L^2(\Omega_+)$ *we have that*

$$\frac{\epsilon\delta^N}{|Y|}\int_{\Sigma\times\mathbb{R}^N_+} |\mathcal{T}_{\epsilon,\delta}(\phi)(x',z)|^2 dx'dz \le \int_{\Sigma^\epsilon_+} |\phi|^2 dx.$$

Proof. Note that

$$\frac{\epsilon\delta^N}{2|Y|}\int_{\Sigma\times\mathbb{R}^N_+} |\mathcal{T}_{\epsilon,\delta}(\phi)(x',z)|^2 dx'dz \le \frac{\delta^N}{|Y|}\int_{\hat{\Sigma}^\epsilon_+\times\mathbb{R}^N_+} |\mathcal{T}_{\epsilon,\delta}(\phi)(x',z)|^2 dxdz$$

and apply Theorem 3.1. □

From Corollary 3.2 we can obtain that for $\phi \in H^1(\Omega_+)$ we have:

$$\frac{\epsilon\delta^N}{|Y|}\int_{\Sigma\times\mathbb{R}^N_+} |\mathcal{T}_{\epsilon,\delta}(\nabla_x\phi)(x',z)|^2 dx'dz \le 2\int_{\Sigma^\epsilon_+} |\nabla_x\phi|^2 dx. \qquad (6)$$

Using $\mathcal{T}_{\epsilon,\delta}(\nabla_x\phi) = \frac{1}{\epsilon\delta}\nabla_z\mathcal{T}_{\epsilon,\delta}(\phi)$, we obtain

$$\frac{\delta^{N-2}}{\epsilon|Y|}\int_{\Sigma\times\frac{1}{\delta}Y_+} [\nabla_z\mathcal{T}_{\epsilon,\delta}(\phi)]^2 dx'dz \le 2\int_{\Sigma^\epsilon_+} |\nabla_x\phi|^2 dx. \qquad (7)$$

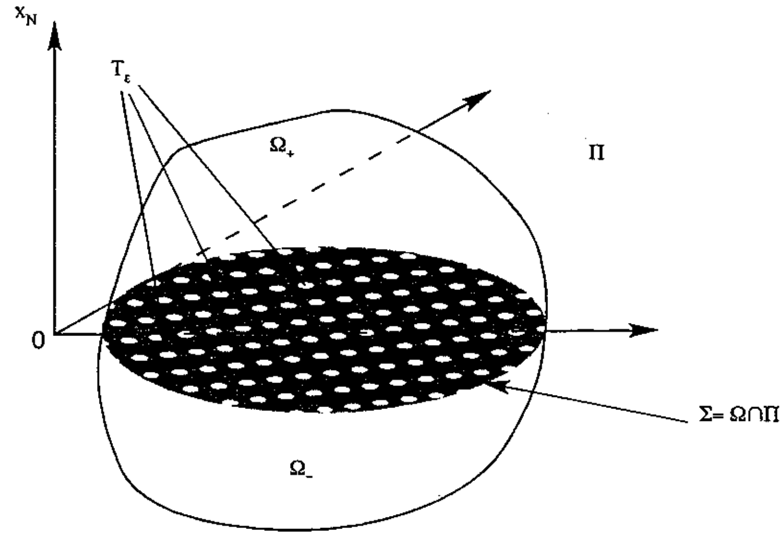

Figure 1: The geometry of the Neumann Sieve

4 Passing to the limit in the Neumann sieve model

In this section we will present a new proof for the limit analysis of the classical Neumann sieve model, (see Figure 1), based on the bl-Unfolding Operator. The Neumann sieve model is ,

$$(\mathcal{P}_1^\epsilon)\begin{cases} \Delta u_\epsilon = f & \text{in } \Omega_\epsilon \\ \dfrac{\partial u_\epsilon}{\partial n^+} = -\dfrac{\partial u_\epsilon}{\partial n^-} = 0 & \text{on } \Sigma\backslash T_\epsilon \\ u_\epsilon = 0 & \text{on } \partial\Omega \end{cases}$$

where $f \in L^2(\Omega)$ and n^+ is the exterior normal to Σ oriented toward Ω_-.

The variational form of $(\mathcal{P}_1^\epsilon)$ is,

$$\int_{\Omega_+\cup\Omega_-} \nabla u_\epsilon \nabla \psi dx = \int_\Omega f\psi \quad \text{for all} \quad \psi \in V_\epsilon. \tag{8}$$

Using u_ϵ as a test function in (8) we can easily see that there exists a constant C independent of any small parameter such that,

$$||u_\epsilon||_V \le C||f||_{L^2(\Omega)}, \tag{9}$$

where we used the Poincare inequality on Ω_+ and Ω_- respectively. From (9) we have that there exists $u_0 \in V$ such that $u_\epsilon \rightharpoonup u_0$ on a subsequence still denoted by ϵ. The first limit equation and the first part of the interface condition can be obtained in a classical way (see [3]). Indeed, consider $\psi \in \mathcal{D}(\Omega)$ and let ψ be a test function in problem $(\mathcal{P}_1^\epsilon)$. Then we have:

$$\int_{\Omega_\epsilon} \nabla u_\epsilon \nabla \psi(x) dx = \int_{\Omega_\epsilon} \psi(x) f dx. \tag{10}$$

Next we have:

$$\int_{\Omega_+} \nabla u_\epsilon^+ \nabla \psi(x) dx + \int_{\Omega_-} \nabla u_\epsilon^- \nabla \psi(x) dx = \int_{\Omega_\epsilon} \psi(x) f dx.$$

Now we can pass to the limit in the above equality and obtain the first limit equation of the unfolded formulation for the limit problem,

$$\sum_{i \in \{+,-\}} \int_{\Omega_i} \nabla_x u_0 \nabla_x \psi(x) dx = \int_\Omega \psi f dx \text{ for all } \psi \in \mathcal{D}(\Omega). \tag{11}$$

We can see that equation (11) does not offer the complete information about the interface condition we expect for u_0 on Σ. From the fact that (11) is verified for all $\psi \in \mathcal{D}(\Omega)$ we can easily obtain ,

$$\frac{\partial u_0^+}{\partial n^+} = -\frac{\partial u_0^-}{\partial n^-}. \tag{12}$$

But, in the limit, we would expect a relation between the jump of u_0 and it's normal derivatives on Σ. In order to obtain this interface condition, we will use the bl-Unfolding Operator.

First we prove a few fundamental convergence results.

Proposition 4.1. *Let $v_\epsilon \rightharpoonup v_0$ in V and define $M_{Y_+}^\epsilon(v_\epsilon^+)(x') \doteq \frac{\delta^N}{|Y_+^\delta|} \int_{\frac{1}{\delta}Y_+^\delta} \mathcal{T}_{\epsilon,\delta}(v_\epsilon^+)(x', z) dz$. Then we have*

$$M_{Y_+}^\epsilon(v_\epsilon^+) \to v_0^+ \text{ strongly in } L^2(\Sigma).$$

The same result holds for v_ϵ^-.

Proof. After an obvious change of variables, we have that for any $\phi \in \mathcal{D}(\Omega)$

$$\int_\Sigma M_{Y_+}^\epsilon(v_\epsilon^+)(x')\phi(x') dx' = \frac{1}{|Y_+|} \int_{\Sigma \times Y_+} \mathcal{T}_\epsilon(v_\epsilon^+)(x', y)\phi(x') dx' dy. \tag{13}$$

The conclusion now follows from the following two Lemmas.

Lemma 4.2. *let $z_\epsilon \rightharpoonup z$ in $H^1(\Omega)$. Then if we define the strip $S_\epsilon = \{x \in \mathbb{R}^N ; |x_N| < \epsilon\}$, we have*

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{S_\epsilon} |z_\epsilon - z|^2 dx = 0.$$

Proof. For $|t| < \epsilon$ we have

$$z_\epsilon(x', t) = z_\epsilon(x', 0) + \int_0^t \frac{\partial z_\epsilon}{\partial x_N}(x', s) ds.$$

Therefore, from Minkowski inequality we obtain,

$$z_\epsilon^2(x',t) \le 2\left(z_\epsilon^2(x',0) + t\int_0^t \frac{\partial z_\epsilon^2}{\partial x_N}(x',s)ds\right),$$

and integrating now over S_ϵ and using that $|t| < \epsilon$ we have,

$$\frac{1}{\epsilon}\int_{-\epsilon}^{\epsilon}\int_\Sigma z_\epsilon^2(x',t)dx'dt \le 2\int_{\{x_N=0\}} z_\epsilon^2(x',0)dx' + \epsilon\int_{-\epsilon}^{\epsilon}\int_\Sigma \frac{\partial z_\epsilon^2}{\partial x_N}(x',s)ds$$

and using the bound of z_ϵ in $H^1(\Omega)$ we arrive to the conclusion of the lemma. □

Lemma 4.3. *One has*

$$||\mathcal{T}_\epsilon(v_\epsilon^+) - v_0^+||_{L^2(\Sigma\times Y_+)} \to 0$$

and the similar result is true for v_ϵ^-.

Proof. Let $\{v_0^k\}_{k\in\mathbb{N}} \in C^\infty(\Omega_+)$ be such that $v_0^k \to v_0^+$ strongly in $H^1(\Omega_+)$. Then we have

$$||\mathcal{T}_\epsilon(v_\epsilon^+) - v_0^+||_{L^2(\Sigma\times Y_+)} \le ||\mathcal{T}_\epsilon(v_\epsilon^+ - v_0^+)||_{L^2(\Sigma\times Y_+)} + ||\mathcal{T}_\epsilon(v_0^+ - v_0^k)||_{L^2(\Sigma\times Y_+)} +$$
$$+ ||\mathcal{T}_\epsilon(v_0^k) - v_0^k||_{L^2(\Sigma\times Y_+)} + ||v_0^k - v_0^+||_{L^2(\Sigma\times Y_+)} \le$$
$$\le \left(\frac{c}{\epsilon}\int_{\Sigma_+^\epsilon} |v_\epsilon^+ - v_0^+|^2\right)^{\frac{1}{2}} + \left(\frac{c}{\epsilon}\int_{\Sigma_+^\epsilon} |v_0^+ - v_0^k|^2\right)^{\frac{1}{2}} +$$
$$+ ||\mathcal{T}_\epsilon(v_0^k) - v_0^k||_{L^2(\Sigma\times Y_+)} + ||v_0^k - v_0^+||_{L^2(\Sigma\times Y_+)},$$

where we used Corollary 3.2. Using Lemma 4.2 and property 6 in Theorem 2.1 we can pass to the limit in the above inequality and obtain the desired result. □

Finally we can use Lemma 4.3 and pass to the limit in (13) to prove the statement of the Proposition 4.1. □

Proposition 4.4. *Let* $v_\epsilon \in V_\epsilon$ *such that* $v_\epsilon \rightharpoonup v_0$ *in* V. *Then, there exists* $U^+ \in L^2(\Sigma, K_2(\mathbb{R}_+^N))$ *and* $U^- \in L^2(\Sigma, K_2(\mathbb{R}_-^N))$ *such that, up to a subsequence still denoted by* ϵ, δ, *we have*

$$\frac{\delta^{\frac{N-2}{2}}}{\epsilon^{\frac{1}{2}}}\mathcal{T}_{\epsilon,\delta}\left(v_\epsilon^+ - M_{Y_+}^\epsilon(v_\epsilon^+)\right) \rightharpoonup U^+ \quad in \quad L^2(\Sigma, L^{2^*}(\mathbb{R}_+^N))$$

$$\frac{\delta^{\frac{N-2}{2}}}{\epsilon^{\frac{1}{2}}}\nabla_z\mathcal{T}_{\epsilon,\delta}(v_\epsilon^+) \rightharpoonup \nabla_z U^+ \quad in \quad L^2(\Sigma, L^2_{loc}(\mathbb{R}^N))$$

and the same results hold for v_ϵ^- *and* U^-.

Proof. Indeed, using the fact that $\mathcal{T}_{\epsilon,\delta}(M^{\epsilon}_{Y_+}(v^+_\epsilon))(x,z) = M^{\epsilon}_{Y_+}(v^+_\epsilon)$ for any $(x,z) \in \Sigma \times \frac{1}{\delta}Y^{\delta}_+$, the Poincare-Writinger inequality and (7) we have,

$$||\frac{\delta^{\frac{N-2}{2}}}{\epsilon^{\frac{1}{2}}}\mathcal{T}_{\epsilon,\delta}(v^+_\epsilon - M^{\epsilon}_{Y_+}(v^+_\epsilon)||^2_{L^2(\Sigma,L^{2^*}(\mathbb{R}^N_+))} \leq$$

$$\leq \frac{\delta^{N-2}}{\epsilon}||\nabla_z \mathcal{T}_{\epsilon,\delta}(v^+_\epsilon)||^2_{L^2(\Sigma\times\frac{1}{\delta}Y^{\delta}_+)} \leq c\int_{\Sigma^{\epsilon}_+}|\nabla_x v_\epsilon|^2 dx \leq c||v_\epsilon||^2_V$$

and this shows that there exists $U^+ \in L^2(\Sigma, L^{2^*}(\mathbb{R}^N_+))$ such that,

$$\frac{\delta^{\frac{N-2}{2}}}{\epsilon^{\frac{1}{2}}}\mathcal{T}_{\epsilon,\delta}\left(v^+_\epsilon - M^{\epsilon}_{Y_+}(v^+_\epsilon)\right) \rightharpoonup U^+ \text{ in } L^2(\Sigma, L^{2^*}(\mathbb{R}^N_+)). \tag{14}$$

From (6) and (7) we have that there exists $V^+ \in L^2(\Sigma\times\mathbb{R}^N_+, \mathbb{R}^N)$ such that,

$$\delta^{\frac{N}{2}}\epsilon^{\frac{1}{2}}\mathcal{T}_{\epsilon,\delta}(\nabla_x v^+_\epsilon) \rightharpoonup V^+ \text{ in } L^2(\Sigma\times\mathbb{R}^N_+, \mathbb{R}^N). \tag{15}$$

For every $\phi \in \mathcal{D}(\Sigma \times \mathbb{R}^N, \mathbb{R}^N)$, from property 3 in Theorem 2.1, we have

$$\int_{\Sigma\times\mathbb{R}^N_+}\delta^{\frac{N}{2}}\epsilon^{\frac{1}{2}}\mathcal{T}_{\epsilon,\delta}(\nabla_x v^+_\epsilon)\phi dx'dz = \int_{\Sigma\times\frac{1}{\delta}Y^{\delta}_+}\frac{\delta^{\frac{N-2}{2}}}{\epsilon^{\frac{1}{2}}}\nabla_z\mathcal{T}_{\epsilon,\delta}(v^+_\epsilon)\phi dx'dz.$$

Passing to the limit when $\epsilon \to 0$ we get

$$\frac{\delta^{\frac{N-2}{2}}}{\epsilon^{\frac{1}{2}}}\nabla_z\mathcal{T}_{\epsilon,\delta}(v^+_\epsilon) \rightharpoonup V^+ \text{ in } L^2(\Sigma, L^2_{loc}(\mathbb{R}^N_+)). \tag{16}$$

Similar results hold for U^- and V^-.
Next we have,

$$\int_{\Sigma\times\frac{1}{\delta}Y^{\delta}_+}\frac{\delta^{\frac{N-2}{2}}}{\epsilon^{\frac{1}{2}}}\nabla_z\mathcal{T}_{\epsilon,\delta}(v^+_\epsilon)\phi dx'dz = \int_{\Sigma\times\frac{1}{\delta}Y^{\delta}_+}\frac{\delta^{\frac{N-2}{2}}}{\epsilon^{\frac{1}{2}}}\nabla_z\mathcal{T}_{\epsilon,\delta}\left(v^+_\epsilon - M^{\epsilon}_{Y_+}(v^+_\epsilon)\right)\phi dx'dz =$$

$$= -\int_{\Sigma\times\frac{1}{\delta}Y^{\delta}_+}\frac{\delta^{\frac{N-2}{2}}}{\epsilon^{\frac{1}{2}}}\mathcal{T}_{\epsilon,\delta}\left(v^+_\epsilon - M^{\epsilon}_{Y_+}(v^+_\epsilon)\right)\nabla_z\phi dx'dz.$$

Using (14) and (16) we pass to the limit in the above equality, and obtain,

$$\int_{\Sigma\times\mathbb{R}^N} V^+\phi dx'dz = -\int_{\Sigma\times\mathbb{R}^N} U^+\nabla_z\phi dx'dz = \int_{\Sigma\times\mathbb{R}^N}\nabla_z U^+\phi dx'dz$$

and this together with (15) show that $\nabla_z U^+ \in L^2(\Sigma, L^2(\mathbb{R}^N_+))$.

□

From the Proposition 4.1 and Proposition 4.4 for u_ϵ^+ and u_ϵ^- we have that there exists $U^+ \in L^2(\Sigma, K^2(\mathbb{R}_+^N))$ and $U^- \in L^2(\Sigma, K^2(\mathbb{R}_-^N))$ such that

$$U^+(x',z) - U^-(x',z) = -k[u_0] \text{ for a.e. } (x',z) \in \Sigma \times T \tag{17}$$

where k has been defined in (2) and we know (5).

Now we define the test functions needed to characterize the contribution of the holes T_ϵ to the limit problem. Define the spaces

$$X^+ = \{v \in C^\infty(\mathbb{R}_+^N);\ v = c \doteq v(T) \text{ on } T;\ supp(v) \subset\subset \frac{1}{\delta}Y\}$$

$$X^- = \{v \in C^\infty(\mathbb{R}_-^N);\ v = c \doteq v(T) \text{ on } T;\ supp(v) \subset\subset \frac{1}{\delta}Y\}.$$

Let $v^+ \in X^+$ and $v^- \in X^-$. Consider the following two sequences,

$$v_\epsilon^+(x) = \begin{cases} v^+(T) - v^+\left(\dfrac{\left\{\frac{x}{\epsilon}\right\}_Y}{\delta}\right) & \text{if } 0 < x_N < \dfrac{\epsilon}{2} \\ v^+(T) & \text{otherwise} \end{cases} \tag{18}$$

and similarly defined v_ϵ^-. We will consider next the extension by zero of v_ϵ^+ and v_ϵ^- to the hole space $\mathbb{R}^N$ still denoted by v_ϵ^+ and v_ϵ^- respectively. It is clear from the definition that $[v_\epsilon^+] = [v_\epsilon^-] = 0$ on T_ϵ. Then obviously $v_\epsilon^+ \rightharpoonup v^+(T)$ weakly in $H^1(\Omega_+)$ and $v_\epsilon^- \rightharpoonup v^-(T)$ weakly in $H^1(\Omega_-)$.

For $\psi \in \mathcal{D}(\Omega)$ and v_ϵ^+ as above use $\phi(x) = \psi(x)v_\epsilon^+(x)$ as a test function in problem $(\mathcal{P}_1^\epsilon)$. We have:

$$\int_{\Omega_+} \nabla_x u_\epsilon \nabla_x \psi(x) v_\epsilon^+(x)dx + \int_{\Omega_+} \nabla_x u_\epsilon \psi(x) \nabla_x v_\epsilon^+ dx = \int_{\Omega_+} f\psi(x)v_\epsilon^+(x)dx. \tag{19}$$

For the first term of the left hand side of (19) we can pass to the limit and obtain :

$$\int_{\Omega_+} \nabla_x u_\epsilon^+ \nabla_x \psi(x) v_\epsilon^+(x)dx \to v^+(T)\int_{\Omega_+} \nabla_x u_0^+ \nabla_x \psi dx. \tag{20}$$

For the second term of the left hand side of (19) we use:

$$\int_{\Omega_+} \nabla_x u_\epsilon \psi(x) \nabla_x v_\epsilon^+ dx = \int_{\Sigma_+^\epsilon} \nabla_x u_\epsilon \psi(x) \nabla_x v_\epsilon^+ dx \overset{\mathcal{T}_{\epsilon,\delta}}{=}$$

$$= \frac{\epsilon\delta^N}{\delta\epsilon|Y|}\int_{\tilde{\Sigma}_+ \times \mathbb{R}_+^N} \mathcal{T}_{\epsilon,\delta}(\nabla_x u_\epsilon^+)\mathcal{T}_{\epsilon,\delta}(\psi)(-\nabla_z v^+)dx'dz =$$

$$= \frac{\delta^{N-2}}{\epsilon|Y|}\int_{\tilde{\Sigma}_+ \times \mathbb{R}_+^N} \nabla_z \mathcal{T}_{\epsilon,\delta}(u_\epsilon^+)\mathcal{T}_{\epsilon,\delta}(\psi)(-\nabla_z v^+)dx'dz \tag{21}$$

where $\tilde{\Sigma}_+ = \tilde{\Sigma}_+^\epsilon \cap \Pi$.

Using the straightforward inequality

$$||\mathcal{T}_{\epsilon,\delta}(\psi) - \psi||_{L^\infty(\Sigma\times\frac{1}{\delta}Y)} \leq c\epsilon||\nabla_x\psi||_{L^\infty(\Omega)^N}$$

we obtain

$$\mathcal{T}_{\epsilon,\delta}(\psi)\nabla_z v^+ \to \psi\nabla_z v^+ \text{ strongly } L^2(\Sigma\times\mathbb{R}^N). \tag{22}$$

From (16), (22) and (21) we have for $\epsilon \to 0$:

$$\int_{\Omega_+} \nabla_x u_\epsilon^+ \psi(x)\nabla_x v_\epsilon^+ dx \longrightarrow -\frac{k}{|Y|}\int_{\Sigma\times\mathbb{R}_+^N} \nabla_z U^+\psi(x)\nabla_z v^+ dx' dz. \tag{23}$$

So from (19), (20), (23) we have,

$$v^+(T)\int_{\Omega_+} \nabla_x u_0^+ \nabla_x\psi dx - \frac{k}{|Y|}\int_{\Sigma\times\mathbb{R}_+^N} \nabla_z U^+\psi(x)\nabla_z v^+ dx' dz = v^+(T)\int_{\Omega_+} f\psi dx. \tag{24}$$

Similarly, if we choose $\phi(x) = \psi(x)v_\epsilon^-(x)$ as a test function in problem $(\mathcal{P}_1^\epsilon)$ we obtain,

$$v^-(T)\int_{\Omega_-} \nabla_x u_0^- \nabla_x\psi dx - \frac{k}{|Y|}\int_{\Sigma\times\mathbb{R}_-^N} \nabla_z U^-\psi(x)\nabla_z v^- dx' dz = v^-(T)\int_{\Omega_-} f\psi dx. \tag{25}$$

Note that equation (11) can be rewritten as,

$$\Delta u_0 = f \text{ on } \Omega_+ \cup \Omega_- , \tag{26}$$

and if we consider $\psi \in \mathcal{D}(\Omega)$ as a test function in the equation (26), we obtain,

$$\int_{\Omega_+} \nabla_x u_0 \nabla_x\psi dx - \int_\Sigma \frac{\partial u_0^+}{\partial n^+}\psi dx' = \int_{\Omega_+} f\psi dx. \tag{27}$$

$$\int_{\Omega_-} \nabla_x u_0 \nabla_x\psi dx - \int_\Sigma \frac{\partial u_0^-}{\partial n^-}\psi dx' = \int_{\Omega_-} f\psi dx. \tag{28}$$

Using (27) and (24), we obtain the second equation in the unfolded formulation for the limit problem, i.e,

$$\frac{k}{|Y|}\int_{\Sigma\times\mathbb{R}_+^N} \nabla_z U^+\psi(x)\nabla_z v^+(z) dx' dz = v^+(T)\int_\Sigma \frac{\partial u_0^+}{\partial n^+}\psi dx' \tag{29}$$

for all $\psi \in \mathcal{D}(\Omega)$ and all $v^+ \in X^+$. Similarly, from (28) and (25) we can write the third equation in the unfolded formulation for the limit problem, i.e,,

$$\frac{k}{|Y|}\int_{\Sigma\times\mathbb{R}_-^N} \nabla_z U^-\psi(x)\nabla_z v^-(z) dx' dz = v^-(T)\int_\Sigma \frac{\partial u_0^-}{\partial n^-}\psi \tag{30}$$

for all $\psi \in \mathcal{D}(\Omega)$ and all $v^- \in X^-$.

Now we can give the unfolded formulation of the limit problem for the classical Neumann Sieve model:

Theorem 4.5. *The unfolded formulation for the limit of the problems* $(\mathcal{P}_1^\epsilon)$ *is:*
-Find $(u_0, U^+, U^-) \in V \times L^2(\Sigma, K^2(\mathbb{R}_+^N)) \times L^2(\Sigma, K^2(\mathbb{R}_-^N))$ *such that for all* $\psi \in \mathcal{D}(\Omega)$, $v^+ \in X^+$ *and* $v^- \in X^-$ *we have*

$$1. \sum_{i\in\{+,-\}} \int_{\Omega_i} \nabla_x u_0 \nabla_x \psi(x) dx = \int_\Omega \psi f dx$$

$$2. \frac{k}{|Y|}\int_{\Sigma\times\mathbb{R}_+^N} \nabla_z U^+ \psi(x) \nabla_z v^+(z) dx' dz = v^+(T) \int_\Sigma \frac{\partial u_0^+}{\partial n^+}\psi$$

$$3. \frac{k}{|Y|}\int_{\Sigma\times\mathbb{R}_-^N} \nabla_z U^- \psi(x) \nabla_z v^-(z) dx' dz = v^-(T) \int_\Sigma \frac{\partial u_0^-}{\partial n^-}\psi$$

where U^+ *and* U^- *verifies (17).*

We will prove next that the above problem is well posed and as a consequence we will show that Theorem 4.5 gives (1) as the problem satisfied by u_0. Indeed, from the second and the third equations in Theorem 4.5, and using (17) we can easily prove the existence and uniqueness for $U^+ \in L^2(\Sigma, K^2(\mathbb{R}_+^N))$ and $U^- \in L^2(\Sigma, K^2(\mathbb{R}_-^N))$. Therefore, to prove the well-posedness of the unfolded problem, we need to prove the existence and uniqueness for $u_0 \in V$.

Let $v^-(z', z_N) = -v^+(z', -z_N)$ and consider $\tilde{U}^-$ the extension of U^- by reflexion with respect to the hyperplane Σ, i.e.,

$$\tilde{U}^-(z) = \begin{cases} U^-(z', z_N) & \text{if } z_N < 0 \\ U^-(z', -z_N) & \text{otherwise} . \end{cases} \tag{31}$$

Using v^- as above in (30) and subtracting (30) from (29), after an obvious change of variable we obtain,

$$\frac{k}{|Y|}\int_{\Sigma\times\mathbb{R}_+^N} \nabla_z (U^+ - \tilde{U}^-) \psi(x) \nabla_z v^+(z) dx' dz = 2v^+(T) \int_\Sigma \frac{\partial u_0^+}{\partial n^+}\psi \tag{32}$$

where we used (12). One can immediately see that (32) implies,

$$\frac{\partial u_0^+}{\partial n^+} \in L^2(\Sigma).$$

Now consider the following capacity problem:

$$\begin{cases} \theta^+ \in H_T^{1,*}(\mathbb{R}_+^N) \\ \displaystyle\int_{\mathbb{R}_+^N} \nabla_z \theta^+ \nabla_z \Psi dz = \Psi(T) \\ \text{for all } \Psi \in H_T^{1,*}(\mathbb{R}^N) \end{cases} \tag{33}$$

where $H_T^{1,*}(\mathbb{R}_+^N)$ has been defined in Section 2. It is well known that the problem (33) is well possed. We use the following notation:

$$U \doteq U^+ - \tilde{U}^- \text{ on } \mathbb{R}_+^N, \quad U \in L^2(\Sigma, K^2(\mathbb{R}_+^N)).$$

From (29) and the cell problem (33) we obtain,

$$U(x',z)=\frac{2|Y|}{k}\frac{\partial u_0^+}{\partial n^+}(x')\theta^+(z) \quad \text{on} \quad \Sigma\times\mathbb{R}^N_+.$$

Now using (17) and (12) we have

$$-k[u_0]=\frac{2|Y|}{k}\frac{\partial u_0^+}{\partial n^+}\theta^+(T)=-\frac{2|Y|}{k}\frac{\partial u_0^-}{\partial n^-}\theta^+(T).$$

Note that, from the symmetry of (33) with respect to Σ we have,

$$\theta^+(T)=\frac{1}{cap_2(T,\mathbb{R}^N_+\cup(T\times\{0\}))}=\frac{2}{cap_2(T,\mathbb{R}^N)}.$$

So the final limit problem can be written as (1), i.e,

$$\begin{cases} -\Delta u_0=f & \text{on} \quad \Omega_+\cup\Omega_- \\ \dfrac{\partial u_0^+}{\partial n^-}=\dfrac{\partial u_0^-}{\partial n^-}=\dfrac{k^2}{4|Y|}[u_0]cap_2(T,\mathbb{R}^N) & \text{on} \quad \Sigma. \end{cases} \tag{34}$$

5 The thick Neumann Sieve.

In this section we will present an extension of the classical Neumann Sieve model to the case when the Sieve has a certain thickness $h(\epsilon)\approx\epsilon$ and $\epsilon>0$ is a small parameter. We will use the same notations, with the same meaning as in the previous section, otherwise specified. We also need some new notations for this model. We recall that for an arbitrary set $A\subset\mathbb{R}^N$ we defined by $A_+\equiv A\cap\mathbb{R}^N_+$ and similarly A_-. Also without loss of generality we will assume again, that $Y=]-\frac{1}{2},\frac{1}{2}[^N$ and that $0\in T\subset\Pi$.

We will introduce next the class of admissible sets, which can be used to describe our thick sieve,

Definition 5.1. *Let $S^*_\delta=\frac{1}{\delta}\overline{Y}\setminus\{x\in\mathbb{R}^N;\ |x_N|=\frac{1}{2\delta};\ |x'|<\frac{1}{2\delta}\}$. Then we say that the set F is an admissible set and denote that by, $F\in\mathcal{F}$, if:*

i) F, unbounded open subset of $\mathbb{R}^N$

ii) F symmetric with respect to all the hyperplans passing through the origin and which are orthogonal to one of the vectors in the canonical basis of $\mathbb{R}^N$.

iii) F is such that $\overline{F}\cap\frac{1}{\delta}\overline{Y}\subset S^*_\delta$ for any $0<\delta<<1$

iv) F_+ and F_- are unbounded, $F\cap T=\emptyset$ and $F=F_+\cup F_-\cup\{\Pi\cap F\}$.

As an example, in Figure 2 and Figure 3 we present the 2D and respectively the 3D geometry of a set F in the class $\mathcal{F}$.

Let $Y^\delta=(Y_+\setminus\delta F_+)\cup(Y_-\setminus\delta F_-)\cup\delta T$. Let $I_\epsilon=\{\xi\in\mathbb{Z}^{N-1}\times\{0\};\ \epsilon\xi+\epsilon Y^\delta\subset\Omega\}$ and define $F^\epsilon=\bigcup_{\xi\in I_\epsilon}\{(\epsilon\xi+\epsilon\delta F)\cap(\epsilon\xi+\epsilon Y)\}$. Let $\Omega^\epsilon_+=\Omega_+\setminus F^\epsilon_+$ and Ω^ϵ_- be similarly defined.

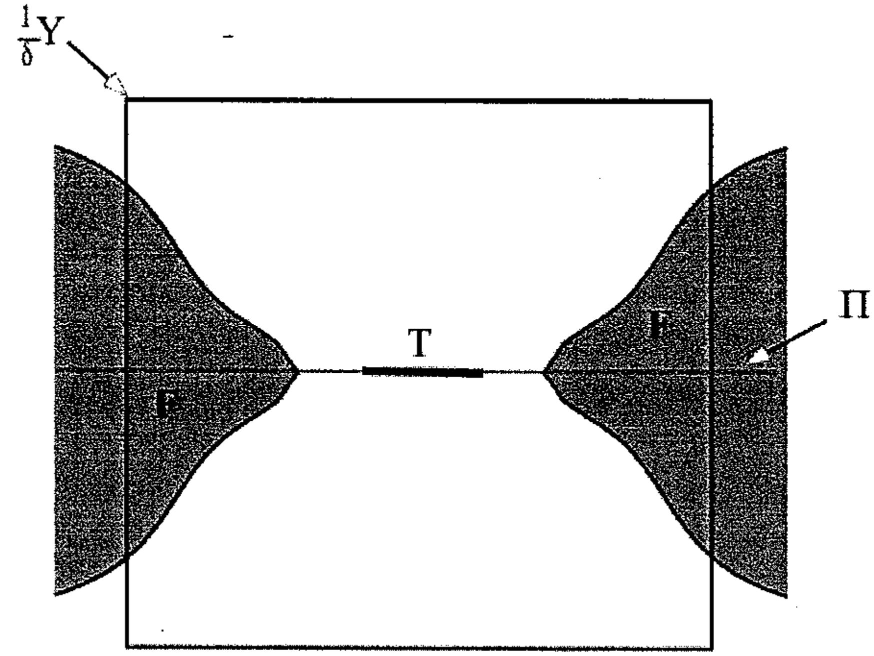

Figure 2: The 2D geometry of the set F

Let $T \subset Y_N$ and let the holes in the sieve, T_ϵ, be as before, i.e.,

$$T_\epsilon = \left[\bigcup_{\xi \in \mathbf{Z}^{N-1} \times \{0\}} \{\xi\epsilon + \epsilon\delta T\} \right] \cap \Omega.$$

Define $\Omega_\epsilon = \Omega_+^\epsilon \cup \Omega_-^\epsilon \cup T_\epsilon$ and $\Sigma_\epsilon = \Omega_\epsilon \cap \{x; |x_N| < \frac{\epsilon}{2}\}$. Consider also $H_\epsilon = \Sigma \setminus \{T_\epsilon \cup F^\epsilon\}$

In the case when F has a sufficiently smooth boundary, the thick Neumann Sieve problem, (see Figure 4 below), can be stated as follows:

$$(\mathcal{P}_2^\epsilon) \begin{cases} -\Delta u_\epsilon = f & \text{in } \Omega_\epsilon \\ \dfrac{\partial u_\epsilon}{\partial n^+} = -\dfrac{\partial u_\epsilon}{\partial n^-} = 0 & \text{on } \partial F^\epsilon \cup H_\epsilon \\ u_\epsilon = 0 & \text{on } \partial\Omega \end{cases}$$

where $f \in L^2(\Omega)$ and n^+, n^- denote the exterior normals to Ω_+^ϵ and Ω_-^ϵ respectively.

The variational formulation is

$$\int_{\Omega_\epsilon} \nabla u_\epsilon \nabla \psi = \int_{\Omega_\epsilon} f\psi \text{ for all } \psi \in V_\epsilon \tag{35}$$

where the space V_ϵ is as in Section 4. With the new notations introduced above let B_ϵ be as in Section 4 and define $\tilde{\Sigma}^\epsilon = \bigcup_{\xi \in I_\epsilon} \{\epsilon\xi + \epsilon Y^\delta\}$, $\tilde{\Omega}_\epsilon^* = \tilde{\Sigma}^\epsilon \cup \tilde{\Omega}_\epsilon'$ and $\tilde{\Omega}_+ = \tilde{\Omega}_\epsilon^* \cap \mathbb{R}_+^N$. Similarly as in (4) using the new notations of this section, we can define the Unfolding operator for the thick sieve, i.e.,

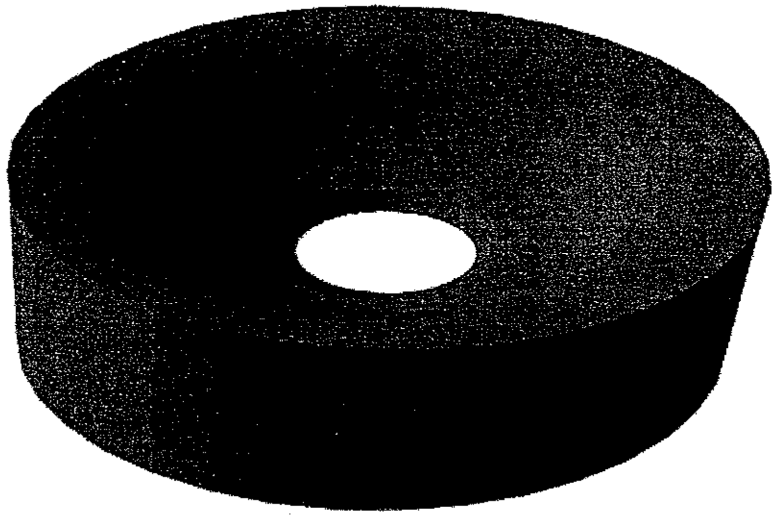

Figure 3: The 3D geometry of the set F

$\mathcal{T}_{\epsilon,\delta} : L^2(\tilde{\Omega}^*_\epsilon) \to L^2(\Omega \times \mathbf{R}^N)$ with

$$\mathcal{T}_{\epsilon,\delta}(\phi)(x,z) = \begin{cases} \phi(\epsilon\left[\frac{x'}{\epsilon}\right]_Y + \epsilon\delta z) & \text{if } \delta z \in Y^\delta \text{ and } |x_N| < \frac{\epsilon}{2} \\ 0 & \text{if } \delta z \in \mathbb{R}^N \setminus Y^\delta \text{ or } |x_N| > \frac{\epsilon}{2}. \end{cases} \tag{36}$$

Next, in the same way as in the proofs of Theorem 3.1 and Corollary 3.2 can be proved that similar results hold for this model, i.e.,

Theorem 5.2. *We have that*

$$\frac{\delta^N}{|Y^\delta|} \int_{\tilde{\Omega}_+ \times \frac{1}{\delta} Y^\delta_+} \mathcal{T}_{\epsilon,\delta}(\phi)(x,z)\, dxdz = \frac{1}{2} \int_{\Sigma^+_\epsilon} \phi\, dx \quad \textit{for all } \phi \in L^2(\Omega_+)$$

and

Corollary 5.3. *We have*

$$\frac{\epsilon\delta^N}{|Y^\delta|} \int_{\Sigma \times \frac{1}{\delta} Y^\delta_+} |\mathcal{T}_{\epsilon,\delta}(\phi)(x',z)|^2 dx'dz \leq \int_{\Sigma^+_\epsilon} |\phi|^2 dx \quad \textit{for all } \phi \in L^2(\Omega_+)$$

In what follows we are going to need the following definition, (see [7]):

Definition 5.4. *Let $k \in \mathbb{Z}$ with $k \geq 1$ and $p \geq 1$. Let D_ϵ be a given sequence of open and bounded sets in $\mathbb{R}^N_+$, and $\epsilon > 0$ a small parameter. We say that D_ϵ has the uniform extension property with respect to ϵ in the Sobolev space $W^{k,p}$, if there exist a sequence of linear and continuous operators $E_\epsilon : W^{k,p}(D_\epsilon) \to W^{k,p}(\mathbb{R}^N_+)$ such that*

$$\sup_\epsilon ||E_\epsilon||_{\mathcal{O}} < +\infty.$$

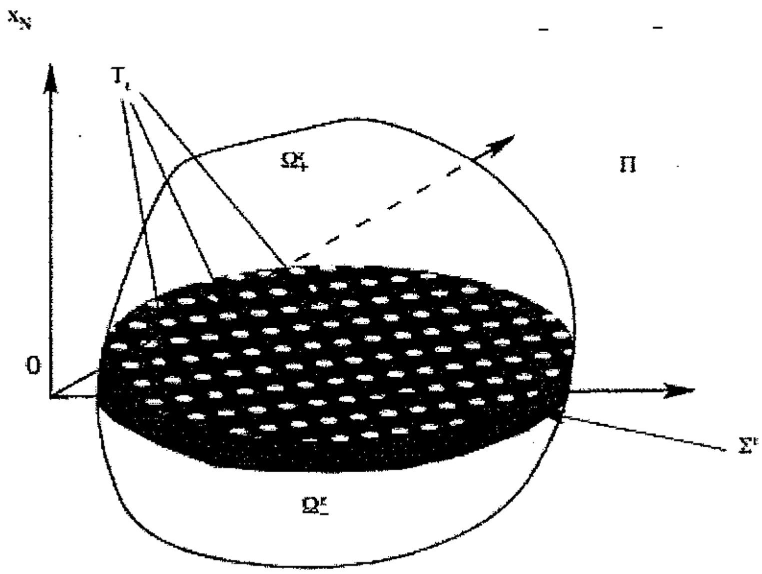

Figure 4: The geometry of the Thick Neumann Sieve

Next, following the same arguments as in the previous section and considering the same test functions we obtain the unfolded formulation of the limit problem for the thick Neumann Sieve model, i.e.,

Theorem 5.5. *Assume that Ω_+^ϵ and Ω_-^ϵ have the uniform extension property with respect to ϵ and that $F \in \mathcal{F}$. Then the unfolded formulation of the limit for the problem $(\mathcal{P}_2^\epsilon)$ is*

Find $(u_0, U^+, U^-) \in V \times L^2(\Sigma, K^2(\mathbb{R}_+^N \setminus F^+)) \times L^2(\Sigma, K^2(\mathbb{R}_-^N \setminus F^-))$ such that for all $\psi \in \mathcal{D}(\Omega)$, $v^+ \in X^+$ and $v^- \in X^-$ we have

$$1.\ \sum_{i\in\{+,-\}} \int_{\Omega_i} \nabla_x u_0 \nabla_x \psi(x) dx = \int_\Omega \psi f dx$$

$$2.\ \frac{k}{|Y|}\int_{\Sigma\times\mathbb{R}_+^N\setminus F^+} \nabla_z U^+ \psi(x) \nabla_z v^+(z) dx' dz = v^+(T) \int_\Sigma \frac{\partial u_0^+}{\partial n^+}\psi$$

$$3.\ \frac{k}{|Y|}\int_{\Sigma\times\mathbb{R}_-^N\setminus F^-} \nabla_z U^- \psi(x) \nabla_z v^-(z) dx' dz = v^-(T) \int_\Sigma \frac{\partial u_0^-}{\partial n^-}\psi \tag{37}$$

where U^+ and U^- verifies (17) and u_0 is the weak limit of the sequence of uniform extensions of u_ϵ.

Proof. The first equation of the limit problem is obtained identically as in (11), Section 4. For the other two equations, the arguments are similar as those used in the previous section with only a few technical extra difficulties. Indeed consider again, $v^+ \in X^+$ and $v^- \in X^-$. Let v_ϵ^+ and v_ϵ^- be as in (18). For $\psi \in \mathcal{D}(\Omega)$ and v_ϵ^+ as above similarly as in Section 4 use $\phi(x) = \psi(x)v_\epsilon^+(x)$ as a test function in problem $(\mathcal{P}_2^\epsilon)$. We obtain:

$$\int_{\Omega_+} \nabla_x u_\epsilon \nabla_x \psi(x) v_\epsilon^+(x) dx + \int_{\Omega_+} \nabla_x u_\epsilon \psi(x) \nabla_x v_\epsilon^+ dx = \int_{\Omega_+} f\psi(x) v_\epsilon^+(x) dx. \tag{38}$$

All the limit arguments used in Section 4 to pass to the limit in (19) can be used here too, except the limit analysis for the first term above. In this case we cannot pass to the limit directly. We have,

$$\int_{\Omega_+} \nabla_x u_\epsilon \nabla_x \psi(x) v_\epsilon^+(x) dx = \int_{\Sigma_+^\epsilon} \nabla_x u_\epsilon \nabla_x \psi(x) v_\epsilon^+(x) dx + v^+(T) \int_{\{x_N > \frac{\epsilon}{2}\}} \nabla_x u_\epsilon \nabla_x \psi(x) dx \tag{39}$$

and for the first integral of the right hand side, using Cauchy inequality, we obtain

$$\int_{\Sigma_+^\epsilon} \nabla_x u_\epsilon \nabla_x \psi(x) v_\epsilon^+(x) dx \leq (\int_{\Sigma_+^\epsilon} |\nabla_x u_\epsilon|^2 dx)^{\frac{1}{2}} (\int_{\Sigma_+^\epsilon} |\nabla_x \psi(x) v_\epsilon^+(x)|^2 dx)^{\frac{1}{2}} \leq$$

$$\leq c||f||_{L^2(\Omega)} |\nabla_x \psi(x)|_{L^\infty} ||v^+||_{L^\infty} \cdot o(\epsilon^{\frac{1}{2}}).$$

For the second term of the right hand side of (39) we can pass to the limit directly and obtain a similar statement as in (20). The analysis is done in the same way as in the previous section and we finally obtain the last two equations in Theorem 5.5. □

For a corollary we give the characterization of a class of admissible sets F such that the domains Ω_+^ϵ and Ω_-^ϵ we do have the uniform extension property.

Corollary 5.6. *Let $F \in \mathcal{F}$ and assume that the boundary of F^+ is the graph of a Liptschitz function. Assume also that F^+ does not intersect Σ at right angles. Then the domains Ω_+^ϵ and Ω_-^ϵ have the uniform extension property with respect to ϵ and the Theorem 5.5 holds.*

Proof. Note that by symmetry we have that the boundary of F^- is also the graph of a Liptschitz function and does not intersect Σ at right angles. We can extend outside $\overline{\Omega}_+$ and $\overline{\Omega}_-$ by zero because of the homogenuous Dirichlet boundary condition on $\partial\Omega$. Next we only need to prove that the domains $R_+^N \setminus F_+^\epsilon$ and $\mathbb{R}_-^N \setminus F_-^\epsilon$ have the uniform extension property. We can see that this is a consequence of the fact that these domains are special Liptschitz domains in the sense of Stein (see [19]), and therefore there exist a sequence of linear continuous extension operators uniformly bounded in the operator norm by a constant depending on the Liptschitz constant of the domains (see Stein [19], Thm. 5',page 181), which in our case is the Lipschitz constant of the boundary of F. □

Using the same arguments as in Section 4, we show that the unfolded problem (37) is well posed and as consequence we obtain the limit problem satisfied by u_0 in a similar way.

First we have,

$$U^+ - \tilde{U}^- = 2\frac{|Y|}{k}\frac{\partial u_0^+}{\partial n^+}\theta^+(z) \text{ on } \Sigma \times \{\mathbb{R}_+^N \setminus F^+\}. \tag{40}$$

where $\tilde{U}^- \in L^2(\Sigma, K_2(\mathbb{R}^N \setminus F))$ is the extension of U^- by reflexion with respect to Σ and $U^+ \in L^2(\Sigma, K_2(\mathbb{R}_+^N \setminus F^+))$, $U^- \in L^2(\Sigma, K_2(\mathbb{R}_-^N \setminus F^-))$ are the weak limits of $\mathcal{T}_{\epsilon,\delta}(u_\epsilon^+)$

and $\mathcal{T}_{\epsilon,\delta}(u_\epsilon^-)$ respectively and where θ^+ is the solution of the following capacity problem:

$$\begin{cases} \theta^+ \in H_T^{1,*}(\mathbb{R}_+^N \setminus F^+) \\ \displaystyle\int_{\mathbb{R}_+^N \setminus F^+} \nabla_z \theta^+ \nabla_z \Psi dz = \Psi(T) \\ \text{for all } \; \Psi \in H_T^{1,*}(\mathbb{R}_+^N \setminus F^+) \end{cases} \tag{41}$$

where $H_T^{1,*}(\mathbb{R}_+^N \setminus F^+)$ has been defined in Section 2. Note that U^+ and U^-, similarly as in Section 4, verify condition (17).

From (40), the cell problem (41) and (17) we obtain the limit problem for u_0, i.e,

$$\begin{cases} -\Delta u_0 = f & \text{on } \; \Omega_+ \cup \Omega_- \\ \dfrac{\partial u_0^+}{\partial n^-} = \dfrac{\partial u_0^-}{\partial n^-} = \dfrac{k^2}{2|Y|\theta^+(T)}[u_0] & \text{on } \; \Sigma \end{cases} \tag{42}$$

where from the symmetry,

$$\frac{1}{\theta^+(T)} = cap_2(T, (\mathbb{R}_+^N \setminus F^+) \cup (T \times \{0\})) = \frac{cap_2(T, (\mathbb{R}^N \setminus F))}{2}.$$

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