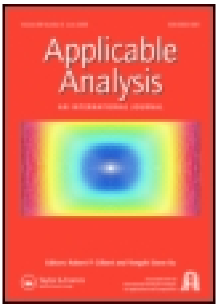

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Asymptotic analysis of second-order boundary layer correctors

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In this article we extend the ideas presented in Onofrei and Vernescu [*Asymptotic Anal.* 54 (2007), pp. 103–123] and introduce suitable second-order boundary layer correctors, to study the H^1-norm error estimate for the classical problem of homogenization, i.e.

$$\begin{cases} -\nabla \cdot \left(A\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon(x) \right) = f & \text{in } \Omega, \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Previous second-order boundary layer results assume either smooth enough coefficients (which is equivalent to assuming smooth enough correctors χ_j, $\chi_{ij} \in W^{1,\infty}$), or smooth homogenized solution u_0, to obtain an estimate of order $O(\epsilon^{\frac{3}{2}})$. For this we use some ideas related to the periodic unfolding method proposed by Cioranescu et al. [*C. R. Acad. Sci. Paris, Ser. I* 335 (2002), pp. 99–104]. We prove that in two dimensions, for non-smooth coefficients and general data, one obtains an estimate of order $O(\epsilon^{\frac{3}{2}})$. In three dimensions the same estimate is obtained assuming $\chi_j, \chi_{ij} \in W^{1,p}$, with $p > 3$.

Keywords: homogenization; error estimates; nonsmooth coefficients

AMS Subject Classifications: 35J15; 35B27

1. Introduction

This article is dedicated to the study of error estimates for the classical problem in homogenization using suitable boundary layer correctors.

Let $\Omega \in \mathbb{R}^N$, denote a convex bounded domain with a sufficiently smooth boundary. Consider also the unit cube $Y = (0,1)^N$. It is well-known that for $A \in L^\infty(Y)^{N\times N}$, symmetric and Y-periodic with $m|\xi|^2 \le A_{ij}(y)\xi_i\xi_j \le M|\xi|^2$, for any $\xi \in \mathbb{R}^N$, the solutions of

$$\begin{cases} -\nabla \cdot \left(A\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon(x) \right) = f & \text{in } \Omega, \\ u_\epsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

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have the property that [1–4],

$$u_\epsilon \rightharpoonup u_0 \quad \text{in } H_0^1(\Omega),$$

where u_0 verifies

$$\begin{cases} -\nabla \cdot (\mathcal{A}^{\text{hom}} \nabla u_0(x)) = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

with

$$\mathcal{A}_{ij}^{\text{hom}} = M_Y\left(A_{ij}(y) + A_{ik}(y) \frac{\partial \chi_j}{\partial y_k} \right) \tag{3}$$

where $M_Y(\cdot) = \frac{1}{|Y|} \int_Y \cdot \, \mathrm{d}y$ and $\chi_j \in W_{\text{per}}(Y) = \{\chi \in H^1_{\text{per}}(Y) | M_Y(\chi) = 0\}$ are the solutions of the local problem

$$-\nabla_y \cdot (A(y)(\nabla \chi_j + e_j)) = 0. \tag{4}$$

Here e_j represent the canonical basis in $\mathbb{R}^N$. In this article, ∇ and $(\nabla \cdot)$ denote the full gradient and divergence operators respectively, and ∇_x, $(\nabla_x \cdot)$ and ∇_y, $(\nabla_y \cdot)$ denote the gradient and the divergence in the slow and fast variable respectively.

Remark 1 Throughout this article, we denote by Φ the continuous extension of a given function $\Phi \in W^{p,m}(\Omega)$ with $p, m \in \mathbb{Z}$, to the space $W^{p,m}(\mathbb{R}^N)$. With minimal assumption on the smoothness of Ω a stable extension operator can be constructed [5, Ch. VI, 3.1].

The formal asymptotic expansion corresponding to the above results can be written as

$$u_\epsilon(x) = u_0(x) + \epsilon w_1\left(x, \frac{x}{\epsilon}\right) + \cdots,$$

where

$$w_1\left(x, \frac{x}{\epsilon}\right) = \chi_j\left(\frac{x}{\epsilon}\right) \frac{\partial u_0}{\partial x_j}. \tag{5}$$

We make the observation that the Einstein summation convention will be used and that the letter C will denote a constant independent of any other parameter, unless otherwise specified.

A classical result [1–3,6], states that with additional regularity assumptions on the local problem solutions χ_j or on u_0, one has

$$\left\| u_\epsilon(\cdot) - u_0(\cdot) - \epsilon w_1\left(\cdot, \frac{\cdot}{\epsilon}\right) \right\|_{H^1(\Omega)} \leq C\epsilon^{\frac{1}{2}}. \tag{6}$$

Without any additional assumptions a similar result has been recently proved by Griso [7], using the Periodic Unfolding method developed in [8], i.e.

$$\left\| u_\epsilon(\cdot) - u_0(\cdot) - \epsilon \chi_j\left(\frac{\cdot}{\epsilon}\right) Q_\epsilon\left(\frac{\partial u_0}{\partial x_j}\right) \right\|_{H^1(\Omega)} \leq C\epsilon^{\frac{1}{2}} \|u_0\|_{H^2(\Omega)}, \tag{7}$$

with

$$x \in \tilde{\Omega}_\epsilon, \quad Q_\epsilon(\phi)(x) = \sum_{i_1,..,i_N} M_Y^\epsilon(\phi)(\epsilon\xi + \epsilon i) \bar{x}_{1,\xi}^{i_1} \cdot \ldots \bar{x}_{N,\xi}^{i_N}, \qquad \xi = \left[\frac{x}{\epsilon}\right]$$

for $\phi \in L^2(\Omega)$, $i=(i_1,\dots,i_N) \in \{0,1\}^N$ and

$$\bar{x}^{i_k}_{k,\xi} = \begin{cases} \dfrac{x_k - \epsilon\xi_k}{\epsilon} & \text{if } i_k = 1 \\ 1 - \dfrac{x_k - \epsilon\xi_k}{\epsilon} & \text{if } i_k = 0 \end{cases} \qquad x \in \epsilon(\xi + Y),$$

where

$$M^{\epsilon}_{Y}(\phi) = \frac{1}{\epsilon^N}\int_{\epsilon\xi+\epsilon Y} \phi(y)\mathrm{d}y \quad \text{and} \quad \tilde{\Omega}_\epsilon = \bigcup_{\xi\in\mathbb{Z}^N} \left\{\epsilon\xi + \epsilon Y; (\epsilon\xi+\epsilon Y)\cap\Omega \neq \emptyset\right\}.$$

In order to improve the error estimates in (6), boundary layer terms have been introduced as solutions to

$$-\nabla\cdot\left(A\left(\frac{x}{\epsilon}\right)\nabla\theta_\epsilon\right) = 0 \quad \text{in } \Omega, \quad \theta_\epsilon = w_1\left(x, \frac{x}{\epsilon}\right) \quad \text{on } \partial\Omega. \tag{8}$$

Assuming $A \in C^\infty(Y)$, symmetric and Y-periodic matrix and a sufficiently smooth homogenized solution u_0 it has been proved in [4] (see also [6]) that

$$\left\| u_\epsilon(\cdot) - u_0(\cdot) - \epsilon w_1\left(\cdot, \frac{\cdot}{\epsilon}\right) + \epsilon\theta_\epsilon(\cdot)\right\|_{H^1_0(\Omega)} \leq C\epsilon, \tag{9}$$

$$\left\| u_\epsilon(\cdot) - u_0(\cdot) - \epsilon w_1\left(\cdot, \frac{\cdot}{\epsilon}\right) + \epsilon\theta_\epsilon(\cdot)\right\|_{L^2(\Omega)} \leq C\epsilon^2. \tag{10}$$

Moskow and Vogelius [9] proved the above estimates assuming $A \in C^\infty(Y)$, Y-periodic matrix and $u_0 \in H^2(\Omega)$ or $u_0 \in H^3(\Omega)$ for (9) or (10) respectively. Inequality (9) is proved in [10] for the case when $A \in L^\infty(Y)$ and $u_0 \in W^{2,\infty}(\Omega)$.

Sarkis and Versieux [11] showed that the estimates (9) and respectively (10) still holds in a more general setting, when one has $u_0 \in W^{2,p}(\Omega)$, $\chi_j \in W^{1,q}_{\text{per}}(Y)$ for (9), and $u_0 \in W^{3,p}(\Omega)$, $\chi_j \in W^{1,q}_{\text{per}}(Y)$ for (10), where, in both cases, $p > N$ and $q > N$ satisfy $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$. In [11] the constants in the right-hand side of (9) and (10) are proportional to $\|u_0\|_{W^{2,p}(\Omega)}$, and $\|u_0\|_{W^{3,p}(\Omega)}$ respectively.

In order to improve the error estimate in (9) and (10), one needs to consider the second-order boundary layer corrector, φ_ϵ defined as the solution of,

$$-\nabla\cdot\left(A\left(\frac{x}{\epsilon}\right)\nabla\varphi_\epsilon\right) = 0 \ \text{ in } \Omega, \quad \varphi_\epsilon(x) = \chi_{ij}\left(\frac{x}{\epsilon}\right)\frac{\partial^2 u_0}{\partial x_i \partial x_j} \quad \text{on } \partial\Omega, \tag{11}$$

where $\chi_{ij} \in W_{\text{per}}(Y)$ are solution of the following local problems:

$$\nabla_y\cdot(A\nabla_y\chi_{ij}) = b_{ij} + \mathcal{A}^{\text{hom}}_{ij}, \tag{12}$$

with $\mathcal{A}^{\text{hom}}$ defined by (2), $M_Y(b_{ij}(y)) = -\mathcal{A}^{\text{hom}}_{ij}$, and $b_{ij} = -A_{ij} - A_{ik}\frac{\partial\chi_j}{\partial y_k} - \frac{\partial}{\partial y_k}(A_{ik}\chi_j)$.

For the case when $u_0 \in W^{3,\infty}(\Omega)$ and $\chi_{ij} \in W^{1,\infty}(Y)$, with the help of φ_ϵ defined in (11), Allaire and Amar [10] proved the following result:

$$\left\| u_\epsilon(\cdot) - u_0(\cdot) - \epsilon w_1\left(\cdot, \frac{\cdot}{\epsilon}\right) + \epsilon\theta_\epsilon(\cdot) - \epsilon^2\chi_{ij}\left(\frac{\cdot}{\epsilon}\right)\frac{\partial^2 u_0}{\partial x_i\partial x_j}\right\|_{H^1(\Omega)} \leq C\epsilon^{\frac{3}{2}}\|u_0\|_{W^{3,\infty}(\Omega)}. \tag{13}$$

This result shows that with the help of the second-order correctors one can essentially improve the order of the estimate (9). In the general case of non-smooth periodic coefficients, i.e. $A \in L^\infty(Y)$ and $u_0 \in H^2(\Omega)$, inspired by Griso's idea, we proved in [12] that

$$\left\| u_\epsilon(\cdot) - u_0(\cdot) - \epsilon\chi_j\left(\frac{\cdot}{\epsilon}\right)Q_\epsilon\left(\frac{\partial u_0}{\partial x_j}\right) + \epsilon\beta_\epsilon(\cdot)\right\|_{H_0^1(\Omega)} \leq C\epsilon\|u_0\|_{H^2(\Omega)}, \tag{14}$$

with β_ϵ defined by

$$-\nabla\cdot\left(A\left(\frac{x}{\epsilon}\right)\nabla\beta_\epsilon\right) = 0 \text{ in } \Omega, \quad \beta_\epsilon = u_1\left(x, \frac{x}{\epsilon}\right) \text{ on } \partial\Omega, \tag{15}$$

where $u_1(x, \frac{x}{\epsilon}) \doteq \chi_j(\frac{x}{\epsilon})Q_\epsilon(\frac{\partial u_0}{\partial x_j})$.

When $u_0 \in W^{3,p}(\Omega)$ with $p > N$ we also proved in [12] that

$$\left\| u_\epsilon(\cdot) - u_0(\cdot) - \epsilon\chi_j\left(\frac{\cdot}{\epsilon}\right)\frac{\partial u_0}{\partial x_j} + \epsilon\theta_\epsilon(\cdot)\right\|_{L^2(\Omega)} \leq C\epsilon^2\|u_0\|_{W^{3,p}(\Omega)}. \tag{16}$$

In this article, we present a refinement of (13) for the case of non-smooth coefficients and general data. To do this, we start by describing the asymptotic behaviour of φ_ϵ defined at (11). The key difference between the case of smooth coefficients, and the nonsmooth case discussed in this article is that in the former, by means of the maximum principle or Avellaneda's compactness results [13], it can be proved that the second-order boundary layer corrector φ_ϵ is bounded in $L^2(\Omega)$ and is of order $O(\frac{1}{\sqrt{\epsilon}})$ in $H^1(\Omega)$, while in the latter one cannot use the aforementioned techniques to describe the asymptotic behaviour of φ_ϵ in $L^2(\Omega)$ or $H^1(\Omega)$. Thus, one needs to carefully address the question of the asymptotic behaviour of φ_ϵ with respect to ϵ.

First, we can easily observe that $\epsilon\varphi_\epsilon$ can be interpreted as the solution of an elliptic problem with variable periodic coefficients and with weakly convergent data in $H^{-1}(\Omega)$. For this class of problems a result of Tartar [14] (see also [15]) implies

$$\epsilon\varphi_\epsilon \overset{\epsilon}{\rightharpoonup} 0 \quad \text{in } H^1(\Omega).$$

As a consequence of Proposition 2.2, we obtain that for $u_0 \in H^3(\Omega)$ and $\chi_j, \chi_{ij} \in W^{1,p}_{\text{per}}(Y)$, for some $p > N$, we have

$$\|\epsilon\varphi_\epsilon\|_{H^1(\Omega)} \leq C\epsilon^{\frac{1}{2}}\|u_0\|_{H^3(\Omega)}. \tag{17}$$

Using (17) we are able to prove that for $u_0 \in H^3(\Omega)$ and $\chi_j, \chi_{ij} \in W^{1,p}_{\text{per}}$ with $p > N$ we have

$$\left\| u^\epsilon(\cdot) - u_0(\cdot) - \epsilon\chi_j\left(\frac{\cdot}{\epsilon}\right)\frac{\partial u_0}{\partial x_j} + \epsilon\theta_\epsilon(\cdot) - \epsilon^2\chi_{ij}\left(\frac{\cdot}{\epsilon}\right)\frac{\partial^2 u_0}{\partial x_i\partial x_j}\right\|_{H^1(\Omega)} \leq C\epsilon^{\frac{3}{2}}\|u_0\|_{H^3(\Omega)}. \tag{18}$$

Remark 2 states that in two dimensions due to a Meyer-type regularity for the solutions of the cell problems, χ_j, χ_{ij}, estimate (18) holds only assuming $u_0 \in H^3(\Omega)$.

2. A fundamental result

In this section we analyse the asymptotic behaviour with respect to ϵ of the solutions to a certain class of elliptic problems with highly oscillating coefficients and boundary data. The main result is stated in Proposition 2.2, but we will first present a technical Lemma which will be useful in what follows.

LEMMA 2.1 *Let Φ be such that $\Phi \in W^{1,p}_{\mathrm{per}}(Y)$ with $p > N$, and let $\psi \in H^1(\Omega)$. Then we have*

$$\int_{\Omega} \left| \nabla_y \Phi\left(\frac{x}{\epsilon}\right)\right|^2 (\psi(x) - M^{\epsilon}_Y(\psi)(x))^2 \mathrm{d}x \le C\epsilon^2 \|\Phi\|^2_{W^{1p}(Y)} \|\psi\|^2_{H^1(\Omega)}, \tag{19}$$

where $M^{\epsilon}_Y(\phi) = \frac{1}{\epsilon^N}\int_{\epsilon\xi+\epsilon Y} \phi(y)\mathrm{d}y$.

Proof Let $\tilde{\Omega}_\epsilon \subset \mathbb{R}^N$ be the smallest union of integer translates of ϵY that covers Ω, i.e.

$$\tilde{\Omega}_\epsilon \doteq \bigcup_{\xi \in Z_\epsilon} (\xi\epsilon + \epsilon Y),$$

where

$$Z_\epsilon \doteq \left\{ \xi \in \mathbb{Z}^N,\ (\xi\epsilon + \epsilon Y) \cap \Omega \neq \emptyset \right\}.$$

We start by recalling that there exists a linear and continuous extension operator $\mathcal{P}\colon H^1(\Omega) \to H^1(\tilde{\Omega}_\epsilon)$, with the continuity constant independent of ϵ (see [7,16] for details). In the rest of this section, without having to specify it every time, every function in $H^1(\Omega)$ will be extended trough $\mathcal{P}$ to $H^1(\tilde{\Omega}_\epsilon)$. Next we proceed with the proof of the lemma. We have

$$\begin{aligned}
\int_{\Omega} \left| \nabla_y \Phi\left(\frac{x}{\epsilon}\right)\right|^2 (\psi(x) - M^{\epsilon}_Y(\psi)(x))^2 \mathrm{d}x &\le \int_{\tilde{\Omega}_\epsilon} \left| \nabla_y \Phi\left(\frac{x}{\epsilon}\right)\right|^2 (\psi(x) - M^{\epsilon}_Y(\psi)(x))^2 \mathrm{d}x \\
&\le \sum_{\xi \in Z_\epsilon} \int_{\xi\epsilon+\epsilon Y} \left| \nabla_y \Phi\left(\frac{x}{\epsilon}\right)\right|^2 (\psi(x) - M^{\epsilon}_Y(\psi)(x))^2 \mathrm{d}x \\
&\le \sum_{\xi \in Z_\epsilon} \epsilon^N \int_Y |\nabla_y \Phi|^2 (\psi(\xi\epsilon + \epsilon y) - M^{\epsilon}_Y(\psi)(\xi\epsilon+\epsilon y))^2 \mathrm{d}y.
\end{aligned} \tag{20}$$

Let $\psi(\xi\epsilon + \epsilon y) = z_\xi(y)$. Using this in (20) we obtain,

$$\begin{aligned}
&\int_{\Omega} \left| \nabla_y \Phi\left(\frac{x}{\epsilon}\right)\right|^2 (\psi(x) - M^{\epsilon}_Y(\psi)(x))^2 \mathrm{d}x \\
&\quad \le \sum_{\xi \in Z_\epsilon} \epsilon^N \int_Y |\nabla_y \Phi|^2 \left(z_\xi(y) - \frac{1}{|Y|}\int_Y z_\xi(s)\mathrm{d}s \right)^2 \mathrm{d}y \\
&\quad \le \sum_{\xi \in Z_\epsilon} \epsilon^N \|\Phi\|^2_{W^{1p}(Y)} \left\| z_\xi - \frac{1}{|Y|}\int_Y z_\xi(s)\mathrm{d}s \right\|^2_{L^{\frac{2p}{p-2}}(Y)}.
\end{aligned} \tag{21}$$

Note that $\nabla_y z_\xi = \epsilon \nabla_x \psi(\xi\epsilon + \epsilon y)$. Next we will recall now a very important inequality [17, Chap. 2] to be used for our estimates. For any $p > N$ we have

$$\|\phi\|_{L^{\frac{2p}{p-2}}(\Omega)} \le c(p)\left(\|\phi\|_{L^2(\Omega)} + \|\nabla\phi\|^{\frac{N}{p}}_{L^2(\Omega)} \|\phi\|^{1-\frac{N}{p}}_{L^2(\Omega)} \right), \tag{22}$$

for any $\phi \in H^1(\Omega)$ and where $c(p)$ is a constant which depends only on q, N, Ω. Then, (22) together with the Poincare–Wirtinger inequality, implies

$$\left\| z_\xi - \frac{1}{|Y|}\int_Y z_\xi(s)\mathrm{d}s \right\|_{L^{\frac{2p}{p-2}}(Y)} \le c_p \|\nabla_y z_\xi\|_{L^2(Y)}. \tag{23}$$

Substituting (23) in (21), we have

$$\begin{aligned}
&\int_\Omega |\nabla_y \Phi\Big(\frac{x}{\epsilon}\Big)|^2(\psi(x) - M_Y^\epsilon(\psi)(x))^2 \mathrm{d}x \\
&\leq c_p \epsilon^N \|\Phi\|^2_{W^{1p}(Y)} \sum_{\xi \in Z_\epsilon} \|\nabla_y z_\xi\|^2_{L^2(Y)} \\
&= c_p \epsilon^{N+2} \|\Phi\|^2_{W^{1p}(Y)} \sum_{\xi \in Z_\epsilon} \int_Y (\nabla_x \psi(\xi\epsilon + \epsilon y))^2 \mathrm{d}y \\
&= c_p \epsilon^2 \|\Phi\|^2_{W^{1p}(Y)} \sum_{\xi \in Z_\epsilon} \int_{\epsilon\xi + \epsilon Y} |\nabla_x \psi|^2 \mathrm{d}x \\
&\leq C\epsilon^2 \|\Phi\|^2_{W^{1p}(Y)} \|\psi\|^2_{H^1(\Omega)},
\end{aligned} \tag{24}$$

where C depends on p only. So the statement of the lemma is proved. ■

Proposition 2.2 *Let $\Omega \subset \mathbb{R}^N$ be bounded convex and with smooth enough boundary. Consider the following problem*:

$$\begin{cases} -\nabla \cdot \Big(A\Big(\frac{x}{\epsilon}\Big)\nabla y_\epsilon\Big) = h & \text{in } \Omega, \\ y_\epsilon = g_\epsilon & \text{on } \partial\Omega \end{cases}, \tag{25}$$

where $h \in L^2(\Omega)$, the coefficient matrix A satisfies the hypothesis of the first section, and we have that there exists $\phi_* \in W^{1,p}_{\text{per}}(Y)$ with $p > N$, and z_ϵ a bounded sequence in $H^1(\Omega)$ such that

$$g_\epsilon(x) = \epsilon\phi_*\Big(\frac{x}{\epsilon}\Big) z_\epsilon(x) \quad \text{a.e. } \Omega. \tag{26}$$

Then there exists $y_* \in H^1_0(\Omega)$ such that

$$y_\epsilon \rightharpoonup y_* \quad \text{in } H^1(\Omega), \tag{27}$$

and y_* satisfies

$$\begin{cases} \nabla \cdot (\mathcal{A}^{\text{hom}} \nabla y_*) = h & \text{in } \Omega, \\ y_* = 0 & \text{on } \partial\Omega, \end{cases} \tag{28}$$

where $\mathcal{A}^{\text{hom}}$ is the classical homogenized matrix defined in (3). *Moreover we have*

$$\Big\| y_\epsilon - y_* - \epsilon\chi_j\Big(\frac{x}{\epsilon}\Big) Q_\epsilon\Big(\frac{\partial y_*}{\partial x_j}\Big)\Big\|_{H^1(\Omega)} \leq C\epsilon^{\frac{1}{2}}\big(1 + \|y_*\|_{H^2(\Omega)}\big), \tag{29}$$

where $\chi_j \in W_{\text{per}}(Y)$ are defined in (4), *Q_ϵ is defined in* (7) *and C depends only on p.*

Proof To prove (27) and (28) Tartar's result concerning problems with weakly converging data in H^{-1} could be used. We prefer to present here a different proof based on the periodic unfolding method developed in [8], which will also imply (29). First, observe that the solution of (25) satisfy,

$$y_\epsilon = y_\epsilon^{(1)} + y_\epsilon^{(2)} + y_\epsilon^{(3)}, \tag{30}$$

where $y_\epsilon^{(1)}, y_\epsilon^{(2)}, y_\epsilon^{(3)}$ satisfy, respectively,

$$\begin{cases} -\nabla\cdot\left(A\left(\frac{x}{\epsilon}\right)\nabla y_\epsilon^{(1)}\right) = h & \text{in } \Omega, \\ y_\epsilon^{(1)} = 0 & \text{on } \partial\Omega, \end{cases} \tag{31}$$

$$\begin{cases} -\nabla\cdot\left(A\left(\frac{x}{\epsilon}\right)\nabla y_\epsilon^{(2)}\right) = 0 & \text{in } \Omega, \\ y_\epsilon^{(2)} = \epsilon\Phi_*\left(\frac{x}{\epsilon}\right)Q_\epsilon(z_\epsilon) & \text{on } \partial\Omega, \end{cases} \tag{32}$$

$$\begin{cases} -\nabla\cdot\left(A\left(\frac{x}{\epsilon}\right)\nabla y_\epsilon^{(3)}\right) = 0 & \text{in } \Omega, \\ y_\epsilon^{(3)} = \epsilon\Phi_*\left(\frac{x}{\epsilon}\right)(z_\epsilon - Q_\epsilon(z_\epsilon)) & \text{on } \partial\Omega. \end{cases} \tag{33}$$

First, note that from Theorem 4.1 in [17], stated here in (7), we have

$$\left\| y_\epsilon^{(1)}(x) - y_*(x) - \epsilon\chi_j\left(\frac{x}{\epsilon}\right)Q_\epsilon\left(\frac{\partial y_*}{\partial x_j}\right)\right\|_{H^1(\Omega)} \le C\epsilon^{\frac{1}{2}}\| y_*\|_{H^2(\Omega)}. \tag{34}$$

From [17] (see the two estimates before Theorem 4.1 there), by using an interpolation inequality, we immediately arrive at,

$$\| y_\epsilon^{(2)}\|_{H^1(\Omega)} \le C\epsilon^{\frac{1}{2}}\|\Phi_*\|_{H^1(Y)}\ \|z_\epsilon\|_{L^2(\Omega)}. \tag{35}$$

Next, we recall the following estimate from [17]:

$$\left\|\nabla_y\Phi_*\left(\frac{\cdot}{\epsilon}\right)(Q_\epsilon u - M_Y^\epsilon u)\right\|_{L^2(\Omega)} \le C\epsilon\|\Phi_*\|_{H^1(Y)}\|u\|_{H^1(\Omega)}, \tag{36}$$

for any $u \in H^1(\Omega)$. Then for $y_\epsilon^{(3)}$ we obtain,

$$\begin{aligned}
\| y_\epsilon^{(3)}\ \|_{H^1(\Omega)} &\le C\left\|\epsilon\Phi_*\left(\frac{x}{\epsilon}\right)(z_\epsilon - Q_\epsilon(z_\epsilon))\right\|_{H^1(\Omega)} \\
&= C\left\|\epsilon\Phi_*\left(\frac{x}{\epsilon}\right)(z_\epsilon - Q_\epsilon(z_\epsilon))\right\|_{L^2(\Omega)} + C\left\|\nabla_y\Phi_*\left(\frac{x}{\epsilon}\right)(z_\epsilon - Q_\epsilon(z_\epsilon))\right\|_{L^2(\Omega)} \\
&\quad + C\left\|\epsilon\Phi_*\left(\frac{x}{\epsilon}\right)\nabla_x(z_\epsilon - Q_\epsilon(z_\epsilon))\right\|_{L^2(\Omega)} \le \epsilon^2\|z_\epsilon\|_{H^1(\Omega)}\|\Phi_*\|_{W^{1p}(Y)} \\
&\quad + C\left\|\nabla_y\Phi_*\left(\frac{\cdot}{\epsilon}\right)(z_\epsilon - Q_\epsilon(z_\epsilon))\right\|_{L^2(\Omega)} + \epsilon\|\Phi_*\|_{W^{1p}(Y)}\|z_\epsilon\|_{H^1(\Omega)} \\
&\le C\left\|\nabla_y\Phi_*\left(\frac{\cdot}{\epsilon}\right)(z_\epsilon - M_Y^\epsilon z_\epsilon)\right\|_{L^2(\Omega)} + C\left\|\nabla_y\Phi_*\left(\frac{\cdot}{\epsilon}\right)(Q_\epsilon z_\epsilon - M_Y^\epsilon z_\epsilon)\right\|_{L^2(\Omega)} \\
&\quad + C\epsilon\|\Phi_*\|_{W^{1p}(Y)}\|z_\epsilon\|_{H^1(\Omega)} \le C\epsilon\|\Phi_*\|_{W^{1p}(Y)}\|z_\epsilon\|_{H^1(\Omega)},
\end{aligned} \tag{37}$$

where C depends only on p and where we used triangle inequality in the fourth line above and we used Lemma 2.1 and (36), respectively, to estimate the first and the second terms in the fifth line. Substituting (34), (35), (37) in (30), we obtain the statement of the proposition. ■

3. Boundary layer error estimates

In this section, for the case of L^∞ coefficients, with the only assumptions that $\chi_j, \chi_{ij} \in W^{1,p}_{\text{per}}(Y)$ for some $p > N$ and $u_0 \in H^3(\Omega)$, we show that the left-hand side of (13) is of order $\epsilon^{\frac{3}{2}}$. Indeed, we have the following theorem.

THEOREM 3.1 *Let $A \in L^\infty(Y)$ and $u_0 \in H^3(\Omega)$. If there exists $p > N$ such that $\chi_j, \chi_{ij} \in W^{1,p}_{\mathrm{per}}(Y)$ then we have*

$$\left\| u_\epsilon(\cdot) - u_0(\cdot) - \epsilon w_1\left(\cdot, \frac{\cdot}{\epsilon}\right) + \epsilon\theta_\epsilon(\cdot) - \epsilon^2 \chi_{ij}\left(\frac{\cdot}{\epsilon}\right) \frac{\partial^2 u_0}{\partial x_i \partial x_j} \right\|_{H^1(\Omega)} \leq C\epsilon^{\frac{3}{2}} \|u_0\|_{H^3(\Omega)}.$$

Proof As we did before, for the sake of simplicity, we will assume $N = 3$, the two-dimensional case being similar. First, we will consider the problem with the coefficients A replaced by their mollified version A^n, described bellow (see also [18], Corollary B.1), and then conclude with a limiting argument. The new coefficients A^n are given as follows.

Let $m_n \in C^\infty$ be the standard mollifying sequence, i.e. $0 < m_n \leq 1$, $\int_{\mathbb{R}^N} m_n \, \mathrm{d}z = 1$, $sppt(m_n) \subset B(0, \frac{1}{n})$. Let $A^n(y) = (m_n * A)(y)$, where A has been defined Section 1 (see (1)). We have:

$$\begin{aligned} &1.\ A^n \text{ is an } Y \text{ periodic matrix} \\ &2.\ |A^n|_{L^\infty} < |A|_{L^\infty} \\ &3.\ A^n \to A \ \text{ in } \ L^p \ \text{ for any } \ p \in (1, \infty). \end{aligned} \tag{38}$$

Note that from (38) and the properties of A we have that $c|\xi|^2 \leq A^n_{ij}(y)\xi_i\xi_j \leq C|\xi|^2$ for all $\xi \in \mathbb{R}^N$. Next, for any $i, j \in \{1, 2, 3\}$ let $\chi^n_{ij} \in W_{\mathrm{per}}(Y)$ be the solutions of

$$\nabla_y \cdot (A^n \nabla_y \chi^n_{ij}) = b^n_{ij} - M_Y(b^n_{ij}), \tag{39}$$

where

$$b^n_{ij} = -A^n_{ij} - A^n_{ik} \frac{\partial \chi^n_j}{\partial y_k} - \frac{\partial}{\partial y_k}(A^n_{ik}\chi^n_j),$$

and $M_Y(\cdot)$ is the average on Y. We have that [18, Corollary B.8]

$$|\nabla_y \chi^n_{ij}|_{L^2(Y)} < C \quad \text{and} \quad \chi^n_{ij} \rightharpoonup \chi_{ij} \ \text{ in } W_{\mathrm{per}}(Y), \quad \forall i, j \in \{1, \ldots, , N\},$$

where

$$\int_Y A(y) \nabla_y \chi_{ij} \nabla_y \psi \, \mathrm{d}y = (b_{ij} - M_Y(b_{ij}), \psi)_{(W_{\mathrm{per}}(Y), (W_{\mathrm{per}}(Y))')}$$

for any $\psi \in W_{\mathrm{per}}(Y)$ and with

$$b_{ij} = -A_{ij} - A_{ik} \frac{\partial \chi_j}{\partial y_k} - \frac{\partial}{\partial y_k}(A_{ik}\chi_j).$$

We define

$$u^n_2(x, y) = \chi^n_{ij}(y) \frac{\partial^2 u_0}{\partial x_j \partial x_i}(x),$$

$$(v^n_*(x, y))_k = A^n_{ki}(y)\chi^n_j(y) \frac{\partial^2 u_0}{\partial x_j \partial x_i}(x) + A^n_{kl}(y) \frac{\partial \chi^n_{ij}}{\partial y_l} \frac{\partial^2 u_0}{\partial x_j \partial x_i}. \tag{40}$$

Following the same ideas as in [19], we can show that $\nabla_x \cdot M_Y(v_*^n) = 0$. Let

$$R_{ki}^j = M_Y\left(A_{ki}^n \chi_j^n + A_{kl}^n \frac{\partial \chi_{ij}^n}{\partial y_l}\right);\ (C^n(y))_{ij} = A_{ij}^n(y) + A_{ik}^n(y)\frac{\partial \chi_j^n}{\partial y_k}, \quad \mathcal{A}_n^{\text{hom}} = M_Y(C^n(y)).$$

Consider $\alpha_{ij}^n \in [L^2(Y)]^3$ defined by

$$\alpha_{ij}^n = \begin{pmatrix} A_{1i}^n \chi_j^n + A_{1l}^n \frac{\partial \chi_{ij}^n}{\partial y_l} - R_{1i}^j, \\ A_{2i}^n \chi_j^n + A_{2l}^n \frac{\partial \chi_{ij}^n}{\partial y_l} - R_{2i}^j, \\ A_{3i}^n \chi_j^n + A_{3l}^n \frac{\partial \chi_{ij}^n}{\partial y_l} - R_{3i}^j \end{pmatrix} + \beta_{ij}^n, \tag{41}$$

with

$$\begin{aligned} \beta_{1j}^n &= (0, -\phi_{3j}^n, \phi_{2j}^n)^T, \\ \beta_{2j}^n &= (\phi_{3j}^n, 0, -\phi_{1j}^n)^T \qquad \text{for } j \in \{1,2,3\}, \\ \beta_{3j}^n &= (-\phi_{2j}^n, \phi_{1j}^n, 0)^T, \end{aligned} \tag{42}$$

where T denotes the transpose. The functions $\phi_{ij}^n \in W_{\text{per}}(Y)$ were defined in [12], as solutions of

$$\text{curl}_y \phi_l^n = B_l^n \quad \text{and} \quad \text{div}_y \phi_l^n = 0, \tag{43}$$

where $B^n(y) = C^n(y) - \mathcal{A}_n^{\text{hom}}$ and B_l^n denotes the vector $B_l^n = (B_{il}^n)_i \in [L_{\text{per}}^2(Y)]^N$. It was observed in [12] that for every $1 \in \{1, 2, \ldots, N\}$

$$\phi_l^n \rightharpoonup \phi_l \ \text{ in } [W_{\text{per}}(Y)]^N \ \text{ where } \ \text{curl}_y \phi_l = B_l \ \text{ and } \text{div}_y \phi_l = 0. \tag{44}$$

The conditions on χ_j, χ_{ij} and Remark 3.11 in [19] imply that $\|\phi_l\|_{W^{1p}(Y)} < C$. Next, using the symmetry of the matrix A, we observe that the vectors α_{ij}^n defined above are divergence free with zero average over Y. This implies that there exists $\psi_{ij}^n \in \left[W_{\text{per}}(Y)\right]^3$ (see Theorem 3.4, [19] adapted for the periodic case), so that

$$\text{curl}_y \psi_{ij}^n = \alpha_{ij}^n \quad \text{and} \quad \text{div } \psi_{ij}^n = 0 \quad \text{for any } i,j \in \{1,2,3\}. \tag{45}$$

By using simple limiting arguments (see Corollary B.4 and Corollary B.8 in [18]) together with (44) in the definition of α_{ij}^n above, we obtain

$$\alpha_{ij}^n \rightharpoonup \alpha_{ij} \quad \text{in } [L^2(Y)]^3, \tag{46}$$

where the form of α_{ij} is identical with that of α_{ij}^n and can be obviously obtained from (46). Using the above convergence result and Theorem 3.9 from [19] adapted to the periodic case, we obtain

$$\psi_{ij}^n \rightharpoonup \psi_{ij}, \text{ in } \quad W_{\text{per}}(Y) \ \text{ for any } \quad i,j \in \{1,2,3\},$$

and ψ_{ij} satisfy

$$\text{curl}_y \psi_{ij} = \alpha_{ij} \quad \text{and} \quad \text{div}_y \psi_{ij} = 0 \ \text{ for } i,j \in \{1,2,3\}. \tag{47}$$

The hypothesis on χ_j and χ_{ij} implies that α_{ij} defined at (46) belongs to the space $[L^p(Y)]^3$ and for all pairs (i, j) with $i, j \in \{1, 2, 3\}$ we have

$$\|\alpha_{ij}\|_{[L^p(Y)]^3} \le C(\|\beta_{ij}\|_{[L^p(Y)]^3} + \|\chi_j\|_{L^p(Y)} + \|\chi_{ij}\|_{W^{1p}(Y)}) \le C. \tag{48}$$

Inequality (48) and Remark 3.11 in [19] imply that

$$\|\psi_{ij}\|_{[W^{1p}(Y)]^3} \leq C \quad \text{for } i,j \in \{1,2,3\}. \tag{49}$$

Define $p(x,y) = \psi_{ij}(y)\frac{\partial^2 u_0}{\partial x_i \partial x_j}(x)$ and $v_2(x,y) = \mathrm{curl}_x p(x,y)$. We can see that $p \in H^1(\Omega, H^1_{\mathrm{per}}(Y))$ and $v_2 \in L^2(\Omega, H^1_{\mathrm{per}}(Y))$. Obviously, we have that $\nabla_x \cdot v_2 = 0$ in the sense of distributions (see [9]). Next, using (40) we observe that $\nabla_x \cdot M_Y(v_*) = 0$ where v_* is such that

$$v_*^n \rightharpoonup v_* \quad \text{weakly in } L^2(\Omega, L^2_{\mathrm{per}}(Y)).$$

We have,

$$(v_*(x,y))_k = A_{ki}(y)\chi_j(y)\frac{\partial^2 u_0}{\partial x_j \partial x_i}(x) + A_{kl}(y)\frac{\partial \chi_{ij}}{\partial y_l}\frac{\partial^2 u_0}{\partial x_j \partial x_i}. \tag{50}$$

Using this and the fact that

$$\begin{aligned}\int_{\Omega\times Y}(\nabla_y \cdot v_2)\Phi(x,y)\mathrm{d}x\,\mathrm{d}y &= \int_{\Omega\times Y}(\nabla_y \cdot \mathrm{curl}_x p(x,y))\Phi(x,y)\mathrm{d}x\,\mathrm{d}y \\ &= -\int_{\Omega\times Y}(\nabla_x \cdot \mathrm{curl}_y p(x,y))\Phi(x,y)\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

for any smooth function $\Phi \in \mathcal{D}(\Omega; \mathcal{D}(Y))$, one can immediately see that

$$\nabla_y \cdot v_2 = -\nabla_x \cdot v_*, \tag{51}$$

in the sense of distributions. Let $p^n(x,y) = \psi^n_{ij}(y)\frac{\partial^2 u_0}{\partial x_i \partial x_j}(x)$ and $v^n_2(x,y) = \mathrm{curl}_x p^n(x,y)$. Consider ψ^n_ϵ and ξ^n_ϵ defined as follows:

$$\begin{aligned} w^n_1(x,y) &= \chi^n_j(y)\frac{\partial u_0}{\partial x_j}(x), \\ r^n_0(x,y) &= A^n(y)\nabla_x u_0 + A^n(y)\nabla_y w^n_1(x,y). \end{aligned} \tag{52}$$

$$\psi^n_\epsilon(x) = u^n_\epsilon(x) - u_0(x) - \epsilon w^n_1\left(x, \frac{x}{\epsilon}\right) - \epsilon^2 u^n_2\left(x, \frac{x}{\epsilon}\right). \tag{53}$$

$$\xi^n_\epsilon(x) = A^n\left(\frac{x}{\epsilon}\right)\nabla u^n_\epsilon - r^n_0\left(x, \frac{x}{\epsilon}\right) - \epsilon v^n_*\left(x, \frac{x}{\epsilon}\right) - \epsilon^2 v^n_2\left(x, \frac{x}{\epsilon}\right). \tag{54}$$

Note that

$$A^n\left(\frac{x}{\epsilon}\right)\nabla\psi^n_\epsilon(x) - \xi^n_\epsilon(x) = \epsilon^2\left(v^n_2\left(x, \frac{x}{\epsilon}\right) - A^n\left(\frac{x}{\epsilon}\right)\nabla_x u^n_2\left(x, \frac{x}{\epsilon}\right)\right). \tag{55}$$

We have the following lemma

LEMMA 3.2

$$\text{(i)} \quad \|\psi^n_\epsilon\|_{W^{1,1}(\Omega)} < C \quad \textit{and} \quad \|\xi^n_\epsilon\|_{L^1(\Omega)} < C,$$

and there exists $\psi_\epsilon \in W^{1,1}(\Omega)$ *and* $\xi_\epsilon \in L^1(\Omega)$ *such that*

$$\psi^n_\epsilon \overset{n}{\rightharpoonup} \psi_\epsilon,\ \nabla\psi^n_\epsilon \overset{n}{\rightharpoonup} \nabla\psi_\epsilon,\ \xi^n_\epsilon \overset{n}{\rightharpoonup} \xi_\epsilon, \quad \textit{weakly-* in the sense of measures.}$$

Also we have

$$\psi_\epsilon(x) = u_\epsilon(x) - u_0(x) - \epsilon w_1\Big(x, \frac{x}{\epsilon}\Big) - \epsilon^2 u_2\Big(x, \frac{x}{\epsilon}\Big),$$

$$\xi_\epsilon(x) = A\Big(\frac{x}{\epsilon}\Big)\nabla u_\epsilon - r_0\Big(x, \frac{x}{\epsilon}\Big) - \epsilon v_*\Big(x, \frac{x}{\epsilon}\Big) - \epsilon^2 v_2\Big(x, \frac{x}{\epsilon}\Big).$$

(ii) *Moreover*, $\xi_\epsilon \in L^2(\Omega)$, $\psi_\epsilon \in H^1(\Omega)$ *and we have*

$$A\Big(\frac{x}{\epsilon}\Big)\nabla\psi_\epsilon(x) - \xi_\epsilon(x) = \epsilon^2\Big(v_2\Big(x, \frac{x}{\epsilon}\Big) - A\Big(\frac{x}{\epsilon}\Big)\nabla_x u_2\Big(x, \frac{x}{\epsilon}\Big)\Big), \tag{56}$$

with

$$\nabla \cdot \xi_\epsilon(x) = 0 \tag{57}$$

in the sense of distributions.

Proof Using the fact that, for any $i, j \in \{1, 2, 3\}$, $\chi_j^n, \chi_{ij}^n \in W_{\text{per}}(Y)$ and $\psi_{ij}^n \in [W_{\text{per}}(Y)]^3$ are bounded functions in this spaces, from the definition one can immediately see that

$$\|\psi_\epsilon^n\|_{W^{1,1}(\Omega)} < C \quad \text{and} \quad \|\xi_\epsilon^n\|_{L^1(\Omega)} < C.$$

Recall that

$$\chi_j^n \rightharpoonup \chi_j,\ \chi_{ij}^n \rightharpoonup \chi_{ij} \ \text{ in } W_{\text{per}}(Y) \quad \text{and} \quad \psi_{ij}^n \rightharpoonup \psi_{ij} \ \text{ in } [W_{\text{per}}(Y)]^3.$$

Using the above convergence results and simple limiting arguments presented in the Appendix in [18] the statement (i) in Lemma 3.2 follows immediately. Next, observe that $\chi_j, \chi_{ij} \in W^{1,p}_{\text{per}}(Y)$, with $p > 3$ imply

$$\psi_\epsilon \in H^1(\Omega). \tag{58}$$

To prove (58) it is enough to see that

$$\Big\|u_2\Big(\cdot, \frac{\cdot}{\epsilon}\Big)\Big\|_{H^1(\Omega)} \le \epsilon^2\|\chi_{ij}\|_{L^\infty(Y)}\|u_0\|_{H^2(\Omega)} + \epsilon\|\chi_{ij}\|_{W^{1p}(Y)}\|u_0\|_{H^3(\Omega)} + \epsilon^2\|\chi_{ij}\|_{L^\infty(Y)}\|u_0\|_{H^3(\Omega)},$$

the rest of the necessary estimates being trivial. Similarly, from the definition of r_0, v_* and v_2 and the hypothesis $\chi_j, \chi_{ij} \in W^{1,p}_{\text{per}}(Y)$, with $p > 3$, we see that $\xi_\epsilon \in L^2(\Omega)$. Next note that we immediately have

$$A^n\Big(\frac{x}{\epsilon}\Big)\nabla\psi_\epsilon^n \overset{n}{\rightharpoonup} A\Big(\frac{x}{\epsilon}\Big)\nabla\psi_\epsilon \ \text{ weakly-* in the sense of measures.} \tag{59}$$

Relation (56) follows immediately from (55), (59), relations (38) and a limit argument based on the convergence results obtained at (i). Recall that in the smooth case it is known from [9] that

$$\nabla \cdot \xi_\epsilon^n = 0.$$

This is equivalent to

$$\int_\Omega \xi_\epsilon^n \nabla\Phi(x)\,\mathrm{d}x = 0 \quad \text{for any } \Phi \in \mathcal{D}(\Omega).$$

Using the fact that $\xi_\epsilon \in L^2(\Omega)$, and that we have

$$\xi_\epsilon^n \overset{n}{\rightharpoonup} \xi_\epsilon \quad \text{weakly-* in the sense of measures,}$$

we obtain (57). We make the remark that a different proof for (57) can be found in [11]. ■

We observe that $\chi_j, \chi_{ij} \in W^{1,p}_{\text{per}}(Y)$, with $p > 3$, implies $\psi_{ij} \in W^{1,p}_{\text{per}}(Y)$. Using this we obtain

$$\begin{aligned}\left\| \nabla_x u_2\left(x, \frac{x}{\epsilon}\right)\right\|_{L^2(\Omega)} &\leq \|\chi_{ij}\|_{L^\infty(Y)} \left\| \nabla_x \frac{\partial^2 u_0}{\partial x_j \partial x_i}\right\|_{L^2(\Omega)} \\ &\leq \|\chi_{ij}\|_{W^{1,p}(Y)} \|u_0\|_{H^3(\Omega)} \\ &\leq C\|u_0\|_{H^3(\Omega)}. \end{aligned} \tag{60}$$

$$\begin{aligned}\left\| v_2\left(x, \frac{x}{\epsilon}\right)\right\|_{L^2(\Omega)} &\leq C\|\psi_{ij}\|_{L^\infty(Y)} \left\| \nabla_x \frac{\partial^2 u_0}{\partial x_i \partial x_j}\right\|_{L^2(\Omega)} \\ &\leq C\sum_{i,j} \|\psi_{ij}\|_{W^{1,p}(Y)} \|u_0\|_{H^3(\Omega)} \\ &\leq C\|u_0\|_{H^3(\Omega)}, \end{aligned} \tag{61}$$

where, in (61) above we used (49). Similarly as in [9] substituting (60), (61) in (51), we arrive at

$$\left\| A\left(\frac{x}{\epsilon}\right)\nabla\psi_\epsilon(x) - \xi_\epsilon(x)\right\|_{L^2(\Omega)} \leq C\epsilon^2 \|u_0\|_{H^3(\Omega)}.$$

Consider the second boundary layer φ_ϵ defined as a solution of

$$\nabla \cdot \left(A\left(\frac{x}{\epsilon}\right)\nabla\varphi_\epsilon\right) = 0 \ \text{ in } \Omega, \quad \varphi_\epsilon = u_2\left(x, \frac{x}{\epsilon}\right) \ \text{ on } \partial\Omega. \tag{62}$$

Using (58) and similar arguments as in [9], we obtain

$$\left\| u_\epsilon(x) - u_0(x) - \epsilon w_1\left(x, \frac{x}{\epsilon}\right) + \epsilon\theta_\epsilon(x) - \epsilon^2 u_2\left(x, \frac{x}{\epsilon}\right) + \epsilon^2\varphi_\epsilon\right\|_{H^1_0(\Omega)} \leq C\epsilon^2\|u_0\|_{H^3(\Omega)}. \tag{63}$$

Next we make the observation that without any further regularity assumption on u_0 or on the matrix of coefficients A, one cannot make use of neither Avellaneda compactness result nor the maximum principle to obtain a L^2 or H^1 bound for φ_ϵ. In fact, in [13] it is presented an example where a solution of (62) would blow up in the L^2 norm. By the unboundedness of φ_ϵ in L^2, we can still make the observation that using a result due to Tartar [14] (see also [15, Section 8.5]) concerning the limit analysis of the classical homogenization problem in the case of weakly convergent data in $H^{-1}(\Omega)$ together with a few elementary computations we can obtain

$$\epsilon\varphi_\epsilon \overset{\epsilon}{\rightharpoonup} 0 \quad \text{in } H^1(\Omega).$$

Then applying Proposition 2.2 with $h = 0$, $y_\epsilon = \epsilon\varphi_\epsilon$, $\phi_*(y) = \chi_{ij}(y)$, $z_\epsilon(x) = z(x) = \frac{\partial^2 u_0}{\partial x_i \partial x_j}$, we obtain that

$$\|\epsilon\varphi_\epsilon\|_{H^1(\Omega)} \leq C\epsilon^{\frac{1}{2}}\|u_0\|_{H^3(\Omega)}. \tag{64}$$

Substituting (64) in (63) we have

$$\left\| u_\epsilon(x) - u_0(x) - \epsilon w_1\left(x, \frac{x}{\epsilon}\right) + \epsilon\theta_\epsilon(x) - \epsilon^2 \chi_{ij}\left(\frac{x}{\epsilon}\right) \frac{\partial^2 u_0}{\partial x_i \partial x_j} \right\|_{H^1(\Omega)} \leq C\epsilon^{\frac{3}{2}} \|u_0\|_{H^3(\Omega)} \tag{65}$$

and this concludes the proof of Theorem 3.1. ■

Remark 1 It has been shown in [20] that the assumptions $\chi_j, \chi_{ij} \in W^{1,p}_{\text{per}}(Y)$ for some $p > N$ are implied by the conditions that the BMO semi-norm norm of the coefficients matrix a is small enough (see [20] for the precise statement). In a different work by Lin and Vogelius [21], it has been shown that one can have $\chi_j, \chi_{ij} \in W^{1,\infty}_{\text{per}}(Y)$ in the case of piecewise discontinuous matrix of coefficients when the discontinuities occur on certain smooth interfaces (see [21] for the precise statement). It is clear that the lack of smoothness in the matrix A and the fact that we only assume $u_0 \in H^3(\Omega)$ would not allow one to use neither Avellaneda compactness principle nor the maximum principle to obtain bounds for φ_ϵ in L^2 or H^1.

Remark 2 For $N = 2$ we could use a Meyers-type regularity result and prove that there exists $p > 2$ such that $\chi_j, \chi_{ij} \in W^{1,p}_{\text{per}}(Y)$. Therefore Theorem 3.1 holds true in this case in the very general conditions that $u_0 \in H^3(\Omega)$ and $A \in L^\infty(Y)$.

4. Conclusions and future work

In this article we studied the question of H^1 error estimates associated to the problem (2). We proved in Theorem 3.1 an $O(\epsilon^{3/2})$ estimate by assuming that the homogenized solution u_0 belongs to $H^3(\Omega)$ and that the cell problems solutions χ_i, χ_{ij} belong to $W_per^{1,p}(Y)$ with some $p > N$. In Remark 2 we made the important observation that in two dimensions there exists a $p > 2$ such that χ_i, χ_{ij} are in $W_p\text{er}^{1,p}(Y)$ and thus the only assumption needed for the estimate of Theorem 3.1 to be true will be $u_0 \in H^3(\Omega)$. If we look at the term in the estimate 66 we can observe that this condition, i.e. $u_0 \in H^3(\Omega)$, is the most natural hypothesis for $\chi_{ij}(\frac{x}{\epsilon})\frac{\partial^2 u_0}{\partial x_i \partial x_j}$ to exists in $H^1(\Omega)$. So in a sense we cannot expect a weaker assumption on u_0 as long as we desire an H^1 estimate of the form (65). On the other hand, if we assume Ω convex with sufficiently smooth boundary, together with smooth enough data, say, $f \in H^1(\Omega)$, by using the fact that u_0 solves a homogeneous Dirichlet problem with constant coefficients in Ω classical elliptic regularity theory implies that $u_0 \in H^3(\Omega)$. Following the results in [22] we believe that in some special situations, our estimate can be proved for more general Ω (e.g. convex polyhedron) and we plan to explore this in a forthcoming paper. With a different application in mind, in [18], we also showed how our estimate can be used for the generalization of the results obtained in [9] to the case of non-smooth coefficients.

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