J. Math. Anal. Appl. 355 (2009) 180–194

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Mathematical analysis of the waves propagation through a rectangular material structure in space–time

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ARTICLE INFO

Article history:
Received 17 March 2008
Available online 22 January 2009
Submitted by T. Witelski

Keywords:
Dynamic materials
Wave propagation
Focusing energy

ABSTRACT

We consider propagation of waves through a spatio-temporal doubly periodic material structure with rectangular microgeometry in one spatial dimension and time. Both spatial and temporal periods in this dynamic material are assumed to be the same order of magnitude. Mathematically the problem is governed by a standard wave equation $(\rho u_t)_t - (k u_z)_z = 0$ with variable coefficients. We consider a checkerboard microgeometry where variables cannot be separated. The rectangles in a space–time checkerboard are assumed filled with materials differing in the values of phase velocities $\sqrt{\frac{k}{\rho}}$ but having equal wave impedance $\sqrt{k\rho}$. The difference between dynamic materials and classical static composites is that in the former case the design variables will also be time dependent. Within certain parameter ranges, the formation of distinct and stable limiting characteristic paths, i.e., limit cycles, was observed in [K.A. Lurie, S.L. Weekes, Wave propagation and energy exchange in a spatio-temporal material composite with rectangular microstructure, J. Math. Anal. Appl. 314 (2006) 286–310]; such paths attract neighboring characteristics after a few time periods. The average speed of propagation along the limit cycles remains the same throughout certain ranges of structural parameters, and this was called in [K.A. Lurie, S.L. Weekes, Wave propagation and energy exchange in a spatio-temporal material composite with rectangular microstructure, J. Math. Anal. Appl. 314 (2006) 286–310] a plateau effect. Based on numerical evidence, it was conjectured in [K.A. Lurie, S.L. Weekes, Wave propagation and energy exchange in a spatio-temporal material composite with rectangular microstructure, J. Math. Anal. Appl. 314 (2006) 286–310] that a checkerboard structure is on a plateau if and only if it yields stable limit cycles and that there may be energy concentrations over certain time intervals depending on material parameters. In the present work we give a more detailed analytic characterization of these phenomena and provide a set of sufficient conditions for the energy concentration that was predicted numerically in [K.A. Lurie, S.L. Weekes, Wave propagation and energy exchange in a spatio-temporal material composite with rectangular microstructure, J. Math. Anal. Appl. 314 (2006) 286–310].

1. Introduction

In this paper we continue the work on the novel paradigm of dynamic composites. These are formations assembled from materials which are distributed on a microscale both in space and time. This material concept takes into consideration

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.01.031

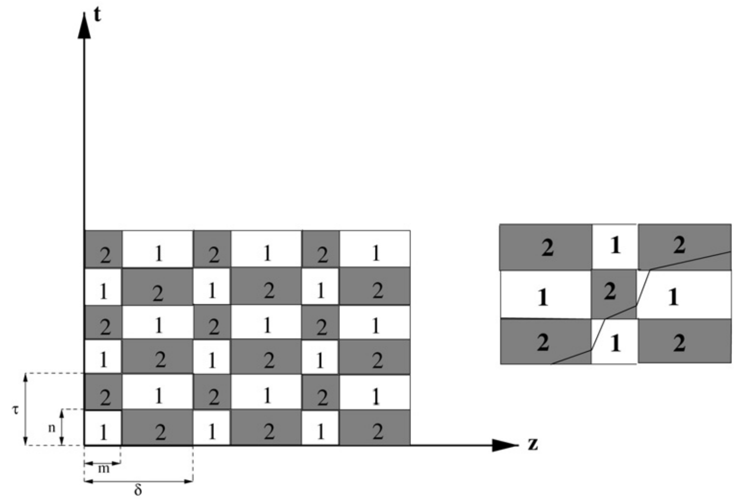

Fig. 1. Space–time rectangular microstructure.

the inertial, elastic, electromagnetic and other material properties that affect the dynamic behavior of various mechanical, electrical and environmental systems. In static or non-smart applications, the design variables such as material density and stiffness, yield force and other structural parameters are position dependent but invariant in time. When it comes to dynamic applications, we also need temporal variability in the material properties in order to adequately match the changing environment. To this end, in dynamic material design, dynamic materials will take up the role played by classical composites in static material design. Such materials demonstrate a number of unusual effects not achievable through conventional composites; some of those effects are discussed in [2] and in references therein.

There also is a substantial theoretical difference between dynamic material composites and their static counterparts. While the latter allow for a standard homogenization procedure [4–6] practically regardless of their structural microgeometry, the dynamic material assemblages often fail to demonstrate this property. To the best of our knowledge, the laminates in space–time represent the only dynamic structures for which a standard homogenization operation received a rigorous justification. Dynamic laminate is defined here as a periodic spatial pattern of different material properties: (k_1, ρ_1) and (k_2, ρ_2); this pattern moves at some constant velocity. Formally, this procedure was successfully carried out for laminates a decade ago in [1]. Since then, homogenization of laminates in space–time has been accomplished by two other approaches, one of them based on the Floquet theory; all three methods are summarized in [2]. A rigorous mathematical foundation for them was established relatively recently. In the case when the laminate pattern remains "*subsonic*," i.e. it is moving at the speed less than the phase velocity $\sqrt{k/\rho}$ in either of the material constituents, the weak compactness of the set of field components was established in [7]. It was based on the boundedness of total energy in the frame where the interfaces in a laminate remain immovable (i.e, strictly static). When the laminate pattern is "*supersonic*," i.e. moving at the speed that exceeds both of the phase velocities $\sqrt{k/\rho}$, the required compactness was established in [8]. This time it follows from the boundedness of the net momentum flux density defined as the integral of such density *over time* t (the total energy is defined as minus the integral of the same integrand taken *over the coordinate* z). The time integration is carried out in the frame in which the interfaces in a laminate appear to be strictly temporal. As mentioned above, for dynamic material formations more general that laminates, there may be no room for a standard homogenization procedure based on the weak compactness. Specifically, in [3] it was found that the energy in a checkerboard may, for certain parameter ranges, exhibit exponential growth in time. It is certainly important to better specify conditions that lead up to such growth. Particularly the growth was observed in connection with the limit cycles that arise in a checkerboard within appropriate ranges of material parameters. This and other related issues are discussed in the following sections.

The paper is organized as follows. In the first part we analytically describe the so-called "plateaux effect," numerically observed in [3]. In the second part, we give sufficient conditions that ensure the energy growth, and, to some extent, characterize a relative measure of the growth situations as opposed to the bounded energy cases.

2. Analytic characterization of the limit cycles and plateau zones

In this section, we certify analytically the numerical observation made in [3, Section 4], regarding the formation of limit cycles and the existence of the so-called plateau zones for problem (2.1) below. The units of space and time are so chosen that the periods of the assemblage along the z and t axes are dimensionless.

As in [3], we consider a doubly-periodic distribution in the (z, t)-plane, i.e., in the rectangle $(0, \delta) \times (0, \tau)$ (the periodicity cell). Material 1 occupies the region formed by the rectangles $\{(0, m) \times (0, n)\} \cup \{(m, \delta) \times (n, \tau)\}$, with $0 < m < \delta$ and $0 < n < \tau$, and the rest of the cell is occupied by material 2, see Fig. 1. Material i is uniform with parameters ρ_i and k_i

which play the role of the density and stiffness, respectively. In this structure we consider the wave motion governed in each material by the linear second order equation

$$(\rho u_t)_t - (k u_z)_z = 0, \tag{2.1}$$

with ρ, k taking the values ρ_i, k_i, within material i. The above equation can be reformulated as

$$\rho u_t = v_z, \qquad k u_z = v_t. \tag{2.2}$$

Continuity of u and v across the material boundaries is imposed. As commonly used in the literature, $\gamma_i = \sqrt{k_i \rho_i}$ and $a_i = \sqrt{\frac{k_i}{\rho_i}}$ denote, respectively, the impedance and the phase speed. We work in the same setting as in [3] and assume that the two materials have the same wave impedance, i.e., $\gamma_1 = \gamma_2 \doteq \gamma$. Hence the two constituent materials will differ in their phase speeds a_i alone. Unless stated otherwise, we will assume throughout the paper that $a_2 > a_1$. The system (2.2) can then be reduced to two independent first order equations

$$R_t + a R_z = 0, \qquad L_t - a L_z = 0,$$

for the Riemann invariants $R = u - \frac{v}{\gamma}$, $L = u + \frac{v}{\gamma}$. Without loss of generality, we will consider the right-going R-waves only, by assuming that $L = 0$ (hence $R = 2u$) throughout the structure. Every R-wave propagating through a material gives birth to only one secondary wave that travels into the adjacent material after it enters it through a horizontal or vertical material interface.

The grid of a checkerboard structure with given δ and τ is defined by horizontal lines $t = i\tau$, $t = n + i\tau$ and vertical lines $z = i\delta$, $z = m + i\delta$ for $i \in \mathbb{N}$, as illustrated in Fig. 1.

Definition 1. Given a $z_1 \in [0, \delta)$, we define $\mathcal{C}_{z_1}$ to be the class of all characteristic paths that originate at the point $(z_1, 0)$ on the z–t checkerboard grid each having the property that the path first intersects a vertical line of the grid, and then a horizontal line, and then a vertical line again with this alternating pattern repeating.

Note that due to the periodicity of the checkerboard structure, it is enough to restrict our discussion and analysis to paths that originate in the first spatial period, i.e. $z_1 \in [0, \delta)$.

Given a characteristic path in $\mathcal{C}_{z_1}$, we denote by z_j the z-coordinate of the point of intersection of the path with the jth horizontal line of the grid.

Definition 2. Given $q \in \mathbb{N}$ with $q > 1$, the average speed V_{av}^q over q material periods of a characteristic path belonging to the class $\mathcal{C}_{z_1}^v$ is defined by

$$V_{av}^q \doteq \frac{\sum_{i=1}^q \frac{z_{2i+1} - z_1}{i\tau}}{2q} + \frac{\sum_{i=1}^q \frac{z_{2i+2} - z_2}{i\tau}}{2q}. \tag{2.3}$$

Definition 3. A characteristic path in $\mathcal{C}_{z_1}$ will be called a limit cycle if there exist $p, q \in \mathbb{Z}^+$ with $p \neq q$ both even or both odd such that

$$\frac{z_q - z_p}{[\frac{q-1}{2}] - [\frac{p-1}{2}]} = \delta.$$

A limit cycle is called stable if it attracts neighboring paths, i.e., there exists $\epsilon > 0$ such that for any characteristic path in $\mathcal{C}_{z_1^\epsilon}$ with $|z_1^\epsilon - z_1| < \epsilon$ we have

$$\left|z_j^\epsilon - z_j\right| \to 0 \quad \text{as } j \to \infty.$$

Here, z_j^ϵ denotes the intersection of the path originating at z_1^ϵ with the kth horizontal of the grid. A limit cycle which is not stable will be called unstable.

Define w_j to be the difference between z_j and the z-coordinate of the closest node of the grid located to the left of z_j. Given this, an equivalent definition for limit cycles in the class $\mathcal{C}_{z_1}$ reads:

Definition 4. A characteristic path in $\mathcal{C}_{z_1}$ is a limit cycle if there exist $p, q \in \mathbb{Z}^+$ with $p \neq q$ both even or both odd such that

$$w_p = w_q.$$

A limit cycle is called stable if it attracts neighboring paths, i.e., there exists $\epsilon > 0$ such that for any characteristic path with $|z_1^\epsilon - z_1| < \epsilon$ we have

$$\left|w_j^\epsilon - w_j\right| \to 0 \quad \text{as } j \to \infty.$$

Here, w_j^ϵ is the difference between z_j^ϵ and the z-coordinate of the closest node of the grid located to the left of z_j^ϵ. A limit cycle which is not stable will be called unstable.

We will see in what follows that the limit cycles in the class $\mathcal{C}_{z_1}$, for an arbitrary $z_1 \in [0, \delta]$, all have the same average speed V_{av}^q for any $q \in \mathbb{N}$, $q > 1$.

Next, we will state the main theorems of this section. Henceforth, all limit cycles are assumed to belong to $\mathcal{C}_{z_1}$.

Theorem 5. *For any material parameters $\delta, \tau, m, n, a_1, a_2$, with $a_1 \neq a_2$, we have*:

(i) *If $(0, z_1)$ belongs to material 1 (see Fig. 1), i.e., $0 \leqslant z_1 = w_1 \leqslant m$, then all paths in the class $\mathcal{C}_{z_1}$ are characterized by the following conditions*:

$$\begin{cases} m - a_1 n \leqslant w_{2j+1} \leqslant \dfrac{a_1}{a_2}(\delta - m) - a_1 n + m, \\ \delta - m - a_1(\tau - n) \leqslant w_{2j+2} \leqslant \dfrac{a_1}{a_2} m + \delta - m - a_1(\tau - n), \end{cases} \tag{2.4}$$

for $j = 0, 1, 2, \ldots$, and we have

$$\begin{cases} w_{2j+1} = A_1 \dfrac{1 - (\frac{a_2}{a_1})^{2j}}{1 - (\frac{a_2}{a_1})^2} + \left(\dfrac{a_2}{a_1}\right)^{2j} w_1, \\ w_{2j+2} = a_2\left(n - \dfrac{m}{a_1} + \dfrac{A_1}{a_1} \dfrac{1 - (\frac{a_2}{a_1})^{2j}}{1 - (\frac{a_2}{a_1})^2}\right) + \left(\dfrac{a_2}{a_1}\right)^{2j+1} w_1, \end{cases} \tag{2.5}$$

for $j = 0, 1, 2, \ldots$, where

$$A_1 = a_2\left(\tau - \frac{\delta}{a_1} + \left(\frac{a_2}{a_1} - 1\right)\left(n - \frac{m}{a_1}\right)\right). \tag{2.6}$$

(ii) *If $(0, z_1)$ belongs to material 2 (see Fig. 1), then all the paths in the class $\mathcal{C}_{z_1}$ are characterized by the following conditions*:

$$\begin{cases} \delta - m - a_2 n \leqslant w_{2j+1} \leqslant \dfrac{a_2}{a_1} m - a_2 n + \delta - m, \\ m - a_2(\tau - n) \leqslant w_{2j+2} \leqslant \dfrac{a_2}{a_1}(\delta - m) + m - a_2(\tau - n), \end{cases} \tag{2.7}$$

for $j = 0, 1, 2, \ldots$, and we have

$$\begin{cases} w_{2j+1} = A_2 \dfrac{1 - (\frac{a_1}{a_2})^{2j}}{1 - (\frac{a_1}{a_2})^2} + \left(\dfrac{a_1}{a_2}\right)^{2j} w_1, \\ w_{2j+2} = a_1\left(n - \dfrac{\delta - m}{a_2} + \dfrac{A_2}{a_2} \cdot \dfrac{1 - (\frac{a_1}{a_2})^{2j}}{1 - (\frac{a_1}{a_2})^2}\right) + \left(\dfrac{a_1}{a_2}\right)^{2j+1} w_1, \end{cases} \tag{2.8}$$

for $j = 0, 1, 2, \ldots$, where

$$A_2 = a_1\left(\tau - \frac{\delta}{a_2} + \left(\frac{a_1}{a_2} - 1\right) n + \frac{\delta - m}{a_1}\left(1 - \frac{a_1}{a_2}\right)\right). \tag{2.9}$$

Proof. We will only present the proof of (i), as (ii) follows from similar ideas due to obvious symmetry arguments. Recall that the grid of our rectangular microstructure was defined by the horizontal lines, $t = j\tau$, $t = n + j\tau$, and vertical lines $z = m + j\delta$, $z = m + j\delta$, for $j \in \mathbb{N}$ (see Fig. 1).

We can see from Definition 1 that a trajectory in the class $\mathcal{C}_{z_1}$ whit $(0, z_1)$ in material 1, is characterized by the fact that it always enters material 1 through a horizontal line of the space–time grid and always enters material 2 through a vertical line of the grid. Let T_j denote the t-coordinate of the point of intersection of the path with the $(j+1)$th horizontal line of the grid. Also let t_j be defined as the difference between T_j and the t-coordinate of the closest node of the grid located below T_j.

Next, let p_j be defined as the slope $(\frac{dz}{dt})$ of the segment between z_j and the closest node of the rectangular space–time microstructure going in the N–E direction. Similarly we define q_j to be the slope $(\frac{dz}{dt})$ of the segment between t_j and the closest node of the rectangular space–time microstructure going in the N–E direction. In other words, p_j and q_j are defined such that the conditions

$$\begin{cases} p_j < a_1, \\ a_2 < q_j \quad \text{for any } j \in \mathbb{N}, \end{cases} \tag{2.10}$$

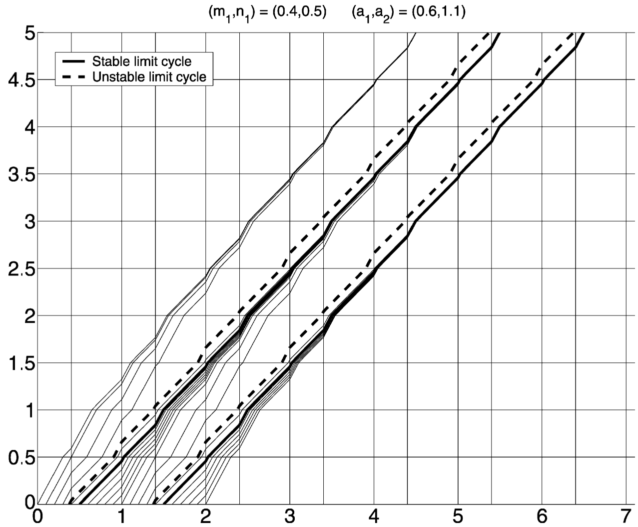

Fig. 2. Stable and unstable limit cycles in the checkerboard structure.

are necessary and sufficient for a characteristic path with phase speeds a_1 and a_2 starting at $(0, z_1)$ to belong to the class $\mathcal{C}_{z_1}$. Using the definitions of w_j, t_j, p_j, q_j and $\mathcal{C}_{z_1}$ together with an induction argument one can immediately see that

$$\begin{cases} p_{2j+1} = \dfrac{m - w_{2j+1}}{n}, \\ q_{2j+1} = \dfrac{\delta - m}{n - t_{2j+1}}, \quad \text{for } j \geqslant 0, \end{cases} \qquad \begin{cases} p_{2j} = \dfrac{\delta - m - w_{2j}}{\tau - n}, \\ q_{2j} = \dfrac{m}{\tau - n - t_{2j}}, \quad \text{for } j \geqslant 1, \end{cases} \tag{2.11}$$

and

$$\begin{cases} t_{2j+1} = \dfrac{m - w_{2j+1}}{a_1}, \\ w_{2j+1} = (\tau - n - t_{2j})a_2, \quad \text{for } j \geqslant 0, \end{cases} \qquad \begin{cases} t_{2j} = \dfrac{\delta - m - w_{2j}}{a_1}, \\ w_{2j} = (n - t_{2j-1})a_2, \quad \text{for } j \geqslant 1. \end{cases} \tag{2.12}$$

By using (2.10), (2.11) and (2.12) we obtain that a set of necessary and sufficient conditions such that a path starting at $(0, z_1)$ in material 1 will belong to the class $\mathcal{C}_{z_1}$ is given by

$$\begin{cases} a_1 > p_{2j+1} = \dfrac{m - w_{2j+1}}{n}, \\ a_2 < q_{2j+1} = \dfrac{\delta - m}{n - \frac{m - w_{2j+1}}{a_1}}, \quad \text{for } j \geqslant 0, \end{cases} \qquad \begin{cases} a_1 > p_{2j} = \dfrac{\delta - m - w_{2j}}{\tau - n}, \\ a_2 < q_{2j} = \dfrac{m}{\tau - n - \frac{\delta - m - w_{2j}}{a_1}}, \quad \text{for } j \geqslant 1. \end{cases} \tag{2.13}$$

Also, by using (2.12) we obtain

$$\begin{cases} w_{2j+1} = a_2\left(\tau - n - \dfrac{\delta - m}{a_1} + \dfrac{a_2}{a_1}n - \dfrac{a_2}{a_1^2}m + \dfrac{a_2}{a_1^2}w_{2j-1}\right), \quad \text{for } j \geqslant 1, \\ w_{2j} = a_2\left(n - \dfrac{m}{a_1} + \dfrac{a_2}{a_1}(\tau - n) - \dfrac{a_2}{a_1^2}(\delta - m) + \dfrac{a_2}{a_1^2}w_{2j-2}\right), \quad \text{for } j \geqslant 2. \end{cases} \tag{2.14}$$

By solving the linear recurrences in (2.14) above together with conditions (2.13) we obtain the statement (i) of the theorem. □

It has been numerically observed in [3] that, in the case of a square lattice, i.e., $\delta = \tau$, there are two limit cycles per material period, one stable and the other unstable. For example, when $m = 0.4$, $n = 0.5$, $a_1 = 0.6$, $a_2 = 1.1$, the origination points for the stable limit cycles correspond to $z_1 = 0.4953$, and for the unstable limit cycle, we have $z_1 = 0.375$. Fig. 2 clearly shows that the path pattern is the same for each period of origination.

We will now formalize analytically the conditions under which, in the case of a general rectangular lattice, there will be limit cycles in the class $\mathcal{C}_{z_1}$, both unstable and stable.

Proposition 6. *Assume that $a_2 > a_1$. Then:*

(i) *There exists a unique stable limit cycle per material period in the class $\mathcal{C}_{z_1}$, with the point of origination in material 2, i.e., $0 \leqslant z_1 = w_1 + m \leqslant \delta$, and this cycle is characterized by the following conditions*

$$\begin{cases} \delta - m - a_2 n \leqslant w_1 = \dfrac{A_2 a_2^2}{a_2^2 - a_1^2} \leqslant \dfrac{a_2}{a_1} m - a_2 n + \delta - m, \\ m - a_2(\tau - n) \leqslant w_2 = a_1\left(n - \dfrac{\delta - m}{a_2}\right) + \dfrac{A_2 a_1 a_2}{a_2^2 - a_1^2} \leqslant \dfrac{a_2}{a_1}(\delta - m) + m - a_2(\tau - n), \end{cases} \tag{2.15}$$

where A_2 is defined in (2.9)*. Moreover, the w_j on the stable limit cycle are given by*

$$\begin{cases} w_{2j+1} = w_1 = \dfrac{A_2 a_2^2}{a_2^2 - a_1^2}, \\ w_{2j+2} = w_2 = a_1\left(n - \dfrac{\delta - m}{a_2}\right) + \dfrac{A_2 a_1 a_2}{a_2^2 - a_1^2}, \end{cases} \tag{2.16}$$

for $j = 0, 1, 2, \ldots$. Therefore, the stable limit cycle originates at $z = m + \frac{A_2 a_2^2}{a_2^2 - a_1^2}$ at time 0*.*

(ii) *There exists a unique unstable limit cycle with the origination point in material* 1*, i.e., $0 \leqslant z_1 = w_1 \leqslant m$, and this cycle is characterized by the following conditions*

$$\begin{cases} m - a_1 n \leqslant w_1 = \dfrac{A_1 a_1^2}{a_1^2 - a_2^2} \leqslant \dfrac{a_1}{a_2}(\delta - m) - a_1 n + m, \\ \delta - m - a_1(\tau - n) \leqslant w_2 = a_2\left(n - \dfrac{m}{a_1}\right) + \dfrac{A_1 a_1 a_2}{a_1^2 - a_2^2} \leqslant \dfrac{a_1}{a_2} m + \delta - m - a_1(\tau - n), \end{cases} \tag{2.17}$$

with A_1 defined in (2.6)*. Moreover, the w_j on the unstable limit cycle are given by*

$$\begin{cases} w_{2j+1} = w_1 = \dfrac{A_1 a_1^2}{a_1^2 - a_2^2}, \\ w_{2j+2} = w_2 = a_2\left(n - \dfrac{m}{a_1}\right) + \dfrac{A_1 a_1 a_2}{a_1^2 - a_2^2}, \end{cases} \tag{2.18}$$

for $j = 0, 1, 2, \ldots$. Therefore, the unstable limit cycle originates at $z = \frac{A_1 a_1^2}{a_1^2 - a_2^2}$ at time 0.

When $a_1 > a_2$, we have:

(i) *There exists a unique stable limit cycle per material period in the class $\mathcal{C}_{z_1}$, with the point of origination in material* 1*, i.e., $0 \leqslant z_1 = w_1 \leqslant m$ and this cycle is characterized by conditions* (2.17)*. Moreover, in this case w_j on the stable limit cycle are given by formulae* (2.18)*. Therefore the stable limit cycle originates at $z = \frac{A_1 a_1^2}{a_1^2 - a_2^2}$ at time* 0*.*

(ii) *There exists a unique unstable limit cycle per material period in the class $\mathcal{C}_{z_1}$, with the origination point in material* 2*, i.e., $0 \leqslant z_1 = w_1 + m \leqslant \delta$ and this cycle is characterized by conditions* (2.15)*. Moreover, in this case w_j on the unstable limit cycle are given by formulae* (2.16)*. Therefore the unstable limit cycle originates at $m + \frac{A_2 a_2^2}{a_2^2 - a_1^2}$.*

Proof. We will only prove the statement in the case when $a_2 > a_1$, the other case following by similar arguments. Note that, in this case, i.e., $a_2 > a_1$, from formulas (2.5), (2.6), (2.8) and (2.9) we have

$$(0, z_1) \in \text{material } 2 \implies \begin{cases} w_{2j+1} - w_{2j-1} = \left(\dfrac{a_1}{a_2}\right)^{2j-1}\left(A_2 - w_1\left(1 - \left(\dfrac{a_1}{a_2}\right)^2\right)\right), \\ w_{2j+2} - w_{2j} = \left(\dfrac{a_1}{a_2}\right)^{2j-1}\left(A_2 - w_1\left(1 - \left(\dfrac{a_1}{a_2}\right)^2\right)\right), \end{cases}$$

$$(0, z_1) \in \text{material } 1 \implies \begin{cases} w_{2j+1} - w_{2j-1} = \left(\dfrac{a_2}{a_1}\right)^{2j-1}\left(A_1 - w_1\left(1 - \left(\dfrac{a_2}{a_1}\right)^2\right)\right), \\ w_{2j+2} - w_{2j} = \left(\dfrac{a_2}{a_1}\right)^{2j-1}\left(A_1 - w_1\left(1 - \left(\dfrac{a_2}{a_1}\right)^2\right)\right). \end{cases} \tag{2.19}$$

Recall that from Definition 4, we have that a characteristic path is a limit cycle if there exist $p, q \in \mathbb{N}$, $p \neq q$, both odd or both even, such that $w_p = w_q$. Without losing the generality let us assume $p = q \pmod 2$, with $p > q$. We observe that

$$|w_p - w_q| = \sum_{j=0}^{\frac{p-q}{2}} (w_{p-2j} - w_{p-2j-2}). \tag{2.20}$$

From Definition 4, relations (2.20), (2.19), and the fact that $a_1 \neq a_2$, we conclude that there are only two limit cycles, i.e. periodic paths, in the class $\mathcal{C}_{z_1}$ per time–space period, both having period 1, with one originating in material 2 and given by conditions (2.15) and formulas (2.16), and the other originating in material 1 and given by conditions (2.17) and formulas (2.18). Moreover, the above relations imply

$$\begin{cases} \left| w_{2j+1} - \dfrac{A_2 a_2^2}{a_2^2 - a_1^2} \right| = \left(\dfrac{a_1}{a_2}\right)^{2j} \left| w_1 - \dfrac{A_2 a_2^2}{a_2^2 - a_1^2} \right|, \\ \left| w_{2j} - a_1\left(n - \dfrac{\delta - m}{a_2}\right) - \dfrac{A_2 a_1 a_2}{a_2^2 - a_1^2} \right| = \left(\dfrac{a_1}{a_2}\right)^{2j+1} \left| w_1 - \dfrac{A_2 a_2^2}{a_2^2 - a_1^2} \right|, \end{cases} \tag{2.21}$$

for $(0, z_1)$ in material 2, and, respectively,

$$\begin{cases} \left| w_{2j+1} - \dfrac{A_1 a_1^2}{a_1^2 - a_2^2} \right| = \left(\dfrac{a_2}{a_1}\right)^{2j} \left| w_1 - \dfrac{A_1 a_1^2}{a_1^2 - a_2^2} \right|, \\ \left| w_{2j} - a_2\left(n - \dfrac{m}{a_1}\right) - \dfrac{A_1 a_1 a_2}{a_1^2 - a_2^2} \right| = \left(\dfrac{a_2}{a_1}\right)^{2j+1} \left| w_1 - \dfrac{A_1 a_1^2}{a_1^2 - a_2^2} \right|, \end{cases} \tag{2.22}$$

for $(0, z_1)$ in material 1. From (2.21) and (2.22) we conclude that indeed the path given by (2.15) and (2.16) is a unique stable limit cycle originating in material 2 while the path given by (2.17) and (2.18) is a unique unstable limit cycle originating in material 1. □

By using Theorem 5 and Proposition 6, one arrives at

Corollary 7. *Let $0 \leqslant z_1 \leqslant \delta$. A necessary and sufficient condition for a characteristic line in class $\mathcal{C}_{z_1}$ to be a limit cycle is*

$$V_{av}^q = \frac{\delta}{\tau}, \quad \textit{for all } q \in \mathbb{N}.$$

Proof. Assume without loss of generality that the characteristic line starts in material 1. If this characteristic line is a limit cycle (in this case unstable), then, from Definition 4, and relations (2.19), (2.20) obtained above we have that the only limit cycle starting in material 1 is given by

$$\begin{cases} w_{2j+1} = w_1 = \dfrac{A_1 a_1^2}{a_1^2 - a_2^2}, \\ w_{2j+2} = w_2 = a_2\left(n - \dfrac{m}{a_1}\right) + \dfrac{A_1 a_1 a_2}{a_1^2 - a_2^2}, \end{cases} \tag{2.23}$$

for $j = 0, 1, 2, \ldots$. Moreover, Eqs. (2.23) and (2.3) show that $V_{av}^q = \frac{\delta}{\tau}$ for all $q \in \mathbb{N}$.

For the inverse implication, assume that

$$V_{av}^q = \frac{\delta}{\tau}, \quad \text{for all } q \in \mathbb{N}.$$

Then, by (2.3),

$$0 = V_{av}^{q+1} - V_{av}^q = \frac{\sum_{i=1}^{q+1} \frac{z_{2i+1} - z_1}{i\tau}}{2q+2} + \frac{\sum_{i=1}^{q+1} \frac{z_{2i+2} - z_2}{i\tau}}{2q+2} - \frac{\sum_{i=1}^{q} \frac{z_{2i+1} - z_1}{i\tau}}{2q} - \frac{\sum_{i=1}^{q} \frac{z_{2i+2} - z_2}{i\tau}}{2q}. \tag{2.24}$$

Since we either have $z_q = w_q + [\frac{q-1}{2}]$ for q odd, or $z_q = w_q + [\frac{q-1}{2}] + m$ for q even, from (2.24) we obtain

$$0 = -V_{av}^q \frac{1}{q+1} + \frac{w_{2q+3} - w_1}{2\tau(q+1)^2} + \frac{w_{2q+4} - w_2}{2\tau(q+1)^2} + \frac{\delta}{\tau(q+1)}. \tag{2.25}$$

Next, by using the hypothesis, after obvious simplifications we obtain

$$w_{2q+3} - w_1 + w_{2q+4} - w_2 = 0. \tag{2.26}$$

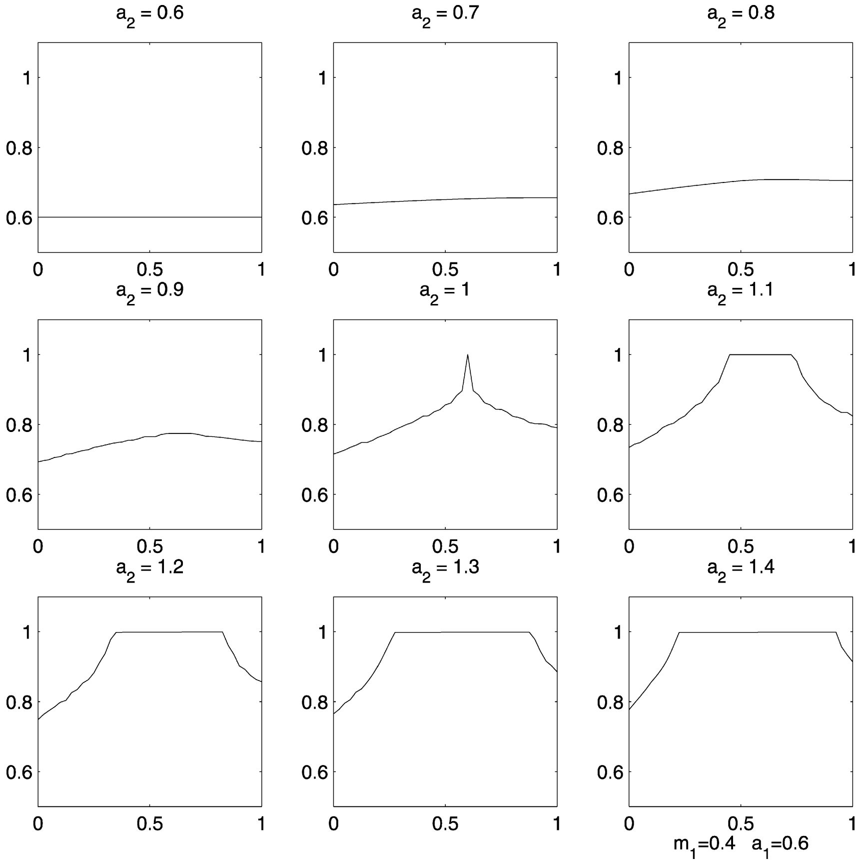

Fig. 3. Plateaux for $a_1 = 0.6$, $m = 0.4$ and a_2 between 0.6 and 1.4.

From (2.5) in Theorem 5 we can observe that unless a given characteristic path in the class $\mathcal{C}_{z_1}$ is a limit cycle (in which case $w_{2q+3} = w_1$, $w_{2q+4} = w_2$), the sequences $\{w_{2j+1}\}$ and $\{w_{2j}\}$ will be both either strictly increasing or strictly decreasing. This contradicts (2.26). Consequently, the given characteristic path corresponding to w_j must be a limit cycle. $\square$

Another numerical observation made in [3] is that if one considers the average speed associated with a path in the composite with given phase speeds a_1, a_2, then there may exist intervals of n for which the average speed is constant for a given m value; these intervals are called "plateaux" and the associated structure is referred to as "being on a plateau." It is conjectured in [3] that a structure is on a plateau if and only if the structure yields stable limit cycles. See Figs. 2 and 3.

By using Theorem 5 and Proposition 6, we can analytically describe this behavior of paths in the class $\mathcal{C}_{z_1}$ for $0 \leqslant z_1 \leqslant \delta$.

Proposition 8. *A structure yields two limit cycles, one stable and the other unstable, if and only if the structure is on a plateau, i.e., the following two pairs of inequalities hold simultaneously*:

$$\begin{cases} \dfrac{a_1\tau + (1 - \frac{a_1}{a_2})m - \delta}{a_1 - a_2} \leqslant n \leqslant \dfrac{a_1\tau + (1 - \frac{a_2}{a_1})m - \delta}{a_1 - a_2}, \\ \dfrac{m - a_2\tau + \frac{a_2}{a_1}(\delta - m)}{a_1 - a_2} \leqslant n \leqslant \dfrac{m - a_2\tau + \frac{a_1}{a_2}(\delta - m)}{a_1 - a_2}. \end{cases} \tag{2.27}$$

Proof. Without loss of generality we consider only the case $a_2 > a_1$ with the origination point in material 1, i.e., $0 \leqslant z_1 = w_1 \leqslant m$. As we see from Proposition 6, the possible (unstable) limit cycle is characterized by (2.17) and given by (2.18). Therefore we need to show that conditions (2.17) are satisfied if and only if (2.27) is true.

The first inequality in $(2.17)_1$ is equivalent to the first inequality in $(2.27)_2$, the second inequality of $(2.17)_1$ is equivalent to the second inequality in $(2.27)_2$, the first inequality in $(2.17)_2$ is equivalent to the second inequality in $(2.27)_1$, and finally the second inequality in $(2.17)_2$ is equivalent to the first inequality in $(2.27)_2$. $\square$

Remark 9. One can check that our formulae predict the exact interval for n when one fixes a_1 and m. For example, if $\delta = \tau = 1$, $a_2 = 1$, $a_1 = 0.6$ and $m = 0.4$, one has $n = 0.6$ to be the only value for which a limit cycle appears; see Fig. 3. We come to this conclusion as we observe that the left-hand side of $(2.27)_1$ and the right-hand side of $(2.27)_2$ both become equal to 0.6 in this case. In general, if $\delta = \tau$ and $a_2 = 1$, then the limit cycles appear only for $n = \tau - m$, and if $a_1 = 1$, they appear only for $n = m$.

3. Conditions on material parameters necessary and sufficient for energy accumulation

The phenomenon of energy accumulation in a time–space checkerboard microstructure originally observed in [3] came up as a consequence of some special kinematics of characteristics, such that the energy is periodically pumped into the wave as it travels through the checkerboard. This happens each time the characteristics (shown in Fig. 2 as broken lines) enter the material with higher phase velocity via the horizontal interface. The energy growth then appears to be exponential, with the exponent proportional to the logarithm of the ratio of higher/lower phase velocity. In the case when this ratio exceeds unity only slightly, the characteristic paths may (in certain circumstances analyzed below) become close to being single straight lines. To preserve the energy accumulation, it is then sufficient that these lines continue to enter the higher phase velocity material from across the horizontal interface. In this section, we will give bounds for a small parameter $\mu > 0$ such that a given microstructure with parameters $a_1^{\mu} = \frac{p}{q} - \mu$ and $a_2^{\mu} = \frac{p}{q} + \mu$ will exhibit a limit cycle in the class C_{z_1}. The results obtained help us understand for which values of $p, q > 0$, a straight line with slope $\frac{p}{q}$ can be viewed as the asymptotic limit $(\mu \to 0)$ of the original paths and we are then able to prescribe a set of *necessary and sufficient* conditions on the material parameters for such situations.

We now state the central theorem of this section.

Theorem 10. *Suppose $\tau \neq 2n$. Let $\mu > 0$ be a small parameter. Consider the microstructure with phase velocities $a_1^{\mu} = \frac{p}{q} - \mu$ and $a_2^{\mu} = \frac{p}{q} + \mu$, respectively. Then there exists $\bar{\mu} > 0$ such that the microstructure will form a limit cycle in the class C_{z_1} for any $\mu \in (0, \bar{\mu}]$ if and only if the following two conditions are simultaneously satisfied*

$$\frac{p}{q} = \frac{\delta}{\tau} \quad \text{and} \quad \begin{cases} \text{if } \tau > 2n, & \text{then } \dfrac{1}{2} < \dfrac{n}{\tau} + \dfrac{m}{\delta} \leqslant \dfrac{3}{2}, \\ \text{if } \tau < 2n, & \text{then } \dfrac{1}{2} \leqslant \dfrac{n}{\tau} + \dfrac{m}{\delta} < \dfrac{3}{2}. \end{cases} \tag{3.1}$$

If the above conditions are satisfied, then the microstructure will form a limit cycle in the class C_{z_1} for all $\mu \in (0, \bar{\mu}]$ with $\bar{\mu}$ given by

$$\bar{\mu} = \begin{cases} \min\left\{\delta \frac{(\frac{m}{\delta} + \frac{n}{\tau} - \frac{1}{2})}{\frac{\tau}{2} - n}, \frac{\delta}{\tau}\right\} & \text{if } \tau > 2n, \\ \min\left\{\delta \frac{(\frac{3}{2} - \frac{m}{\delta} - \frac{n}{\tau})}{n - \frac{\tau}{2}}, \frac{\delta}{\tau}\right\} & \text{if } \tau < 2n. \end{cases} \tag{3.2}$$

In the other cases, when at least one of the above conditions is not satisfied, there exist two positive values, $0 < \mu_1 < \mu_2$, such that the microstructure will exhibit limit cycles in the class C_{z_1} for any $\mu \in [\mu_1, \mu_2]$.

Proof. Recall that in (2.27) a set of necessary and sufficient conditions for the formation of a limit cycle in the class C_{z_1} was established. We first set $a_1^{\mu} = \frac{p}{q} - \mu$ and $a_2^{\mu} = \frac{p}{q} + \mu$ in (2.27) and see for which material parameters these relations stay true if μ is allowed to approach zero. Define $n^* \doteq \tau - n$, $m^* \doteq \delta - m$, and $r = \frac{p}{q}$. With a_1 and a_2 set as above, after simple algebraic manipulations (2.27) becomes

$$\begin{cases} (\tau - 2n)\mu^2 + (\delta - 2nr - 2m)\mu + r(\delta - \tau r) \leqslant 0, \\ (2n - \tau)\mu^2 + (2m - \delta - 2nr + 2\tau r)\mu + r(\delta - \tau r) \geqslant 0, \\ (2n - \tau)\mu^2 + (2m - \delta - 2nr)\mu + r(\tau r - \delta) \leqslant 0, \\ (\tau - 2n)\mu^2 + (\delta - 2nr - 2m + 2\tau r)\mu + r(\tau r - \delta) \geqslant 0. \end{cases} \tag{3.3}$$

Next observe that if we define the functions

$$\begin{aligned} T(x, y, \mu) &= A(x)\mu^2 + B(y)\mu + C + 2xr\mu, \\ L(x, y, \mu) &= A(x)\mu^2 + B(y)\mu + C - 2xr\mu, \end{aligned} \tag{3.4}$$

where $A(x) = \tau - 2x$, $B(y) = \delta - 2y$ and $C = r(\delta - \tau r)$, then the system (3.3) is equivalent to

$$\begin{cases} T(n, m, \mu) \geqslant 0, & L(n, m, \mu) \leqslant 0, \\ T(n^*, m^*, \mu) \geqslant 0, & L(n^*, m^*, \mu) \leqslant 0. \end{cases} \tag{3.5}$$

For any given pair of (x, y), let us denote by $t_{1,2}(x, y)$ and $l_{1,2}(x, y)$ the roots μ of $T(x, y, \mu) = 0$ and $L(x, y, \mu) = 0$, respectively. To simplify the exposition, we introduce the notations

$$t_{1,2}(n,m)=\epsilon_{1,2}, \qquad t_{1,2}\big(n^*,m^*\big)=\epsilon^*_{1,2},$$
$$l_{1,2}(n,m)=\tilde{\epsilon}_{1,2}, \qquad l_{1,2}\big(n^*,m^*\big)=\tilde{\epsilon}^*_{1,2}. \tag{3.6}$$

Observe that without loss of generality we can assume that $C>0$, the opposite case being treated similarly by working with $-T$ and $-L$ instead of T and L. Let $\Delta_T(x,y)=(B(y)+2xr)^2-4A(x)C$ and $\Delta_L(x,y)=(B(y)-2xr)^2-4A(x)C$ be the two discriminants of $T(x,y,\mu)=0$ and $L(x,y,\mu)=0$, respectively.

Note that, for $C>0$ and $A(x)\neq 0$ we have

$$\begin{cases} \big(t_2(x,y)-t_1(x,y)\big)\operatorname{sgn}\big(A(x)\big)>0, \\ \big(l_2(x,y)-l_1(x,y)\big)\operatorname{sgn}\big(A(x)\big)>0. \end{cases} \tag{3.7}$$

With this observation, and following a few standard arguments concerning the sign of a quadratic function, we state the following:

Lemma 11. *Let (x,y) be fixed. Consider the following general system*

$$\begin{cases} T(x,y,\mu)\geqslant 0, \\ L(x,y,\mu)\leqslant 0. \end{cases} \tag{3.8}$$

Then:

1. *If $\Delta_T(x,y)<0$ and $\Delta_L(x,y)<0$, there is no positive μ satisfying* (3.8)*.*
2. *If $\Delta_T(x,y)<0$ and $\Delta_L(x,y)\geqslant 0$, then* (3.8) *is satisfied for*
$$\mu\in\big[\min\{l_1(x,y),l_2(x,y)\},\max\{l_1(x,y),l_2(x,y)\}\big].$$
3. *If $\Delta_T(x,y)\geqslant 0$ and $\Delta_L(x,y)<0$, then* (3.8) *is satisfied for*
$$\mu\in\big[\min\{t_1(x,y),t_2(x,y)\},\max\{t_1(x,y),t_2(x,y)\}\big].$$
4. *If $\Delta_T(x,y)\geqslant 0$ and $\Delta_L(x,y)\geqslant 0$, then* (3.8) *is satisfied for*
$$\begin{cases} \mu\in\big\{\mathbb{R}\setminus\big(t_1(x,y),t_2(x,y)\big)\big\}\cap\big[l_1(x,y),l_2(x,y)\big], & \text{if } A(x)>0, \\ \mu\in\big\{\mathbb{R}\setminus\big(l_2(x,y),l_1(x,y)\big)\big\}\cap\big[t_2(x,y),t_1(x,y)\big], & \text{if } A(x)<0. \end{cases} \tag{3.9}$$

Now we find the two sets of $\mu>0$ for which the first two inequalities of system (3.5) and the last two inequalities in system (3.5) are respectively satisfied. The final range of $\mu>0$ for which the system is satisfied is obtained as the intersection of those two sets.

By comparison arguments between the roots of $T(n^*,m^*,\mu)$, $L(n^*,m^*,\mu)$, $T(n,m,\mu)$, and $L(n,m,\mu)$, we prove

Proposition 12.

(i) *If $A(n)>0$, then we have*
   1. *For $\Delta_T(n,m)<0$ and $\Delta_L(n,m)<0$, there will be no $\mu>0$ to satisfy* (3.5)*.*
   2. *For $\Delta_T(n,m)<0$ and $\Delta_L(n,m)\geqslant 0$, the system* (3.5) *is satisfied for*
   $$\mu\in\big[\tilde{\epsilon}^*_1,\epsilon^*\big]\cap[\tilde{\epsilon}_1,\tilde{\epsilon}_2],$$
   using (3.6)*.*
   3. *For $\Delta_T(n,m)\geqslant 0$ and $\Delta_L(n,m)<0$, the system* (3.5) *is satisfied for*
   $$\mu\in\big[\tilde{\epsilon}^*_1,\epsilon^*\big]\cap[\epsilon,\epsilon_2].$$
   4. *For $\Delta_T(n,m)\geqslant 0$ and $\Delta_L(n,m)\geqslant 0$, the system (3.5) is satisfied for*
   $$\mu\in\big[\tilde{\epsilon}^*_1,\epsilon^*\big]\cap[\tilde{\epsilon}_1,\tilde{\epsilon}_2]\cap\big\{\mathbb{R}\setminus[\epsilon,\epsilon_2]\big\}.$$

(ii) *If $A(n)<0$, then we have*
   1. *For $\Delta_T(n^*,m^*)<0$ and $\Delta_L(n^*,m^*)<0$, there are no $\mu>0$ which satisfy* (3.5)*.*
   2. *For $\Delta_T(n^*,m^*)<0$ and $\Delta_L(n^*,m^*)\geqslant 0$, system* (3.5) *is satisfied for*
   $$\mu\in[\tilde{\epsilon}_1,\epsilon]\cap\big[\tilde{\epsilon}^*_1,\tilde{\epsilon}^*_2\big].$$
   3. *For $\Delta_T(n^*,m^*)\geqslant 0$ and $\Delta_L(n^*,m^*)<0$, system* (3.5) *is satisfied for*
   $$\mu\in[\tilde{\epsilon}_1,\epsilon]\cap\big[\epsilon^*_1,\epsilon^*_2\big].$$

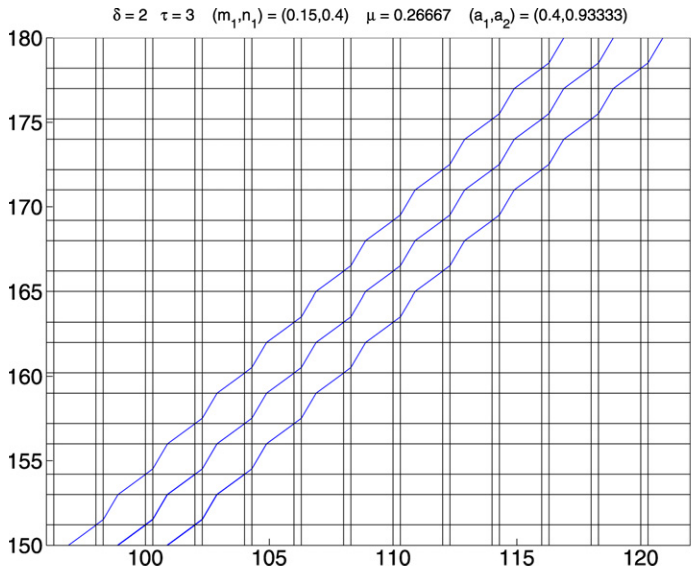

Fig. 4. Characteristic paths when $\mu = 0.4\delta/\tau$.

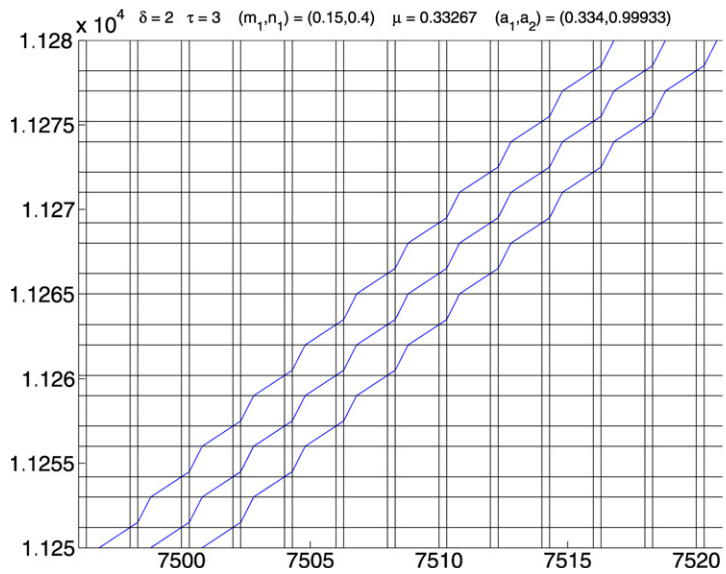

Fig. 5. Characteristic paths when $\mu = 0.499\delta/\tau$.

4. *For* $\Delta_T(n^*, m^*) \geqslant 0$ *and* $\Delta_L(n^*, m^*) \geqslant 0$, *system* (3.5) *is satisfied for*

$$\mu \in [\tilde{\epsilon}_1, \epsilon] \cap \left[\tilde{\epsilon}_1^*, \tilde{\epsilon}_2^*\right] \cap \left\{\mathbb{R} \setminus \left[\epsilon_1^*, \epsilon_2^*\right]\right\}.$$

From Proposition 12, it is immediately seen that for $C > 0$ and $A(n) \neq 0$ the closure of the range of μ satisfying (3.8) does not include 0. Therefore we conclude that the only case for which a line can be viewed as an asymptotic limit of a limit cycle of the class $C_{z_1}^{\nu}$ in a microstructure with $a_1^{\mu} = \frac{p}{q} - \mu$ and $a_2^{\mu} = \frac{p}{q} + \mu$ is when $C = r(\delta - \tau r) = 0$, and this highlights $r = \frac{\delta}{\tau}$ as the only possible slope $\frac{dz}{dt}$ for such a line.

A careful analysis for the case $C = 0$ completes the proof of the theorem. □

Remark 13. The case $A(n) = 0$ is trivial and one can show that in this case no limit cycles are formed in the microstructure when $\mu \to 0$ if $C \neq 0$. On the other hand, if $C = 0$, then one always has limit cycles for arbitrarily small μ. That is, in this case, the limit cycles approach the line with slope $r = \frac{\delta}{\tau}$ as $\mu \to 0$.

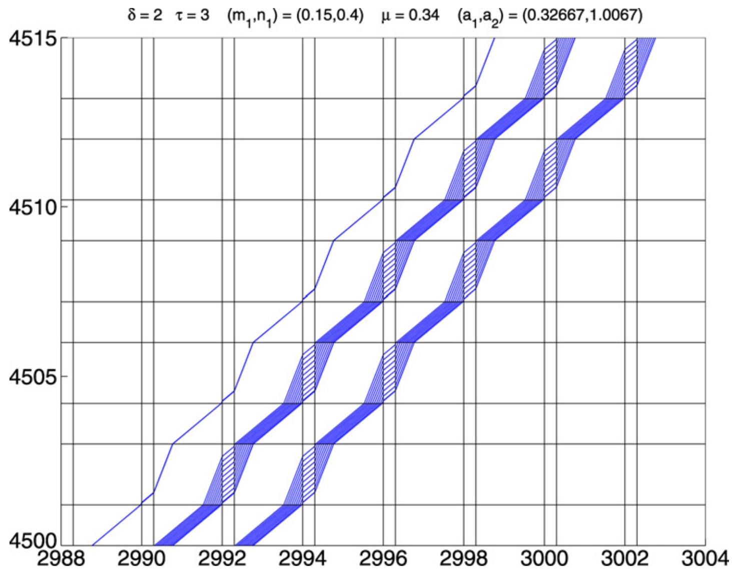

Fig. 6. Characteristic paths when $\mu = 0.51\delta/\tau$.

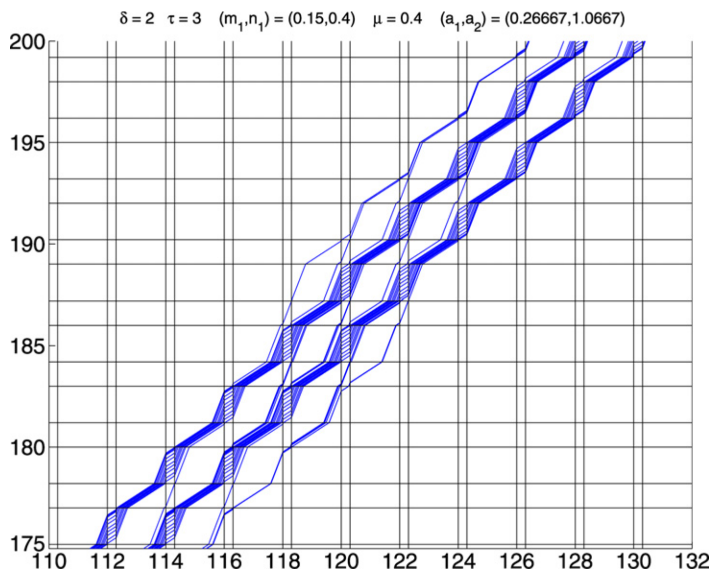

Fig. 7. Characteristic paths when $\mu = 0.6\delta/\tau$.

4. Numerical verification

In this section, we provide numerical support for the theory developed herein. We use $\delta = 2$, $\tau = 3$. The first set of results investigates the checkerboard structure described by $\frac{n}{\tau} = 0.4$ and $\frac{m}{\delta} = 0.15$. Criterion 2 of Theorem 10 is thus satisfied, and according to the theorem and (3.2), there is a critical value

$$\bar{\mu} = 0.5\delta/\tau = 1/3$$

such that, when $a_1 = 2/3 - \mu$ and $a_2 = 2/3 + \mu$, limit cycles with speed $\delta/\tau = 2/3$ for $\mu \in [0, \bar{\mu}]$ develop. The figures in this section show the behavior of paths of right-going information $R = u - v/\gamma$; these paths originated at uniform locations on the interval $[-2, 2]$.

Figs. 4 and 5 show these paths in the cases when μ is chosen in the subcritical zone, taking values $0.4\delta/\tau$ and $0.499\delta/\tau$. It is clearly seen that the paths converge to limit cycles so that information propagates with an overall speed of $\delta/\tau = 2/3$ as predicted. When supercritical values of μ are considered, we find that no limit cycles form. Figs. 6 and 7 illustrate this for $\mu = 0.51\delta/\tau$ and $\mu = 0.6\delta/\tau$.

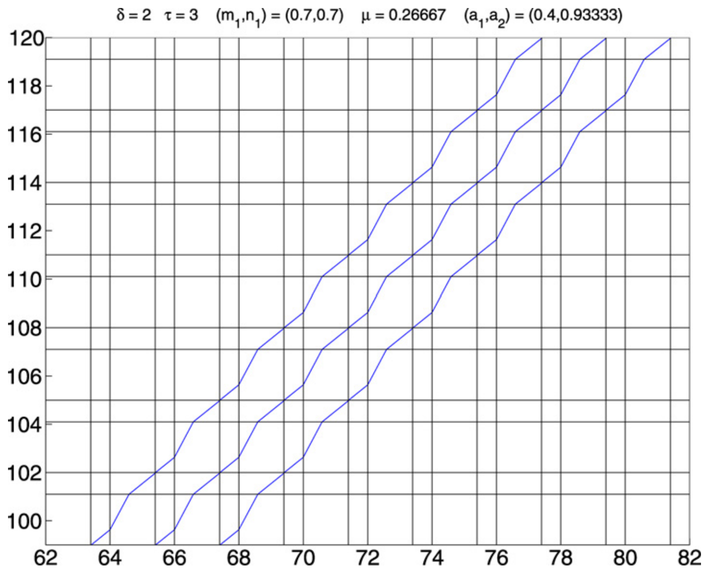

Fig. 8. Characteristic paths when $\mu = 0.4\delta/\tau$.

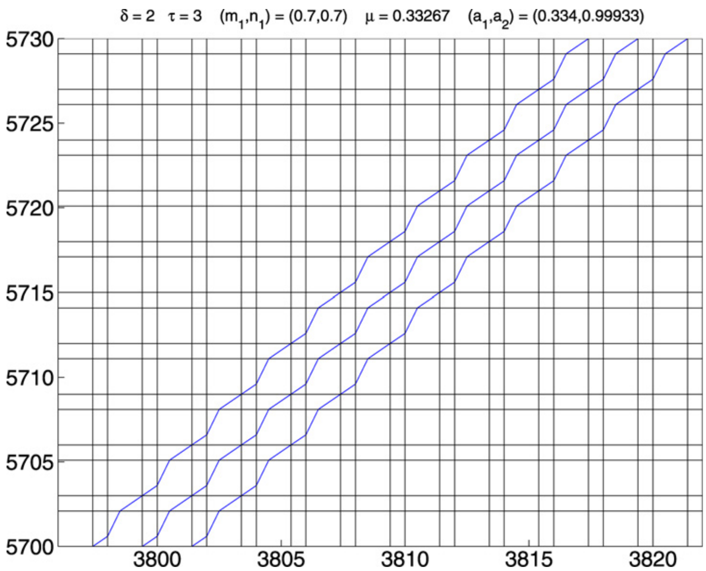

Fig. 9. Characteristic paths when $\mu = 0.499\delta/\tau$.

For the second set of results, we pick $\frac{n}{\tau} = 0.7$. If we also select $\frac{m}{\delta} = 0.7$, then (3.2) again promises limit cycles when $\bar{\mu} = 0.5\delta/\tau = 1/3$. In Figs. 8 and 9, limit cycles with speed 2/3 are clearly seen when $\mu = 0.4\delta/\tau$ and $\mu = 0.499\delta/\tau$. Figs. 10 and 11 illustrate that no limit cycles form when values of μ greater than the critical $\bar{\mu}$ value are used; here, $\mu = 0.51\delta/\tau$ and $\mu = 0.6\delta/\tau$.

5. Conclusions

In the above, we considered paths belonging to the class C_{z_1} introduced in Section 2. The specifics of this class is that every path enters the higher phase velocity (hpv) material 2 via the horizontal (temporal) interface, and leaves it through the vertical (static) interface; see Fig. 1. With this special behavior of characteristics, the wave energy increases by the factor $\frac{a_2}{a_1}$ at each entrance into the hpv-material, and the energy flux remains continuous at each departure from this material. As a result, we create non-stop wave energy accumulation by the factor $(\frac{a_2}{a_1})^2$ per period. The advantage of such an arrangement is obvious: it avoids entrances of the characteristics into the lower phase velocity (lpv) material 1 via the horizontal interface: every such entrance would cause the decrease of energy by the factor $\frac{a_1}{a_2}$. Instead, the characteristics enter the lpv-material through the vertical interface which does not affect the energy due to the continuity of the energy flux. The-

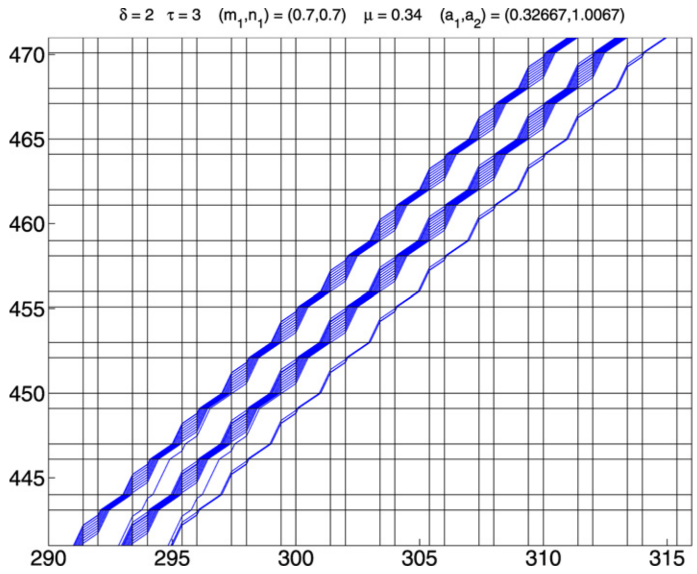

Fig. 10. Characteristic paths when $\mu = 0.51\delta/\tau$.

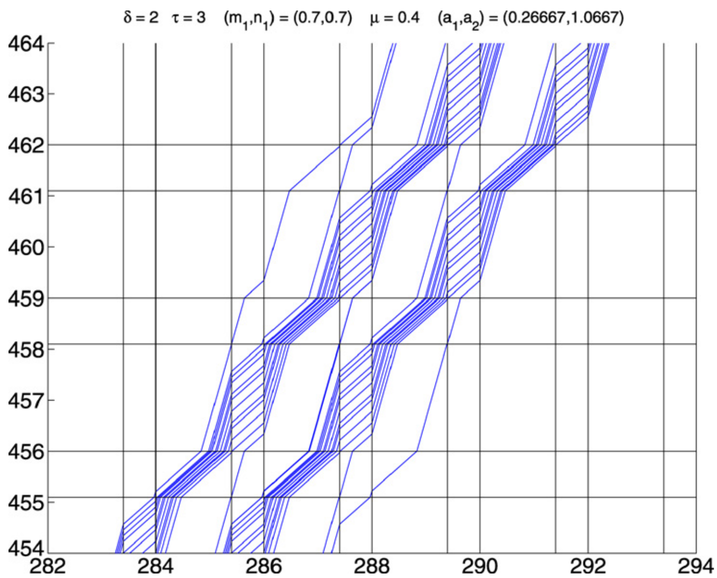

Fig. 11. Characteristic paths when $\mu = 0.6\delta/\tau$.

orem 10 in Section 3 establishes conditions necessary and sufficient for a microstructure with parameters $a_1^\mu = \frac{p}{q} - \mu$ and $a_2^\mu = \frac{p}{q} + \mu$ to exhibit limit cycles in the class C_{z_1} within the range $(0, \bar{\mu}]$ for μ. The closure of the range includes the point $\mu = 0$ which means that the line of slope $\frac{\delta}{\tau}$ may then be viewed as a limit of closed neighboring trajectories that approach it as $\mu \to 0$; the energy carried by the wave blows up in infinite time for all such paths with $\mu \neq 0$. This is the reason why homogenization as classically understood is not possible for this problem, and the study of the limit behavior of characteristics is the instrument through which one can gain information about the wave propagation through a checkerboard structure.

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