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Engineering anisotropy to amplify a long-wavelength field without a limit

L. P. Liu[1] and D. Onofrei[2]

[1]Department of Mathematics, Department of Mechanical and Aerospace Engineering, Rutgers University, Piscataway, NJ 08854, USA
[2]Department of Mathematics, University of Houston, Houston, TX 77204, USA

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Author for correspondence:
L. P. Liu
e-mail: liu.liping@rutgers.edu

We present a simple design of circular or spherical shells capable of amplifying a long-wavelength or static field. This design makes use of only two isotropic materials and is optimally restricted to the prescribed geometric and materials constraint. Furthermore, it is shown that the amplification factor of the structure can be made arbitrarily large as the ratio of the radius of the inner sphere to that of the outer sphere decreases to zero. It is anticipated that the presented design will be useful for high-gain antennae for telecommunications, magnets generating strong, local uniform fields and thermoelectric devices harvesting thermal energy, among other applications.

1. Introduction

The design of composites usually focuses on optimizing (i.e. maximizing or minimizing) the effective properties of composites [1,2]. In many applications, however, it is the local field that one would like to control and optimize. This is clearly the case for mechanical structures where minimizing stress concentration is critical for improving the safety and reliability of the structures [3,4]. Moreover, recent attempts in cloaking devices, as discussed in Pendry *et al.* [5], Leonhardt [6], Milton *et al.* [7] and Norris [8], from the viewpoint of optimal design [9], concern designing a passive structure such that the wave field in a subregion is negligibly small, and henceforth, the presence of a foreign object does not interact with and cannot be detected by external wave fields. The exciting development [10–13] in realizing 'cloaking' for acoustic waves and electromagnetic waves motivates us to ponder on the opposite question of cloaking, i.e. the possibility of 'field amplifiers' such that the wave fields in a subregion are much stronger than

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the external wave fields. It appears to be an easy task to amplify short-wavelength (or high-frequency) waves locally. For instance, the classic convex and concave lens/mirrors can be arranged to focus lights in a small local region. In addition, antennae or microphones are essentially devices that receive and amplify external weak wave fields. Then, the interesting question is whether it is possible to amplify a long-wavelength or static field.

Undoubtedly, the question raised above is of great practical interest. To mention a few, we note that the technology of magnetic resonance imaging requires a highly uniform magnetic field [14] of the order of 0.2–10 T, and hence the use of superconductors and associated cryogenic systems is inevitable. In addition, large magnets generating fields above 10 T cost tens of millions of dollars to build, and are precious tools for investigating a wide range of materials that are the essential building blocks of all modern technologies, including computers, motors, maglev trains, etc. (see [15]). The estimated cost of generating a magnetic field is around US $1 million per tesla [16]. Therefore, there is substantial economic incentive in using a passive structure to amplify an ambient magnetic field. Moreover, in high-precision measurements of magnetic or electric fields, the sensitivity of the measuring devices can be improved as much as a passive structure can amplify the measured fields [17]. Long-distance underwater or in-air communications typically rely on acoustic or electromagnetic waves of long wavelength (of the order of metres for underwater acoustic waves and kilometres for electromagnetic waves in air) [18]. The field amplifier for long-wavelength wave fields can certainly improve the sensitivity of the receivers, the range of communications and lower the power of transmitters. Finally, in the application of thermoelectric materials, the electric energy generated by (per unit volume) thermoelectric materials scales as $|\nabla T|^2$ (T is temperature) [19]. A structure that can amplify $|\nabla T|$ can be used to reduce the thermoelectric materials for a targeted power output and lower the cost of the overall energy production.

From the mathematical viewpoint, to the leading order, a wave field is governed by an elliptic equation for 'normal', non-resonant devices in the long-wavelength limit. As a well-known fact in the theory of elliptic equations, geometric singularities, e.g. a sharp tip or corner, induces singular or unbounded fields at the singularities [20]. However, these unbounded fields are of little interest for the above-mentioned applications as (i) the region of the large field is small and (ii) the field is highly non-uniform and diminishes quickly away from the singular point. Therefore, we focus on passive structures that can amplify an external field *uniformly* in a subregion. Secondly, material singularities, e.g. zero or infinite bulk moduli for acoustic waves, also induce singular fields. Though such singular materials could be realized by dynamic effects such as resonance or composites at a suitable asymptotic limit [21], we shall restrict ourselves to *normal* materials with a positive definite material property tensor. Like the designs of cloaking devices [5,22], the design of our field amplifier also relies on engineering the effective anisotropy of the medium. For explicit calculations, the structures are assumed to be spherically symmetric. We remark that this assumption is not essential and the proposed approach can be implemented, at least numerically, for general geometries.

This paper is organized as follows. The general problem is formulated in §2. We first consider the long-wavelength limit, i.e. the static (or steady-state) case in §3. Simple designs of shells made of two materials that can amplify the applied remote field in a bore region without a limit are presented in §3*b*. In §4, we rigorously show that the designs and results are valid for long- but finite-wavelength fields. We summarize in §5 and provide an outlook.

2. Problem statement

We consider a plane wave propagating in the $\hat{\mathbf{k}}$-direction with the far field given by

$$\nabla\xi(\mathbf{x}) = \mathbf{e}_1 \exp(ik\hat{\mathbf{k}}\cdot\mathbf{x}), \quad \text{if } |\mathbf{x}| \to +\infty, \tag{2.1}$$

where $\hat{\mathbf{k}}$ is a unit vector, $\mathbf{e}_i$ ($i = 1, \ldots, n$) are the canonical bases for the rectangular coordinate system, the far-field amplitude is normalized to be 1, and $\lambda = 2\pi/k$ is the wavelength. Let B_R

denote the ball of radius R centred at the origin, $B_{R_2} \setminus B_{R_1}$ $(0 < R_1 < R_2)$ be the domain where the properties and distribution of the materials are to be designed, and B_{R_1} be the target domain where the field $\nabla\xi$ is preferably strong. The scalar function $\xi : \mathbb{R}^n \to \mathbb{R}$ satisfies

$$\operatorname{div}[\mathbf{A}(\mathbf{x})\nabla\xi(\mathbf{x})] + \varrho(\mathbf{x})k^2\xi(\mathbf{x}) = 0 \quad \text{on } \mathbb{R}^n, \tag{2.2}$$

where the material property tensor $\mathbf{A}(\mathbf{x})$ and $\varrho(\mathbf{x})$ take the values of the ambient medium if $|\mathbf{x}| > R_2$ or $|\mathbf{x}| < R_1$. Without loss of generality, assume that the material properties of the ambient medium satisfy

$$\mathbf{A}(\mathbf{x}) = \mathbf{I} \quad \text{and} \quad \varrho(\mathbf{x}) = 1 \quad \forall |\mathbf{x}| > R_2 \ \& \ |\mathbf{x}| < R_1, \tag{2.3}$$

where $\mathbf{I}$ denotes the the identity matrix in $\mathbb{R}^{n\times n}$.

In general, it can be shown that the gradient field $\nabla\xi$ is by no means bounded on B_{R_1} if the material properties $\mathbf{A}$ and ϱ on our design domain $B_{R_2} \setminus B_{R_1}$ can take arbitrary values of positive tensors and numbers. In reality, the material properties, of course, have limited choices. Subsequently, we shall restrict ourselves to two isotropic materials with the tensor $\mathbf{A}$ given by either $\sigma_0\mathbf{I}$ or $\sigma_1\mathbf{I}$,

$$\mathbf{A}(\mathbf{x}) \in \{\sigma_0\mathbf{I}, \sigma_1\mathbf{I}\} \quad \forall \mathbf{x} \in B_{R_2} \setminus B_{R_1}. \tag{2.4}$$

Below we first consider the long-wavelength limit (i.e. $\lambda \to +\infty$ or zero frequency), and then show that the designs in the long-wavelength limit are also valid for low-frequency waves.

3. The long-wavelength limit

In the long-wavelength limit (i.e. zero frequency), the near field around B_{R_2} is determined by (cf. (4.33) and (2.1))

$$\left.\begin{aligned} &\operatorname{div}[\mathbf{A}(\mathbf{x})\nabla\xi(\mathbf{x})] = 0 \quad \text{on } \mathbb{R}^n \\ \text{and} \quad &(\nabla\xi(\mathbf{x}) - \mathbf{e}_1)\cdot\mathbf{e}_r \to 0 \quad \text{uniformly as } |\mathbf{x}| \to +\infty, \end{aligned}\right\} \tag{3.1}$$

where $\mathbf{e}_r = \mathbf{x}/|\mathbf{x}|$ is the radial direction. We remark that equation (3.1) apparently determines the static response of the dielectric (para/diamagnetic) medium under a far applied electric (magnetic) field, where the tensor $\mathbf{A}$ is interpreted as the dielectric (permeability) tensor. It also describes the steady state of a thermal (electrical) conductivity problem, where the tensor $\mathbf{A}$ is interpreted as the thermal (electrical) conductivity tensor.

(a) Amplification factor of a single anisotropic spherical shell

Our goal is to find $\mathbf{A}(\mathbf{x})$ satisfying the constraint (2.4) such that the gradient field $\nabla\xi$ in B_{R_1} can be as large as possible. Motivated by the calculation of Meyers [23], we assume that the materials in $B_{R_2}\setminus B_{R_1}$ are transverse isotropic with

$$\mathbf{A}(\mathbf{x}) = a_r(r)\mathbf{e}_r \otimes \mathbf{e}_r + a_t(r)(\mathbf{I} - \mathbf{e}_r \otimes \mathbf{e}_r) \quad \forall R_1 < |\mathbf{x}| < R_2, \tag{3.2}$$

where a_r (a_t) is the coefficient in the radial (transverse) direction. Note that the above tensors do not satisfy (2.4) *per se*. Nevertheless, we may assume that material tensor of form (3.2) is the effective tensor achieved by composites of $\sigma_0\mathbf{I}$ and $\sigma_1\mathbf{I}$. From the classic Hashin–Shtrikman bounds
[2,15], we infer that the coefficients (a_r, a_t) in (3.2) necessarily satisfy that, for some volume fraction $\theta \in [0, 1]$,

$$\left.\begin{aligned} &\frac{(n-1)\sigma_0}{a_t - \sigma_0} + \frac{\sigma_0}{a_r - \sigma_0} \le \frac{n\sigma_0 + (1-\theta)(\sigma_1 - \sigma_0)}{\theta(\sigma_1 - \sigma_0)} \\ \text{and} \quad &\frac{(n-1)\sigma_1}{a_t - \sigma_1} + \frac{\sigma_1}{a_r - \sigma_1} \ge \frac{-n\sigma_1 + \theta(\sigma_1 - \sigma_0)}{(1-\theta)(\sigma_1 - \sigma_0)}. \end{aligned}\right\} \tag{3.3}$$

We denote by $\mathcal{C}$ the set of all pairs of coefficients (a_r, a_t) satisfying the above inequalities. Figure 3*a* illustrates this set for the case where $\sigma_0 = 1$, $\sigma_1 = 10$ in three dimensions $(n = 3)$.

We now solve (3.1) explicitly and find the preferred material properties on the design domain $B_{R_2} \backslash B_{R_1}$. By symmetry, we claim that the solution to (3.1) with material properties prescribed by (2.3) and (3.2) is given by

$$\xi = \mathbf{e}_1 \cdot \nabla u = u' \mathbf{e}_r \cdot \mathbf{e}_1, \quad u = u(r) \quad \forall r < R_2. \tag{3.4}$$

To see this, using (3.4), the gradient field is given by

$$\nabla \xi = (\nabla \nabla u)\mathbf{e}_1, \quad \nabla \nabla u = u'' \mathbf{e}_r \otimes \mathbf{e}_r + \frac{u'}{r}(\mathbf{I} - \mathbf{e}_r \otimes \mathbf{e}_r), \tag{3.5}$$

and hence

$$\mathbf{A}(\mathbf{x}) \nabla \xi = \left\{ a_r(r) u''(r) \mathbf{e}_r \otimes \mathbf{e}_r + a_t(r) \frac{u'}{r} [\mathbf{I} - \mathbf{e}_r \otimes \mathbf{e}_r] \right\} \mathbf{e}_1. \tag{3.6}$$

Therefore, equation (3.1) is satisfied if

$$\left.\begin{aligned}
&(a_r(r)u'')' + \frac{n-1}{r}\left[a_r(r)u'' - \frac{a_t(r)}{r}u'\right] = 0 \quad \forall r \in (R_1, R_2),\\
&u''' + \frac{n-1}{r}\left[u'' - \frac{1}{r}u'\right] = 0 \quad \forall r \le R_1 \text{ and } r \ge R_2,\\
&u''(r)|_{r=R_1-} = a_r(r)u''(r)|_{r=R_1+},\\
&u''(r)|_{r=R_2+} = a_r(r)u''(r)|_{r=R_2-}\\
\text{and}\quad &u''(r) \to 1 \quad \text{as } r \to +\infty.
\end{aligned}\right\} \tag{3.7}$$

We observe that the field $\nabla \xi$ determined by solutions to (3.7) are necessarily uniform on B_{R_1} (cf. (3.10)), and that the magnitude of the field is given by $|u''|\,|_{B_{R_1}}$. Therefore, restricted to the above geometric and material constraints, the optimal design problem can be stated as

$$\left.\begin{aligned}
\max\{|u''|\,|_{B_{R_1}} : u \text{ is a solution to (3.7)};\\
\mathbf{A}(\mathbf{x}), \text{ given by (3.2), satisfies (3.3)}\}.
\end{aligned}\right\} \tag{3.8}$$

We remark that the above optimal design problem is difficult for the non-local dependence of the field $|u''||_{B_{R_1}}$ on the material properties $a_r(r)$ and $a_t(r)$, in spite of the fact that (3.7) is merely an ordinary differential equation that, presumably, may be solved in closed form. Subsequently, we consider the case that $a_r(r)$ and $a_t(r)$ are constants independent of r. In this case, we rewrite $(3.7)_1$ and $(3.7)_2$ as $(i = 0, 1, 2)$,

$$u''' + \frac{n-1}{r}\left[u'' - \frac{s_i}{r}u'\right] = 0 \quad \forall r \in [R_i R_{i+1}), \tag{3.9}$$

where $s_0 = s_2 = 1$, $R_0 = 0$, $R_3 = +\infty$ and $s_1 = a_t/a_r > 0$ characterizes the anisotropy of the material in the shell $B_{R_1} \backslash B_{R_0}$. By direct calculations, we find the general solution to (3.9) is given by

$$u'(r) = A_i r^{\alpha_i} + B_i r^{\beta_i} \quad \forall r \in [R_i, R_{i+1}), \; i = 0, 1, 2, \tag{3.10}$$

where A_i, B_i $(i = 0, 1, 2)$ are integration constants, and

$$\left.\begin{aligned}
\alpha_i &= \frac{2 - n + \sqrt{(n-2)^2 + 4(n-1)s_i}}{2} > 0,\\
&\qquad\qquad\qquad\qquad \alpha_i + \beta_i = 2 - n.\\
\beta_i &= \frac{2 - n - \sqrt{(n-2)^2 + 4(n-1)s_i}}{2} < 0,
\end{aligned}\right\} \tag{3.11}$$

The continuity of u' and equations $(3.7)_{3,4,5}$ imply that $B_0 = 0$, $A_2 = 1$ $(i = 1, 2)$

$$A_{i-1} R_i^{\alpha_{i-1}} + B_{i-1} R_i^{\beta_{i-1}} = A_i R_i^{\alpha_i} + B_i R_i^{\beta_i}$$

and

$$a_{i-1} A_{i-1} \alpha_{i-1} R_i^{\alpha_{i-1}-1} + a_{i-1} B_{i-1} \beta_{i-1} R_i^{\beta_{i-1}-1} = a_i A_i \alpha_i R_i^{\alpha_i - 1} + a_i B_i \beta_i R_i^{\beta_i - 1},$$

where $a_0 = a_2 = 1$, $a_1 = a_r$. Thus, we have that

$$\begin{bmatrix} A_i \\ B_i \end{bmatrix} = \mathbf{M}_i \begin{bmatrix} A_{i-1} \\ B_{i-1} \end{bmatrix} \quad (i = 1, 2) \quad \text{and} \quad \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \mathbf{M}_2\mathbf{M}_1 \begin{bmatrix} A_0 \\ B_0 \end{bmatrix}, \tag{3.12}$$

where $(i = 1, 2)$

$$\mathbf{M}_i = \frac{1}{a_i(\beta_i - \alpha_i)} \begin{bmatrix} (-\alpha_{i-1}a_{i-1} + \beta_i a_i)R_i^{\alpha_{i-1}-\alpha_i} & (-\beta_{i-1}a_{i-1} + \beta_i a_i)R_i^{\beta_{i-1}-\alpha_i} \\ (\alpha_{i-1}a_{i-1} - \alpha_i a_i)R_i^{\alpha_{i-1}-\beta_i} & (\beta_{i-1}a_{i-1} - \alpha_i a_i)R_i^{\beta_{i-1}-\beta_i} \end{bmatrix}.$$

Furthermore, as $a_0 = a_2 = 1$ and $s_0 = s_2 = 1$, by (3.11) we have that $\alpha_0 = \alpha_2 = 1$, $\beta_0 = \beta_2 = 1 - n$ and the transfer matrix of the shell $B_{R_2} \setminus B_{R_1}$ is given by

$$\mathbf{T} = \mathbf{M}_2\mathbf{M}_1 = \frac{1}{(\alpha_1 - \beta_1)a_1 n} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix},$$

where $(\rho = R_1/R_2 < 1)$

$$\left.\begin{aligned} T_{11} &= \rho^{n-1}[-(-1+\beta_1 a_1)(-1+\alpha_1 a_1+n)\rho^{\beta_1} \\ &\quad + (-1+\alpha_1 a_1)(-1+\beta_1 a_1+n)\rho^{\alpha_1}], \\ T_{12} &= (-1+\alpha_1 a_1+n)(-1+\beta_1 a_1+n)R_2^{1-n}R_1^{-1}[-\rho^{\beta_1}+\rho^{\alpha_1}], \\ T_{21} &= (-1+\alpha_1 a_1)(-1+\beta_1 a_1)R_1{}^{n-1}R_2[\rho^{\beta_1}-\rho^{\alpha_1}] \\ \text{and} \quad T_{22} &= \rho^{1-n}[(-1+\alpha_1 a_1)(-1+\beta_1 a_1+n)\rho^{-\alpha_1} \\ &\quad - (-1+\beta_1 a_1)(-1+\alpha_1 a_1+n)\rho^{-\beta_1}]. \end{aligned}\right\} \tag{3.13}$$

Noting that $B_0 = 0$, $A_2 = 1$ and the exterior field contributed by the term $B_2 r^{1-n}$ vanishes as $r \to +\infty$, using (3.12), we find the amplification factor f, defined as the ratio of the magnitude of the field on B_{R_1} to that of the far field as $r \to \infty$, is given by

$$f = \frac{(\alpha_1 - \beta_1)a_1 n}{T_{11}} \qquad \left(\text{recall that } a_1 = a_r,\ s_1 = \frac{a_t}{a_r}\right). \tag{3.14}$$

Using (3.11) and $(3.13)_1$ we find that for moderate material properties and geometric factors, e.g. $a_r = 10$, $s_1 = 0.1$, $\rho = 0.1$, the amplification factor $f = 3.76$ in two dimensions, which is a significant effect with regard to the difficulty in enhancing static fields uniformly by passive structures.

Below we write $f = f(a_r, s_1, \rho, n)$ to stress that the amplification factor f depends on the dimension of the space n, material properties (a_r, s_1) and geometric factor ρ and explore the conditions to maximize the amplification factor. By (3.11) and $(3.13)_1$, we show the contours of the amplification factor $f = f(a_r, s_1, \rho, n)$ in figure 1: (*a*) $f = f(a_r, s_1 = 0.1, \rho, n = 2)$; (*b*) $f = f(a_r = 1, s_1, \rho, n = 2)$; (*c*) $f = f(a_r, s_1, \rho = 0.1, n = 2)$; (*d*) $f = f(a_r, s_1 = 0.1, \rho, n = 3)$; (*e*) $f = f(a_r = 1, s_1, \rho, n = 3)$ and (*f*) $f = f(a_r, s_1, \rho = 0.1, n = 3)$. From figure 1*a*,*c* or *d*,*f*, we observe that there exists an optimal a_r such that the amplification factor is maximized for fixed s_1 and ρ. Further, from figure 1*b* or figure 1*e*, we see that the amplification factor $f(a_r, s_1, \rho, n)$ increases as s_1 decreases or if $s_1 < 1$ and ρ decreases. Qualitatively, we also find that the amplification factor increases to infinity if s_1 or ρ approach zero.

(b) A simple design

As mentioned earlier, we will use composites of $\sigma_0\mathbf{I}$, $\sigma_1\mathbf{I}$ to achieve effective tensors prescribed by (3.2). We have denoted by $\mathcal{C}$ the set of all pairs of (a_r, a_t) satisfying (3.3). Direct calculation shows that this set is alternatively given by

$$\mathcal{C} = \left\{(a_r, a_t) : a_r \in [\sigma_0, \sigma_1],\ s_1 = \frac{a_t}{a_r} \in [s_L, s_U]\right\}, \tag{3.15}$$

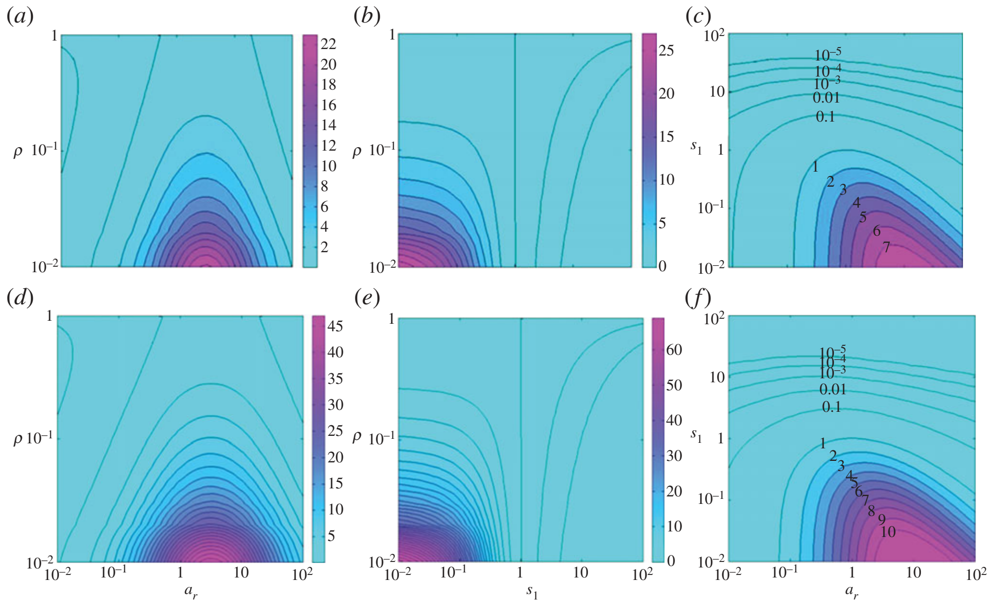

Figure 1. Contour plots of the amplification factor $f = f(a_r, s_1, \rho, n)$ in two and three dimensions. (*a*) $f = f(a_r, s_1 = 0.1, \rho, n = 2)$; (*b*) $f = f(a_r = 1, s_1, \rho, n = 2)$; (*c*) $f = f(a_r, s_1, \rho = 0.1, n = 2)$; (*d*) $f = f(a_r, s_1 = 0.1, \rho, n = 3)$; (*e*) $f = f(a_r = 1, s_1, \rho, n = 3)$ and (*f*) $f = f(a_r, s_1, \rho = 0.1, n = 3)$. (Online version in colour.)

where

$$\left.\begin{aligned} s_L &= \frac{(\sigma_1 + (n-2)a_r)(a_r - \sigma_0)\sigma_0}{a_r[\sigma_0(a_r - \sigma_0)n + a_r(\sigma_1 - a_r) - \sigma_0(\sigma_1 - \sigma_0)]} \\ \text{and} \qquad s_U &= \frac{a_r(\sigma_0 + \sigma_1) - \sigma_0\sigma_1}{a_r^2}. \end{aligned}\right\} \tag{3.16}$$

We remark that the lower boundary of the set $\mathcal{C}$, i.e. $\partial_L\mathcal{C} := \{(a_r, a_t) : s_1 = s_L,\ a_r \in [\sigma_0, \sigma_1]\}$, is also given by

$$\partial_L\mathcal{C} := \left\{(a_r, a_t) : a_t = \frac{(1-\theta)(n-2)\sigma_0 + (1 + \theta(n-2))\sigma_1}{n - 2 + \theta + (1-\theta)\sigma_1/\sigma_0},\ a_r = \theta\sigma_1 + (1-\theta)\sigma_0,\ \theta \in [0,1]\right\}. \tag{3.17}$$

In terms of (a_r, s_1), the set $\mathcal{C}$ is sketched in figure 3*b* for $\sigma_0 = 1$, $\sigma_1 = 10$ in three dimensions, where the curve corresponds to the lower boundary of the set $\mathcal{C}$.

Restricted to effective tensors attainable by composites of $\sigma_0\mathbf{I}$, $\sigma_1\mathbf{I}$, our goal is to find a simple microstructure such that the amplification factor is maximized. To this end, we first note that $s_1 \mapsto f(a_r, s_1, \rho, n)$ is monotonically decreasing. Therefore, the optimal composite must be such that $s_1 = s_L$ or, equivalently, the pair (a_r, a_t) satisfies (3.17) for some $\theta \in [0,1]$. Inserting (3.17) into (3.11) and $(3.13)_1$, using (3.14), we obtain the explicit expression of the amplification factor f as a function of the dimension n, constituent material properties σ_0, σ_1, volume fraction θ and geometric factor ρ,

$$f = \tilde{f}(\sigma_0, \sigma_1, \theta, \rho, n).$$

Upon maximizing the above amplification factor over all $\theta \in [0,1]$, we obtain the optimal volume fraction θ^*, which may be written as

$$\theta^* = \theta^*(\sigma_0, \sigma_1, \rho, n). \tag{3.18}$$

Though there is no fundamental difficulty in finding the (implicit) analytical expression of the optimal volume fraction θ^* and the associated optimal amplification factor

$$f = \tilde{f}^*(\sigma_0, \sigma_1, \rho, n) = \tilde{f}(\sigma_0, \sigma_1, \theta^*(\sigma_0, \sigma_1, \rho, n), \rho, n), \tag{3.19}$$

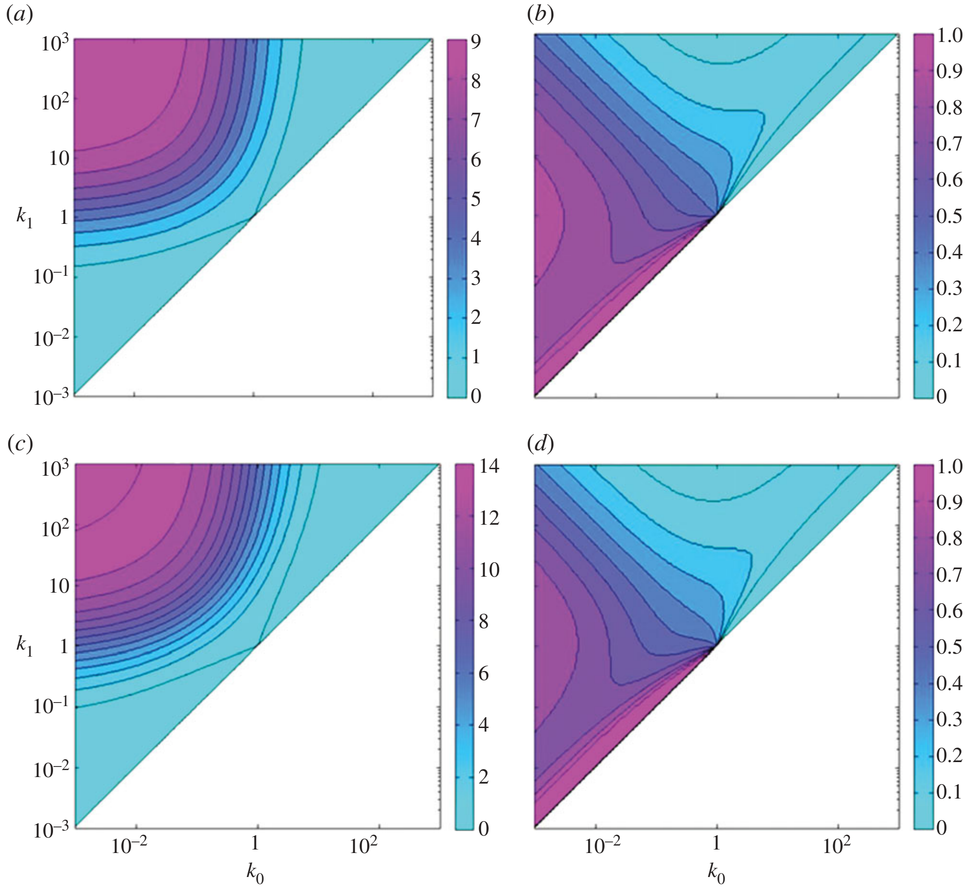

Figure 2. Contour plots of (*a*,*c*) the optimal amplification factor $f=\tilde{f}^*(\sigma_0,\sigma_1,\rho,n)$ and (*b*,*d*) the optimal volume fraction $\theta^*=\theta^*(\sigma_0,\sigma_1,\rho,n)$. (*a*) $f=\tilde{f}^*(\sigma_0,\sigma_1,\rho=0.1,n=2)$; (*b*) $\theta^*=\theta^*(\sigma_0,\sigma_1,\rho=0.1,n=2)$; (*c*) $f=\tilde{f}^*(\sigma_0,\sigma_1,\rho=0.1,n=3)$ and (*d*) $\theta^*=\theta^*(\sigma_0,\sigma_1,\rho=0.1,n=3)$. (Online version in colour.)

the calculation is tedious and we shall resort to numerical solutions. Figure 2*a*,*b* shows the maximum amplification factor $\tilde{f}^*(\sigma_0,\sigma_1,\rho,n)$ and the associated optimal volume fraction $\theta^*(\sigma_0,\sigma_1,\rho,n)$ in two dimensions ($n=2$) for $\rho=0.1$; figure 2*c*,*d* shows the maximum amplification factor $\tilde{f}^*(\sigma_0,\sigma_1,\rho,n)$ and the associated optimal volume fraction $\theta^*(\sigma_0,\sigma_1,\rho,n)$ in two dimensions ($n=3$) for $\rho=0.1$.

Finally, we discuss the microstructure such that composites of $\sigma_0\mathbf{I}$ and $\sigma_1\mathbf{I}$ indeed have the desired properties prescribed by (3.2) with $(a_r,a_t)\in\partial_L\mathcal{C}$. From the attainment conditions discussed in Liu [1], we find that the following simple microstructures realize the material properties prescribed by (3.2) with $(a_r,a_t)\in\partial_L\mathcal{C}$. In two dimensions, consider a simple tapered laminate of $\sigma_0\mathbf{I}$ and $\sigma_1\mathbf{I}$ around the core sphere B_{R_1} such that the overall structure is like a 'spike wheel', as shown in figure 3*c*. Assume the thickness of each laminate is much smaller than R_1, and locally the volume fraction of material $\sigma_1\mathbf{I}$ is θ. By homogenization theory, we see that the effective tensor on the shell $B_{R_2}\setminus B_{R_1}$ is exactly given by (3.2) with $(a_r,a_t)\in\partial_L\mathcal{C}$. In three dimensions, we consider rods of $\sigma_1\mathbf{I}$ standing on the sphere, and the remaining volume of the shell $B_{R_2}\setminus B_{R_1}$ is occupied by the material $\sigma_0\mathbf{I}$. Further, we assume the distances between neighbouring rods are much smaller than R_2 and locally the rods form a periodic lattice. Additionally, we shall assume the cross section of each rod is a periodic E-inclusion or Vigdergauz microstructure. The shape matrix of the periodic E-inclusions is isotropic and the volume fraction is θ. Here, we emphasize that the shapes of these E-inclusions depend not only on the volume fraction and shape matrix, but also on the lattice,

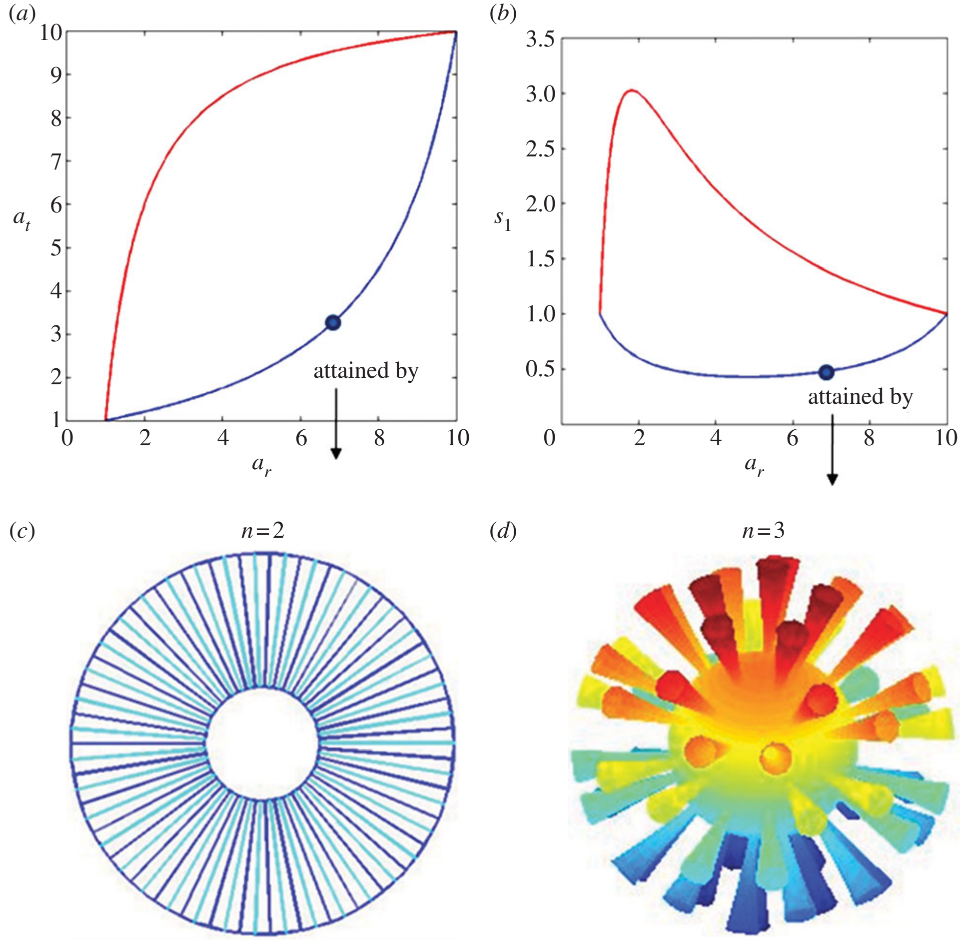

Figure 3. (*a*) The set $\mathcal{C}$ to which the pair (a_r, a_t) necessarily belong to; (*b*) the set of $(a_r, s_1 = a_t/a_r)$ if $(a_r, a_t) \in \mathcal{C}$; (*c*) a two-dimensional microstructure of the composite of $\sigma_0\mathbf{I}$, $\sigma_1\mathbf{I}$ (spike wheel) such that the effective tensor of the composite has the form (3.2) and the pair (a_r, a_t) attain a point on the lower boundary of the set $\mathcal{C}$ and (*d*) a three-dimensional microstructure (hairy sphere) of the composite of $\sigma_0\mathbf{I}$, $\sigma_1\mathbf{I}$ such that the effective tensor of the composite has the form (3.2) and the pair (a_r, a_t) attain a point on the lower boundary of the set $\mathcal{C}$. The tapered rod (hair) on the sphere and its cross section are shown in figure 4*a*,*b*. (Online version in colour.)

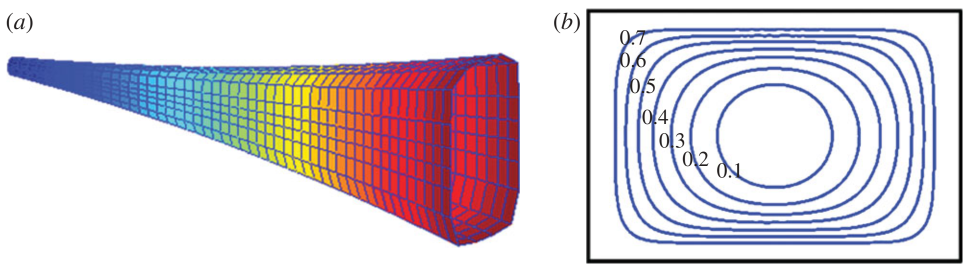

Figure 4. (*a*) The shape of the rods on the hairy sphere in figure 3*d*. (*b*) The cross section of the rod for a unit cell $[0, 1.5] \times [0, 1]$ and various volume fractions. The number on each curve is the volume fraction of the inclusion. (Online version in colour.)

i.e. the unit cell. For example, for a rectangular unit cell $[0, 1.5] \times [0, 1]$, the periodic E-inclusions are shown in figure 4 for various volume fractions. With these rods fabricated, we assemble them around the sphere B_{R_1} such that the final structure looks like a 'hairy sphere'. From the properties of periodic E-inclusions, we can show that the effective tensor on the shell $B_{R_2} \setminus B_{R_1}$ is exactly given by (3.2) with $(a_r, a_t) \in \partial_L \mathcal{C}$ [24].

4. The long-wavelength case

In this section, we will show that the designs presented in the last section are valid for long- but finite-wavelength wave fields. To simplify our exposition, we will assume $\varrho = 1$ in $\mathbb{R}^n$. An incoming plane wave will be modelled in this section by a Neumann condition imposed on a far-field boundary ∂B_R with $R \gg R_2$ ('sphere at infinity'). Thus, the time-harmonic wave field $\zeta(\mathbf{x})$ satisfies

$$\left.\begin{aligned} &\operatorname{div}[\mathbf{A}(\mathbf{x})\nabla\zeta(\mathbf{x})] + k^2\zeta(\mathbf{x}) = 0 \quad \text{in } B_R \\ \text{and} \quad &(\nabla\zeta(\mathbf{x}) - \mathbf{q}(\mathbf{x}))\cdot\mathbf{n} = 0 \quad \text{on } \partial B_R, \end{aligned}\right\} \tag{4.1}$$

where $\mathbf{q}(\mathbf{x}) = \mathbf{e}_1 \cos(k\hat{\mathbf{k}}\cdot\mathbf{x})$ and $\mathbf{n} = \mathbf{e}_r = \mathbf{x}/R$ is the exterior normal to the sphere ∂B_R. We will study the long-wavelength regime corresponding to $kR^{(n+6)/4} \ll 1$. Note that, $kR < kR^{(n+6)/4}$ for all $n \in \mathbb{N}$. We first prove that the discussion of §3*a* remains valid with minor modifications for problem (4.3). Then, in the long-wavelength case (4.1), we show that the solution to (4.3) describes the near-field response of the medium to the leading order. Using the perturbation method, we find that the correction to the solution of (3.1) for finite wavelength is of the order of $kR^{(n+6)/4}$.

We now begin our analysis for problem (4.1) by observing that, upon integrating equation $(4.1)_1$, we obtain

$$\int_{B_R} \zeta(\mathbf{x}) = -\frac{1}{k^2}\int_{\partial B_R} \mathbf{q}\cdot\mathbf{e}_r. \tag{4.2}$$

Consider the following auxiliary problem:

$$\left.\begin{aligned} &-\operatorname{div}[\mathbf{A}(\mathbf{x})\nabla\tilde{\xi}(\mathbf{x})] = 0 \quad \text{in } B_R, \\ &(\nabla\tilde{\xi}(\mathbf{x}) - \mathbf{e}_1)\cdot\mathbf{e}_r = 0 \quad \text{on } \partial B_R \\ \text{and} \quad &\int_{B_R}\tilde{\xi} = \int_{B_R}\zeta. \end{aligned}\right\} \tag{4.3}$$

Following similar arguments as in §3*a*, we claim that the solution of problem (4.3) is given by

$$\tilde{\xi} = \mathbf{e}_1\cdot\nabla v + \int_{B_R}\zeta = v'\mathbf{e}_r\cdot\mathbf{e}_1 + \int_{B_R}\zeta, \tag{4.4}$$

where $v = v(r)$ for $r < R$ has a continuous derivative and satisfies

$$\left.\begin{aligned} &(a_r(r)v'')' + \frac{n-1}{r}\left[a_r(r)v'' - \frac{a_t(r)}{r}v'\right] = 0 \quad \text{for } r \in (R_1, R_2), \\ &v''' + \frac{n-1}{r}\left[v'' - \frac{1}{r}v'\right] = 0 \quad \text{for } r \le R_1 \text{ and } R_2 \le r \le R, \\ &v''(r)|_{r=R_1-} = a_r(r)v''(r)|_{r=R_1+}, \\ &v''(r)|_{r=R_2+} = a_r(r)v''(r)|_{r=R_2-} \\ \text{and} \quad &v''(r) = 1 \quad \text{on } \partial B_R. \end{aligned}\right\} \tag{4.5}$$

Similarly, as in §3, assuming that a_r, a_t are constants, we can solve system (4.5) explicitly and obtain

$$v'(r) = C_i r^{\alpha_i} + D_i r^{\beta_i} \quad \forall r \in (R_i, R_{i+1}), \quad i = 0, 1, 2, \tag{4.6}$$

where $R_0 = 0$, $R_3 = R$, C_i, D_i $(i = 0, 1, 2)$ are integration constants and α_i and β_i are as in (3.11). Then, following very similar arguments as in (3.12) and (3.13), we find that the amplification factor, the ratio of the magnitude of the uniform field in B_{R_1} to that of the far field on ∂B_R, is given by

$$f = \frac{(\alpha_1 - \beta_1)a_1 n}{T_{11} + (1-n)T_{21}R^{-n}}, \tag{4.7}$$

where $T_{11}, T_{21}, \alpha_1, \beta_1, a_1$ are the same as before. We can observe from (4.7) that in the current situation, i.e. when $R \gg 1$, the amplification factor is very close to the amplification factor obtained in (3.14).

In the following, we rigorously prove that the solution $\tilde{\xi}$ to (4.3) is indeed a good approximation of the solution ζ to (4.1) for $R^{(n+6)/4}k \ll 1$. In this regard, we first establish a few estimates concerning the solution $\tilde{\xi}$ to (4.3). Let

$$H^1_*(B_R) = \left\{ u \in H^1(B_R), \int_{B_R} u = 0 \right\}. \tag{4.8}$$

It can easily be observed that the space $H^1_*(B_R)$ is a Hilbert space endowed with the following scalar product:

$$\langle u, v \rangle = \int_{B_R} \nabla u \cdot \nabla v. \tag{4.9}$$

Let us begin by recalling two important results that will be useful in our subsequent analysis. The first result describes the optimal constant for the Poincaré inequality on balls for functions in the space $H^1_*(B_R)$ as obtained by Payne & Weinberger [25],

$$\|\Phi\|_{L^2(B_R)} \le \frac{2R}{\pi} \|\nabla \Phi\|_{L^2(B_R)} \quad \text{for } \Phi \in H^1_*(B_R). \tag{4.10}$$

The second result describes the optimal constant in the trace inequality on balls for functions in the space $H^1_*(B_R)$ as obtained by Auchmuty [26]

$$\|\Phi\|_{L^2(\partial B_R)} \le C\sqrt{R} \|\nabla \Phi\|_{L^2(B_R)} \quad \text{for } \Phi \in H^1_*(B_R), \tag{4.11}$$

where here and subsequently in this section, we will denote by C a generic constant independent of R (and k).

Observing that $\int_{\partial B_R} \mathbf{e}_1 \cdot \mathbf{e}_r = 0$, we note that problem (4.3) admits a unique solution in $H^1(B_R)$. Moreover, from classical elliptic estimates, we have

$$\begin{aligned} \|\nabla\tilde{\xi}\|^2_{L^2(B_R)} &\le C \left| \int_{\partial B_R} (\mathbf{e}_1 \cdot \mathbf{e}_r)\tilde{\xi} \right| = C \left| \int_{\partial B_R} (\mathbf{e}_1 \cdot \mathbf{e}_r) \left(\tilde{\xi} - \frac{1}{|B_R|}\int_{B_R} \tilde{\xi} \right) \right| \\ &\le C \|(\mathbf{e}_1 \cdot \mathbf{e}_r)\|_{L^2(\partial B_R)} \left\| \tilde{\xi} - \frac{1}{|B_R|}\int_{B_R} \tilde{\xi} \right\|_{L^2(\partial B_R)} \\ &\le C |\partial B_R|^{1/2} \left\| \tilde{\xi} - \frac{1}{|B_R|}\int_{B_R} \tilde{\xi} \right\|_{L^2(\partial B_R)} \\ &\le C R^{n/2} \|\nabla\tilde{\xi}\|_{L^2(B_R)}, \end{aligned} \tag{4.12}$$

where, in the last inequality above, we have used (4.11) and the fact that the surface area of the n-dimensional sphere of radius R is given by $|\partial B_R| = R^{n-1}\varsigma_n$, with ς_n denoting the surface area of the n-dimensional unit ball. From (4.12), we deduce

$$\|\nabla\tilde{\xi}\|_{L^2(B_R)} \le O(R^{n/2}). \tag{4.13}$$

Next, recalling that the measure of B_R is given by $|B_R| = R^n \cdot \omega_n$, where ω_n is the volume of the n-dimensional unit ball, the Poincaré–Wirtinger inequality in $H^1(B_R)$ states

$$\left\| \tilde{\xi} - \frac{1}{|B_R|}\int_{B_R} \tilde{\xi} \right\|_{L^2(B_R)} \le \frac{2R}{\pi} \|\nabla\tilde{\xi}\|_{L^2(B_R)}, \tag{4.14}$$

and using this together with $(4.3)_3$ and (4.13), we obtain,

$$\begin{aligned}\|\tilde{\xi}\|_{L^2(B_R)} &\le \frac{2R}{\pi}\|\nabla\tilde{\xi}\|_{L^2(B_R)} + \frac{1}{CR^{n/2}}\left|\int_{B_R}\tilde{\xi}\right| \\ &= \frac{2R}{\pi}\|\nabla\tilde{\xi}\|_{L^2(B_R)} + \frac{1}{CR^{n/2}}\left|\int_{B_R}\zeta\right| \\ &\le O(R^{n/2+1}) + \frac{1}{k^2CR^{n/2}}\left|\int_{\partial B_R}(\mathbf{e}_1\cdot\mathbf{e}_r)\cos(k\hat{\mathbf{k}}\cdot\mathbf{x})\right| \\ &\le O(R^{n/2+1}) + \frac{1}{k^2CR^{\frac{n}{2}+1}}\left|\int_{\partial B_R}(\mathbf{e}_1\cdot\mathbf{e}_r)(1+O(k^2R^2))\right| \\ &\le O(R^{n/2+1}) + O(R^{(n+2)/2}) \\ &\le O(R^{(n+2)/2}),\end{aligned}\tag{4.15}$$

where (4.13) and (4.27) have been used for the second inequality and the third inequality follows from the Taylor expansion for $\mathbf{q}$ (when $kR \ll 1$),

$$|\mathbf{q}(\mathbf{x})\cdot\mathbf{e}_r - \mathbf{e}_1\cdot\mathbf{e}_r| \le O(k^2R^2) \quad \text{for } |\mathbf{x}| = R. \tag{4.16}$$

From (4.15) and (4.13), we have the following estimate:

$$\|\tilde{\xi}\|_{H^1(B_R)} \le O(R^{(n+2)/2}). \tag{4.17}$$

Now we are ready to state the main theorem concerning the long-wavelength case.

Theorem 4.1. *Let ζ and $\tilde{\xi}$ be the solutions to* (4.1) *and* (4.3), *respectively. Then,*

$$\|\zeta - \tilde{\xi}\|_{H^1(B_R)} \le O(k^2R^{(n+6)/2}).$$

Proof. Consider the following problem:

$$\left.\begin{aligned}-\operatorname{div}[\mathbf{A}(\mathbf{x})\nabla z(\mathbf{x})] &= h \quad \text{in } B_R \\ \text{and} \qquad \nabla z(\mathbf{x})\cdot\mathbf{e}_r = g &:= \frac{1}{\varsigma_n R^{n-1}}\int_{B_R} h \quad \text{on } \partial B_R,\end{aligned}\right\}\tag{4.18}$$

for $h \in L^2(B_R)$, where the constant g is chosen such that the following compatibility condition is satisfied:

$$-\int_{\partial B_R} g = \int_{B_R} h. \tag{4.19}$$

It is well known that the problem (4.18) has a unique solution $z \in H^1_*(B_R)$. We can therefore define a linear operator $T: H^1(B_R) \to H^1(B_R)$ such that

$$Th = z.$$

Note that T is linear and bounded. Indeed, assuming $h \in H^1(B_R)$ and multiplying (4.18) by Th and using Green's theorem, the ellipticity of $\mathbf{A}$ and the Cauchy–Schwartz inequality, we obtain

$$\|\nabla Th\|^2_{L^2(B_R)} \le C\|h\|_{L^2(B_R)}\cdot\|Th\|_{L^2(B_R)} + \frac{C}{R^{(n-1)/2}}\|h\|_{L^1(B_R)}\cdot\|Th\|_{L^2(\partial B_R)}. \tag{4.20}$$

Next from the fact that by definition, we have $Th \in H^1_*(B_R)$ and by using (4.10) in (4.20), we get

$$\|\nabla Th\|^2_{L^2(B_R)} \le CR\|h\|_{L^2(B_R)}\cdot\|\nabla Th\|_{L^2(B_R)} + \frac{C}{R^{(n-1)/2}}\|h\|_{L^1(B_R)}\cdot\|Th\|_{L^2(\partial B_R)}. \tag{4.21}$$

By using (4.11) in (4.21), we obtain

$$\|\nabla Th\|^2_{L^2(B_R)} \le CR\|h\|_{L^2(B_R)}\cdot\|\nabla Th\|_{L^2(B_R)} + \frac{C\sqrt{R}}{R^{(n-1)/2}}\|h\|_{L^1(B_R)}\cdot\|\nabla Th\|_{L^2(B_R)}$$

or

$$\begin{aligned}\|\nabla Th\|_{L^2(B_R)} &\le CR\left(\|h\|_{L^2(B_R)} + \frac{1}{R^{n/2}}\|h\|_{L^1(B_R)}\right)\\ &\le CR\big(\|h\|_{L^2(B_R)} + C\,\|h\|_{L^2(B_R)}\big)\\ &\le CR\|h\|_{L^2(B_R)} \le CR\|h\|_{H^1(B_R)}.\end{aligned} \tag{4.22}$$

Using the fact that $Th \in H^1_*(B_R)$, from (4.10) and (4.22), we obtain

$$\|Th\|_{L^2(B_R)} \le CR^2\|h\|_{H^1(B_R)}. \tag{4.23}$$

Combining (4.23) and (4.22), we have

$$\|Th\|_{H^1(B_R)} \le CR^2\|h\|_{H^1(B_R)}. \tag{4.24}$$

Taking now the supremum for all $h \in H^1(B_R)$ with $\|h\|_{H^1(B_R)} = 1$ in (4.24), we obtain

$$\|T\|_{\mathcal{O}} = \sup_{\|h\|_{H^1(B_R)}=1} \|Th\|_{H^1(B_R)} \le CR^2, \tag{4.25}$$

where $\|T\|_{\mathcal{O}}$ denotes the operatorial norm of T. Consider now, problem (4.18) with

$$h = \zeta \quad \text{and} \quad g = g_0 = \frac{1}{k^2|\partial B_R|}\int_{\partial B_R} \mathbf{q}\cdot\mathbf{e}_r, \tag{4.26}$$

where $\mathbf{q}$, ζ were defined in the context of problem (4.1). Indeed, if upon integrating (4.1), we obtain that

$$\int_{B_R} \zeta(\mathbf{x}) = -\frac{1}{k^2}\int_{\partial B_R} \mathbf{q}\cdot\mathbf{e}_r, \tag{4.27}$$

and then we see that relations (4.27) and (4.26) imply that $g = g_0$ and $h = \zeta$ satisfy the compatibility condition (4.19). We continue by observing that the function w defined by

$$w = \zeta - k^2 T\zeta \tag{4.28}$$

solves the following problem:

$$\left.\begin{aligned} &-\operatorname{div}[\mathbf{A}(\mathbf{x})\nabla w(\mathbf{x})] = 0 \quad \text{in } B_R,\\ &\nabla w(\mathbf{x})\cdot\mathbf{e}_r = \mathbf{q}(\mathbf{x})\cdot\mathbf{e}_r - \frac{1}{|\partial B_R|}\int_{\partial B_R}\mathbf{q}\cdot\mathbf{e}_r \quad \text{for } |\mathbf{x}| = R\\ \text{and} \quad &\int_{B_R} w = \int_{B_R}\zeta,\end{aligned}\right\} \tag{4.29}$$

where we used the definition of g_0 introduced in (4.26). We can rewrite problem (4.29) as follows:

$$\left.\begin{aligned} &-\operatorname{div}[\mathbf{A}(\mathbf{x})\nabla w(\mathbf{x})] = 0 \quad \text{in } B_R,\\ &\nabla w(\mathbf{x})\cdot\mathbf{e}_r = \mathbf{e}_1\cdot\mathbf{e}_r + \eta(\mathbf{x}) \quad \text{for } |\mathbf{x}| = R\\ \text{and} \quad &\int_{B_R} w = \int_{B_R}\zeta,\end{aligned}\right\} \tag{4.30}$$

where

$$\eta(\mathbf{x}) = (\mathbf{q}(\mathbf{x}) - \mathbf{e}_1)\cdot\mathbf{e}_r - \frac{1}{|\partial B_R|}\int_{\partial B_R}\mathbf{q}\cdot\mathbf{e}_r \quad \text{for } |\mathbf{x}| = R. \tag{4.31}$$

By using (4.16) in (4.31), we obtain

$$|\eta(\mathbf{x})| \le O(k^2R^2) \quad \text{for } |\mathbf{x}| = R, \tag{4.32}$$

where in equation (4.32), we have also used the fact that $\int_{\partial B_R} \mathbf{e}_1 \cdot \mathbf{e}_r = 0$. From (4.3) and (4.30), we have that $w - \tilde{\xi}$ satisfies the following problem:

$$\left.\begin{aligned} &-\operatorname{div}[\mathbf{A}(\mathbf{x})\nabla(w(\mathbf{x}) - \tilde{\xi}(\mathbf{x}))] = 0 \quad \text{in } B_R,\\ &\nabla(w(\mathbf{x}) - \tilde{\xi}) \cdot \mathbf{e}_r = \eta(\mathbf{x}) \quad \text{for } |\mathbf{x}| = R\\ \text{and}\quad &\int_{B_R} (w - \tilde{\xi}) = 0. \end{aligned}\right\} \tag{4.33}$$

Multiplying by $w - \tilde{\xi}$ in (4.33) and integrating, we obtain

$$\left.\begin{aligned} &\int_{B_R} |\nabla(w - \tilde{\xi})|^2 \le C \int_{\partial B_R} \eta(\mathbf{x})(w - \tilde{\xi}),\\ &\int_{B_R} |\nabla(w - \tilde{\xi})|^2 \le O(k^2R^2)R^{(n-1)/2}\sqrt{R}\left(\int_{B_R} |\nabla(w - \tilde{\xi})|^2\right)^{1/2}\\ \text{and}\quad &\left(\int_{B_R} |\nabla(w - \tilde{\xi})|^2\right)^{1/2} \le O(k^2R^{(n+4)/2}), \end{aligned}\right\} \tag{4.34}$$

where we have used Green's theorem, the ellipticity of $\mathbf{A}$ and (4.11). Using the fact that $w - \tilde{\xi} \in H^1_*(B_R)$ and (4.10) from (4.34), we obtain

$$\left(\int_{B_R} |w - \tilde{\xi}|^2\right)^{1/2} \le O(k^2R^{(n+6)/2}). \tag{4.35}$$

Combining (4.35) with (4.34), we obtain

$$\|w - \tilde{\xi}\|_{H^1(B_R)} \le O(k^2R^{(n+6)/2}). \tag{4.36}$$

From (4.25) and by using the fact that $kR < kR^{(n+6)/4} \ll 1$, we have that $\|k^2T\|_{\mathcal{O}} = O(k^2R^2) \ll 1$. This implies that $I - k^2T : H^1(B_R) \to H^1(B_R)$ is invertible, and we have

$$(I - k^2T)^{-1} = \sum_{i=0}^{\infty} k^{2i}T^i, \tag{4.37}$$

where $T^0 = I$ and $T^i(\cdot) = T^{i-1}(T(\cdot))$ for all $i \ge 1$. From (4.37), (4.36) and (4.28), we conclude that

$$\zeta = (I - k^2T)^{-1}w = \sum_{i=0}^{\infty} k^{2i}T^iw = w + \sum_{i=1}^{\infty} k^{2i}T^iw. \tag{4.38}$$

Next, by using (4.24), we obtain

$$\|T^iw\| \le \|T\|^i_{\mathcal{O}} \cdot \|w\|_{H^1(B_R)} \quad \text{for } i \ge 1. \tag{4.39}$$

From (4.38), (4.36), (4.25) and (4.39), we conclude that

$$\begin{aligned} \|\zeta - \tilde{\xi}\|_{H^1(B_R)} &\le \|w - \tilde{\xi}\|_{H^1(B_R)} + \|w\|_{H^1(B_R)}\|k^2T\|_{\mathcal{O}} \sum_{l=0}^{\infty}(k^2\|T\|_{\mathcal{O}})^l\\ &\le O(k^2R^{(n+6)/2}) + (\|\tilde{\xi}\|_{H^1(B_R)} + O(k^2R^{(n+6)/2}))O(k^2R^2)\frac{1}{1 - k^2\|T\|_{\mathcal{O}}}\\ &\le O(k^2R^{(n+6)/2}) + (\|\tilde{\xi}\|_{H^1(B_R)} + O(k^2R^{(n+6)/2}))O(k^2R^2)\\ &\le O(k^2R^{(n+6)/2}), \end{aligned} \tag{4.40}$$

where we have used (4.17). This then proves the fact that the designs used in the zero-frequency regime also apply to the low-frequency regime in the sense that an incoming uniform field will be almost uniform in B_{R_1} and will get amplified in this region by a factor approximately given by (4.7). ■

5. Summary and discussion

We have provided a recipe for the feasible design of composite material coating around the region B_{R_1} with the desired effect of amplifying a given uniform incoming field from an external source. Our analysis is valid in the static and long- but finite-wavelength regimes. The analysis done in §§3 and 4 also suggests that one can adapt the present results to the context of maximizing the L^2 norm of the potential itself in the region B_{R_1}. Finally, we remark that our analysis in §4 applies to the case of large frequencies with the condition of a sufficiently small control area, i.e. $R \ll 1$.

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