Solutions to the 60th William Lowell Putnam Mathematical Competition
Saturday, December 4, 1999

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A–1 Note that if $r(x)$ and $s(x)$ are any two functions, then

$$\max(r,s) = (r+s+|r-s|)/2.$$

Therefore, if $F(x)$ is the given function, we have

$$\begin{aligned} F(x) &= \max\{-3x-3,0\} - \max\{5x,0\} + 3x + 2 \\ &= (-3x-3+|3x-3|)/2 \\ &\quad - (5x+|5x|)/2 + 3x + 2 \\ &= |(3x-3)/2| - |5x/2| - x + \frac{1}{2}, \end{aligned}$$

so we may set $f(x) = (3x-3)/2$, $g(x) = 5x/2$, and $h(x) = -x + \frac{1}{2}$.

A–2 First solution: First factor $p(x) = q(x)r(x)$, where q has all real roots and r has all complex roots. Notice that each root of q has even multiplicity, otherwise p would have a sign change at that root. Thus $q(x)$ has a square root $s(x)$.

Now write $r(x) = \prod_{j=1}^k (x-a_j)(x-\overline{a_j})$ (possible because r has roots in complex conjugate pairs). Write $\prod_{j=1}^k (x-a_j) = t(x) + iu(x)$ with t,x having real coefficients. Then for x real,

$$\begin{aligned} p(x) &= q(x)r(x) \\ &= s(x)^2 (t(x)+iu(x))(\overline{t(x)+iu(x)}) \\ &= (s(x)t(x))^2 + (s(x)u(x))^2. \end{aligned}$$

(Alternatively, one can factor $r(x)$ as a product of quadratic polynomials with real coefficients, write each as a sum of squares, then multiply together to get a sum of many squares.)

Second solution: We proceed by induction on the degree of p, with base case where p has degree 0. As in the first solution, we may reduce to a smaller degree in case p has any real roots, so assume it has none. Then $p(x) > 0$ for all real x, and since $p(x) \to \infty$ for $x \to \pm\infty$, p has a minimum value c. Now $p(x) - c$ has real roots, so as above, we deduce that $p(x) - c$ is a sum of squares. Now add one more square, namely $(\sqrt{c})^2$, to get $p(x)$ as a sum of squares.

A–3 First solution: Computing the coefficient of x^{n+1} in the identity $(1-2x-x^2)\sum_{m=0}^\infty a_m x^m = 1$ yields the recurrence $a_{n+1} = 2a_n + a_{n-1}$; the sequence $\{a_n\}$ is then characterized by this recurrence and the initial conditions $a_0 = 1, a_1 = 2$.

Define the sequence $\{b_n\}$ by $b_{2n} = a_{n-1}^2 + a_n^2,\ b_{2n+1} = a_n(a_{n-1}+a_{n+1})$. Then

$$\begin{aligned} 2b_{2n+1} + b_{2n} &= 2a_n a_{n+1} + 2a_{n-1}a_n + a_{n-1}^2 + a_n^2 \\ &= 2a_n a_{n+1} + a_{n-1}a_{n+1} + a_n^2 \\ &= a_{n+1}^2 + a_n^2 = b_{2n+2}, \end{aligned}$$

and similarly $2b_{2n} + b_{2n-1} = b_{2n+1}$, so that $\{b_n\}$ satisfies the same recurrence as $\{a_n\}$. Since further $b_0 = 1, b_1 = 2$ (where we use the recurrence for $\{a_n\}$ to calculate $a_{-1} = 0$), we deduce that $b_n = a_n$ for all n. In particular, $a_n^2 + a_{n+1}^2 = b_{2n+2} = a_{2n+2}$.

Second solution: Note that

$$\begin{aligned} &\frac{1}{1-2x-x^2} \\ &\quad = \frac{1}{2\sqrt{2}} \left(\frac{\sqrt{2}+1}{1-(1+\sqrt{2})x} + \frac{\sqrt{2}-1}{1-(1-\sqrt{2})x} \right) \end{aligned}$$

and that

$$\frac{1}{1+(1\pm\sqrt{2})x} = \sum_{n=0}^\infty (1\pm\sqrt{2})^n x^n,$$

so that

$$a_n = \frac{1}{2\sqrt{2}} \left((\sqrt{2}+1)^{n+1} - (1-\sqrt{2})^{n+1} \right).$$

A simple computation (omitted here) now shows that $a_n^2 + a_{n+1}^2 = a_{2n+2}$.

Third solution (by Richard Stanley): Let A be the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$. A simple induction argument shows that

$$A^{n+2} = \begin{pmatrix} a_n & a_{n+1} \\ a_{n+1} & a_{n+2} \end{pmatrix}.$$

The desired result now follows from comparing the top left corner entries of the equality $A^{n+2}A^{n+2} = A^{2n+4}$.

A–4 Denote the series by S, and let $a_n = 3^n/n$. Note that

$$\begin{aligned} S &= \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{1}{a_m(a_m+a_n)} \\ &= \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{1}{a_n(a_m+a_n)}, \end{aligned}$$

where the second equality follows by interchanging m and n. Thus

$$\begin{aligned}2S &= \sum_m \sum_n \left(\frac{1}{a_m(a_m+a_n)} + \frac{1}{a_n(a_m+a_n)} \right) \\ &= \sum_m \sum_n \frac{1}{a_m a_n} \\ &= \left(\sum_{n=1}^{\infty} \frac{n}{3^n} \right)^2.\end{aligned}$$

But

$$\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{3}{4}$$

since, e.g., it's $f'(1)$, where

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{3^n} = \frac{3}{3-x},$$

and we conclude that $S = 9/32$.

A–5 First solution: (by Reid Barton) Let $r_1, \dots, r_{1999}$ be the roots of P. Draw a disc of radius ϵ around each r_i, where $\epsilon < 1/3998$; this disc covers a subinterval of $[-1/2, 1/2]$ of length at most 2ϵ, and so of the 2000 (or fewer) uncovered intervals in $[-1/2, 1/2]$, one, which we call I, has length at least $\delta = (1-3998\epsilon)/2000 > 0$. We will exhibit an explicit lower bound for the integral of $|P(x)|/P(0)$ over this interval, which will yield such a bound for the entire integral.

Note that

$$\frac{|P(x)|}{|P(0)|} = \prod_{i=1}^{1999} \frac{|x-r_i|}{|r_i|}.$$

Also note that by construction, $|x - r_i| \geq \epsilon$ for each $x \in I$. If $|r_i| \leq 1$, then we have $\frac{|x-r_i|}{|r_i|} \geq \epsilon$. If $|r_i| > 1$, then

$$\frac{|x-r_i|}{|r_i|} = |1 - x/r_i| \geq 1 - |x/r_i| \geq= 1/2 > \epsilon.$$

We conclude that $\int_I |P(x)/P(0)|\,dx \geq \delta\epsilon$, independent of P.

Second solution: It will be a bit more convenient to assume $P(0) = 1$ (which we may achieve by rescaling unless $P(0) = 0$, in which case there is nothing to prove) and to prove that there exists $D > 0$ such that $\int_{-1}^{1} |P(x)|\,dx \geq D$, or even such that $\int_0^1 |P(x)|\,dx \geq D$.

We first reduce to the case where P has all of its roots in $[0, 1]$. If this is not the case, we can factor $P(x)$ as $Q(x)R(x)$, where Q has all roots in the interval and R has none. Then R is either always positive or always negative on $[0, 1]$; assume the former. Let k be the largest positive real number such that $R(x) - kx \geq 0$ on $[0, 1]$; then

$$\begin{aligned}\int_{-1}^{1} |P(x)|\,dx &= \int_{-1}^{1} |Q(x)R(x)|\,dx \\ &> \int_{-1}^{1} |Q(x)(R(x)-kx)|\,dx,\end{aligned}$$

and $Q(x)(R(x)-kx)$ has more roots in $[0, 1]$ than does P (and has the same value at 0). Repeating this argument shows that $\int_0^1 |P(x)|\,dx$ is greater than the corresponding integral for some polynomial with all of its roots in $[0, 1]$.

Under this assumption, we have

$$P(x) = c \prod_{i=1}^{1999} (x - r_i)$$

for some $r_i \in (0, 1]$. Since

$$P(0) = -c \prod r_i = 1,$$

we have

$$|c| \geq \prod |r_i^{-1}| \geq 1.$$

Thus it suffices to prove that if $Q(x)$ is a *monic* polynomial of degree 1999 with all of its roots in $[0, 1]$, then $\int_0^1 |Q(x)|\,dx \geq D$ for some constant $D > 0$. But the integral of $\int_0^1 \prod_{i=1}^{1999} |x - r_i|\,dx$ is a continuous function for $r_i \in [0, 1]$. The product of all of these intervals is compact, so the integral achieves a minimum value for some r_i. This minimum is the desired D.

Third solution (by Abe Kunin): It suffices to prove the stronger inequality

$$\sup_{x\in[-1,1]} |P(x)| \leq C \int_{-1}^{1} |P(x)|\,dx$$

holds for some C. But this follows immediately from the following standard fact: any two norms on a finite-dimensional vector space (here the polynomials of degree at most 1999) are equivalent. (The proof of this statement is also a compactness argument: C can be taken to be the maximum of the L1-norm divided by the sup norm over the set of polynomials with L1-norm 1.)

Note: combining the first two approaches gives a constructive solution with a constant that is better than that given by the first solution, but is still far from optimal. I don't know offhand whether it is even known what the optimal constant and/or the polynomials achieving that constant are.

A–6 Rearranging the given equation yields the much more tractable equation

$$\frac{a_n}{a_{n-1}} = 6\,\frac{a_{n-1}}{a_{n-2}} - 8\,\frac{a_{n-2}}{a_{n-3}}.$$

Let $b_n = a_n/a_{n-1}$; with the initial conditions $b_2 = 2, b_3 = 12$, one easily obtains $b_n = 2^{n-1}(2^{n-2}-1)$, and so

$$a_n = 2^{n(n-1)/2}\prod_{i=1}^{n-1}(2^i - 1).$$

To see that n divides a_n, factor n as $2^k m$, with m odd. Then note that $k \le n \le n(n-1)/2$, and that there exists $i \le m-1$ such that m divides $2^i - 1$, namely $i = \phi(m)$ (Euler's totient function: the number of integers in $\{1,\dots,m\}$ relatively prime to m).

B–1 The answer is 1/3. Let G be the point obtained by reflecting C about the line AB. Since $\angle ADC = \frac{\pi-\theta}{2}$, we find that $\angle BDE = \pi - \theta - \angle ADC = \frac{\pi-\theta}{2} = \angle ADC = \pi - \angle BDC = \pi - \angle BDG$, so that E, D, G are collinear. Hence

$$|EF| = \frac{|BE|}{|BC|} = \frac{|BE|}{|BG|} = \frac{\sin(\theta/2)}{\sin(3\theta/2)},$$

where we have used the law of sines in $\triangle BDG$. But by l'Hôpital's Rule,

$$\lim_{\theta\to 0}\frac{\sin(\theta/2)}{\sin(3\theta/2)} = \lim_{\theta\to 0}\frac{\cos(\theta/2)}{3\cos(3\theta/2)} = 1/3.$$

B–2 First solution: Suppose that P does not have n distinct roots; then it has a root of multiplicity at least 2, which we may assume is $x = 0$ without loss of generality. Let x^k be the greatest power of x dividing $P(x)$, so that $P(x) = x^k R(x)$ with $R(0) \ne 0$; a simple computation yields

$$P''(x) = (k^2-k)x^{k-2}R(x) + 2kx^{k-1}R'(x) + x^k R''(x).$$

Since $R(0) \ne 0$ and $k \ge 2$, we conclude that the greatest power of x dividing $P''(x)$ is x^{k-2}. But $P(x) = Q(x)P''(x)$, and so x^2 divides $Q(x)$. We deduce (since Q is quadratic) that $Q(x)$ is a constant C times x^2; in fact, $C = 1/(n(n-1))$ by inspection of the leading-degree terms of $P(x)$ and $P''(x)$.

Now if $P(x) = \sum_{j=0}^n a_j x^j$, then the relation $P(x) = Cx^2P''(x)$ implies that $a_j = Cj(j-1)a_j$ for all j; hence $a_j = 0$ for $j \le n-1$, and we conclude that $P(x) = a_n x^n$, which has all identical roots.

Second solution (by Greg Kuperberg): Let $f(x) = P''(x)/P(x) = 1/Q(x)$. By hypothesis, f has at most two poles (counting multiplicity).

Recall that for any complex polynomial P, the roots of P' lie within the convex hull of P. To show this, it suffices to show that if the roots of P lie on one side of a line, say on the positive side of the imaginary axis, then P' has no roots on the other side. That follows because if $r_1, \dots, r_n$ are the roots of P,

$$\frac{P'(z)}{P(z)} = \sum_{i=1}^n \frac{1}{z - r_i}$$

and if z has negative real part, so does $1/(z - r_i)$ for $i = 1, \dots, n$, so the sum is nonzero.

The above argument also carries through if z lies on the imaginary axis, provided that z is not equal to a root of P. Thus we also have that no roots of P' lie on the sides of the convex hull of P, unless they are also roots of P.

From this we conclude that if r is a root of P which is a vertex of the convex hull of the roots, and which is not also a root of P', then f has a single pole at r (as r cannot be a root of P''). On the other hand, if r is a root of P which is also a root of P', it is a multiple root, and then f has a double pole at r.

If P has roots not all equal, the convex hull of its roots has at least two vertices.

B–3 We first note that

$$\sum_{m,n>0} x^m y^n = \frac{xy}{(1-x)(1-y)}.$$

Subtracting S from this gives two sums, one of which is

$$\sum_{m\ge 2n+1} x^m y^n = \sum_n y^n \frac{x^{2n+1}}{1-x} = \frac{x^3y}{(1-x)(1-x^2y)}$$

and the other of which sums to $xy^3/[(1-y)(1-xy^2)]$. Therefore

$$\begin{aligned} S(x,y) &= \frac{xy}{(1-x)(1-y)} - \frac{x^3y}{(1-x)(1-x^2y)} - \frac{xy^3}{(1-y)(1} \\ &= \frac{xy(1+x+y+xy-x^2y^2)}{(1-x^2y)(1-xy^2)} \end{aligned}$$

and the desired limit is $\lim_{(x,y)\to(1,1)} xy(1+x+y+xy-x^2y^2) = 3$.

B–4 (based on work by Daniel Stronger) We make repeated use of the following fact: if f is a differentiable function on all of $\mathbb{R}$, $\lim_{x\to-\infty} f(x) \ge 0$, and $f'(x) > 0$ for all $x \in \mathbb{R}$, then $f(x) > 0$ for all $x \in \mathbb{R}$. (Proof: if $f(y) < 0$ for some x, then $f(x) < f(y)$ for all $x < y$ since $f' > 0$, but then $\lim_{x\to-\infty} f(x) \le f(y) < 0$.)

From the inequality $f'''(x) \le f(x)$ we obtain

$$f''f'''(x) \le f''(x)f(x) < f''(x)f(x) + f'(x)^2$$

since $f'(x)$ is positive. Applying the fact to the difference between the right and left sides, we get

$$\frac{1}{2}(f''(x))^2 < f(x)f'(x). \qquad (1)$$

On the other hand, since $f(x)$ and $f'''(x)$ are both positive for all x, we have

$$2f'(x)f''(x) < 2f'(x)f''(x) + 2f(x)f'''(x).$$

Applying the fact to the difference between the sides yields

$$f'(x)^2 \leq 2f(x)f''(x). \qquad (2)$$

Combining (1) and (2), we obtain

$$\begin{aligned} \frac{1}{2}\left(\frac{f'(x)^2}{2f(x)}\right)^2 &< \frac{1}{2}(f''(x))^2 \\ &< f(x)f'(x), \end{aligned}$$

or $(f'(x))^3 < f(x)^3$. We conclude $f'(x) < 2f(x)$, as desired.

Note: one can actually prove the result with a smaller constant in place of 2, as follows. Adding $\frac{1}{2}f'(x)f'''(x)$ to both sides of (1) and again invoking the original bound $f'''(x) \leq f(x)$, we get

$$\begin{aligned} \frac{1}{2}[f'(x)f'''(x) + (f''(x))^2] &< f(x)f'(x) + \frac{1}{2}f'(x)f'''(x) \\ &\leq \frac{3}{2}f(x)f'(x). \end{aligned}$$

Applying the fact again, we get

$$\frac{1}{2}f'(x)f''(x) < \frac{3}{4}f(x)^2.$$

Multiplying both sides by $f'(x)$ and applying the fact once more, we get

$$\frac{1}{6}(f'(x))^3 < \frac{1}{4}f(x)^3.$$

From this we deduce $f'(x) < (3/2)^{1/3}f(x) < 2f(x)$, as desired.

I don't know what the best constant is, except that it is not less than 1 (because $f(x) = e^x$ satisfies the given conditions).

B–5 We claim that the eigenvalues of A are 0 with multiplicity $n-2$, and $n/2$ and $-n/2$, each with multiplicity 1. To prove this claim, define vectors $v^{(m)}$, $0 \leq m \leq n-1$, componentwise by $(v^{(m)})_k = e^{ikm\theta}$, and note that the $v^{(m)}$ form a basis for $\mathbb{C}^n$. (If we arrange the $v^{(m)}$ into an $n \times n$ matrix, then the determinant of this matrix is a Vandermonde product which is nonzero.) Now note that

$$\begin{aligned} (Av^{(m)})_j &= \sum_{k=1}^n \cos(j\theta + k\theta)e^{ikm\theta} \\ &= \frac{e^{ij\theta}}{2}\sum_{k=1}^n e^{ik(m+1)\theta} + \frac{e^{-ij\theta}}{2}\sum_{k=1}^n e^{ik(m-1)\theta}. \end{aligned}$$

Since $\sum_{k=1}^n e^{ik\ell\theta} = 0$ for integer ℓ unless $n \mid \ell$, we conclude that $Av^{(m)} = 0$ for $m = 0$ or for $2 \leq m \leq n-1$. In addition, we find that $(Av^{(1)})_j = \frac{n}{2}e^{-ij\theta} = \frac{n}{2}(v^{(n-1)})_j$ and $(Av^{(n-1)})_j = \frac{n}{2}e^{ij\theta} = \frac{n}{2}(v^{(1)})_j$, so that $A(v^{(1)} \pm v^{(n-1)}) = \pm\frac{n}{2}(v^{(1)} \pm v^{(n-1)})$. Thus $\{v^{(0)}, v^{(2)}, v^{(3)}, \dots, v^{(n-2)}, v^{(1)} + v^{(n-1)}, v^{(1)} - v^{(n-1)}\}$ is a basis for $\mathbb{C}^n$ of eigenvectors of A with the claimed eigenvalues.

Finally, the determinant of $I+A$ is the product of $(1+\lambda)$ over all eigenvalues λ of A; in this case, $\det(I+A) = (1+n/2)(1-n/2) = 1 - n^2/4$.

B–6 First solution: Choose a sequence $p_1, p_2, \dots$ of primes as follows. Let p_1 be any prime dividing an element of S. To define p_{j+1} given $p_1, \dots, p_j$, choose an integer $N_j \in S$ relatively prime to $p_1 \cdots p_j$ and let p_{j+1} be a prime divisor of N_j, or stop if no such N_j exists.

Since S is finite, the above algorithm eventually terminates in a finite sequence $p_1, \dots, p_k$. Let m be the smallest integer such that $p_1 \cdots p_m$ has a divisor in S. (By the assumption on S with $n = p_1 \cdots p_k$, $m = k$ has this property, so m is well-defined.) If $m = 1$, then $p_1 \in S$, and we are done, so assume $m \geq 2$. Any divisor d of $p_1 \cdots p_m$ in S must be a multiple of p_m, or else it would also be a divisor of $p_1 \cdots p_{m-1}$, contradicting the choice of m. But now $\gcd(d, N_{m-1}) = p_m$, as desired.

Second solution (from `sci.math`): Let n be the smallest integer such that $\gcd(s, n) > 1$ for all s in n; note that n obviously has no repeated prime factors. By the condition on S, there exists $s \in S$ which divides n.

On the other hand, if p is a prime divisor of s, then by the choice of n, n/p is relatively prime to some element t of S. Since n cannot be relatively prime to t, t is divisible by p, but not by any other prime divisor of n (as those primes divide n/p). Thus $\gcd(s, t) = p$, as desired.