534. For two disjoint oriented curves C_1 and C_2 in three-dimensional space, parametrized by $\vec{v}_1(s)$ and $\vec{v}_2(t)$, define the linking number

$$\operatorname{lk}(C_1, C_2) = \frac{1}{4\pi} \oint_{C_1} \oint_{C_2} \frac{\vec{v}_1 - \vec{v}_2}{\|\vec{v}_1 - \vec{v}_2\|^3} \cdot \left(\frac{d\vec{v}_1}{ds} \times \frac{d\vec{v}_2}{dt} \right) dt\, ds.$$

Prove that if the oriented curves C_1 and $-C_1'$ bound an oriented surface S such that S is to the left of each curve, and if the curve C_2 is disjoint from S, then $\operatorname{lk}(C_1, C_2) = \operatorname{lk}(C_1', C_2)$.

3.4 Equations with Functions as Unknowns

3.4.1 Functional Equations

We will now look at equations whose unknowns are functions. Here is a standard example that we found in B.J. Venkatachala, *Functional Equations: A Problem Solving Approach* (Prism Books PVT Ltd., 2002).

Example. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the functional equation

$$f((x-y)^2) = f(x)^2 - 2xf(y) + y^2.$$

Solution. For $y = 0$, we obtain

$$f(x^2) = f(x)^2 - 2xf(0),$$

and for $x = 0$, we obtain

$$f(y^2) = f(0)^2 + y^2.$$

Setting $y = 0$ in the second equation, we find that $f(0) = 0$ or $f(0) = 1$. On the other hand, combining the two equalities, we obtain

$$f(x)^2 - 2xf(0) = f(0)^2 + x^2,$$

that is,

$$f(x)^2 = (x + f(0))^2.$$

Substituting this in the original equation yields

$$f(y) = \frac{f(x)^2 - f((x-y)^2) + y^2}{2x} = \frac{(x + f(0))^2 - (x - y + f(0))^2 + y^2}{2x}$$
$$= y + f(0).$$

We conclude that the functional equation has the two solutions $f(x) = x$ and $f(x) = x + 1$. □

But we like more the nonstandard functional equations. Here is one, which is a simplified version of a short-listed problem from the 42nd International Mathematical Olympiad. We liked about it the fact that the auxiliary function h from the solution mimics, in a discrete situation, harmonicity—a fundamental concept in mathematics. The solution applies the maximum modulus principle, which states that if h is a harmonic function then the maximum of $|h|$ is attained on the boundary of the domain of definition. Harmonic functions, characterized by the fact that the value at one point is the average of the values in a neighborhood of the point, play a fundamental role in geometry. For example, they encode geometric properties of their domain, a fact made explicit in Hodge theory.

Example. Find all functions $f : \{0, 1, 2, \dots\} \times \{0, 1, 2, \dots\} \to \mathbb{R}$ satisfying

$$f(p,q) = \begin{cases} \frac{1}{2}(f(p+1, q-1) + f(p-1, q+1)) + 1 & \text{if } pq \neq 0, \\ 0 & \text{if } pq = 0. \end{cases}$$

Solution. We see that $f(1,1) = 1$. The defining relation gives $f(1,2) = 1 + f(2,1)/2$ and $f(2,1) = 1 + f(1,2)/2$, and hence $f(2,1) = f(1,2) = 2$. Then $f(3,1) = 1 + f(2,2)/2$, $f(2,2) = 1 + f(3,1)/2 + f(1,3)/2$, $f(1,3) = 1 + f(2,2)/2$. So $f(2,2) = 4$, $f(3,1) = 3$, $f(1,3) = 3$. Repeating such computations, we eventually guess the explicit formula $f(p,q) = pq$, $p, q \geq 0$. And indeed, this function satisfies the condition from the statement. Are there other solutions to the problem? The answer is no, but we need to prove it.

Assume that f_1 and f_2 are both solutions to the functional equation. Let $h = f_1 - f_2$. Then h satisfies

$$h(p,q) = \begin{cases} \frac{1}{2}(h(p+1, q-1) + h(p-1, q+1)) & \text{if } pq \neq 0, \\ 0 & \text{if } pq = 0. \end{cases}$$

Fix a line $p + q = n$, and on this line pick (p_0, q_0) the point that maximizes the value of h. Because

$$h(p_0, q_0) = \frac{1}{2}(h(p_0+1, q_0-1) + h(p_0-1, q_0+1)),$$

it follows that $h(p_0+1, q_0-1) = h(p_0-1, q_0+1) = h(p_0, q_0)$. Shifting the point we eventually conclude that h is constant on the line $p + q = n$, and its value is equal to $h(n, 0) = 0$. Since n was arbitrary, we see that h is identically equal to 0. Therefore, $f_1 = f_2$, the problem has a unique solution, and this solution is $f(p,q) = pq$, $p, q \geq 0$.

And now an example of a problem about a multivariable function, from the same short list, submitted by B. Enescu (Romania).

Increase the left-hand side to $x+\sqrt{k}$; then square both sides. We obtain

$$x^2+k+2x\sqrt{k} \le k+kx^2+1+x^2,$$

which reduces to $0 \le (x\sqrt{k}-1)^2$, and this is obvious. The induction is now c
plete.

535. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x^2-y^2) = (x-y)(f(x)+f(y)).$$

536. Find all complex-valued functions of a complex variable satisfying

$$f(z)+zf(1-z) = 1+z, \quad \text{for all } z.$$

537. Find all functions $f : \mathbb{R}\backslash\{1\} \to \mathbb{R}$, continuous at 0, that satisfy

$$f(x) = f\left(\frac{x}{1-x}\right), \quad \text{for } x \in \mathbb{R}\backslash\{1\}.$$

538. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the inequality

$$f(x+y)+f(y+z)+f(z+x) \ge 3f(x+2y+3z)$$

for all $x, y, z \in \mathbb{R}$.

539. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = x^2-2$ for
numbers x?

540. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+y) = f(x)f(y) - c\sin x \sin y,$$

for all real numbers x and y, where c is a constant greater than 1.

541. Let f and g be real-valued functions defined for all real numbers and satis
functional equation

$$f(x+y)+f(x-y) = 2f(x)g(y)$$

for all x and y. Prove that if $f(x)$ is not identically zero, and if $|f(x)| \le$
x, then $|g(y)| \le 1$ for all y.

542. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the relation

$$3f(2x+1) = f(x)+5x, \quad \text{for all } x.$$

543. Find all functions $f : (0, \infty) \to (0, \infty)$ subject to the conditions
(i) $f(f(f(x))) + 2x = f(3x)$, for all $x > 0$;
(ii) $\lim_{x\to\infty}(f(x) - x) = 0$.

544. Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation

$$g(x - y) = g(x)g(y) + f(x)f(y)$$

for x and y in $\mathbb{R}$, and that $f(t) = 1$ and $g(t) = 0$ for some $t \neq 0$. Prove that f and g satisfy

$$g(x + y) = g(x)g(y) - f(x)f(y)$$

and

$$f(x \pm y) = f(x)g(y) \pm g(x)f(y)$$

for all real x and y.

A famous functional equation, which carries the name of Cauchy, is

$$f(x + y) = f(x) + f(y).$$

We are looking for solutions $f : \mathbb{R} \to \mathbb{R}$.

It is straightforward that $f(2x) = 2f(x)$, and inductively $f(nx) = nf(x)$. Setting $y = nx$, we obtain $f(\frac{1}{n}y) = \frac{1}{n}f(y)$. In general, if m, n are positive integers, then $f(\frac{m}{n}) = mf(\frac{1}{n}) = \frac{m}{n}f(1)$.

On the other hand, $f(0) = f(0) + f(0)$ implies $f(0) = 0$, and $0 = f(0) = f(x) + f(-x)$ implies $f(-x) = -f(x)$. We conclude that for any rational number x, $f(x) = f(1)x$.

If f is continuous, then the linear functions of the form

$$f(x) = cx,$$

where $c \in \mathbb{R}$, are the only solutions. That is because a solution is linear when restricted to rational numbers and therefore must be linear on the whole real axis. Even if we assume the solution f to be continuous at just one point, it still is linear. Indeed, because $f(x + y)$ is the translate of $f(x)$ by $f(y)$, f must be continuous everywhere.

But if we do not assume continuity, the situation is more complicated. In set theory there is an independent statement called the *axiom of choice*, which postulates that given a family of nonempty sets $(A_i)_{i\in I}$, there is a function $f : I \to \cup_i A_i$ with $f(i) \in A_i$. In other words, it is possible to select one element from each set.

Real numbers form an infinite-dimensional vector space over the rational numbers (vectors are real numbers, scalars are rational numbers). A corollary of the axiom of choice (Zorn's lemma) implies the existence of a basis for this vector space. If $(e_i)_{i\in I}$ is this basis, then any real number x can be expressed uniquely as

$$x = r_1 e_{i_1} + r_2 e_{i_2} + \cdots + r_n e_{i_n},$$

where $r_1, r_2, \ldots, r_n$ are nonzero rational numbers. To obtain a solution to Cauchy's equation, make any choice for $f(e_i)$, $i \in I$, and then extend f to all reals in such a way that it is linear over the rationals. Most of these functions are discontinuous. As an example, for a basis that contains the real number 1, set $f(1) = 1$ and $f(e_i) = 0$ for all other basis elements. Then this function is not continuous.

The problems below are all about Cauchy's equation for continuous functions.

545. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous nonzero function, satisfying the equation

$$f(x+y) = f(x)f(y), \quad \text{for all } x, y \in \mathbb{R}.$$

Prove that there exists $c > 0$ such that $f(x) = c^x$ for all $x \in \mathbb{R}$.

546. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+y) = f(x) + f(y) + f(x)f(y), \quad \text{for all } x, y \in \mathbb{R}.$$

547. Determine all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+y) = \frac{f(x)+f(y)}{1+f(x)f(y)}, \quad \text{for all } x, y \in \mathbb{R}.$$

548. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$f(xy) = xf(y) + yf(x), \quad \text{for all } x, y \in \mathbb{R}.$$

549. Find the continuous functions $\phi, f, g, h : \mathbb{R} \to \mathbb{R}$ satisfying

$$\phi(x+y+z) = f(x) + g(y) + h(z),$$

for all real numbers x, y, z.

550. Given a positive integer $n \geq 2$, find the continuous functions $f : \mathbb{R} \to \mathbb{R}$, with the property that for any real numbers $x_1, x_2, \ldots, x_n$,

$$\sum_i f(x_i) - \sum_{i<j} f(x_i + x_j) + \sum_{i<j<k} f(x_i + x_j + x_k) + \cdots$$
$$+ (-1)^{n-1} f(x_1 + x_2 + \cdots + x_n) = 0.$$

We conclude our discussion about functional equations with another instance in which continuity is important. The intermediate value property implies that a one-to-one continuous function is automatically monotonic. So if we can read from a functional equation that a function, which is assumed to be continuous, is also one-to-one, then we know that the function is monotonic, a much more powerful property to be used in the solution.

Example. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $(f \circ f \circ f)(x) = x$ for all $x \in \mathbb{R}$.

Solution. For any $x \in \mathbb{R}$, the image of $f(f(x))$ through f is x. This shows that f is onto. Also, if $f(x_1) = f(x_2)$ then $x_1 = f(f(f(x_1))) = f(f(f(x_2))) = x_2$, which shows that f is one-to-one. Therefore, f is a continuous bijection, so it must be strictly monotonic. If f is decreasing, then $f \circ f$ is increasing and $f \circ f \circ f$ is decreasing, contradicting the hypothesis. Therefore, f is strictly increasing.

Fix x and let us compare $f(x)$ and x. There are three possibilities. First, we could have $f(x) > x$. Monotonicity implies $f(f(x)) > f(x) > x$, and applying f again, we have $x = f(f(f(x))) > f(f(x)) > f(x) > x$, impossible. Or we could have $f(x) < x$, which then implies $f(f(x)) < f(x) < x$, and $x = f(f(f(x))) < f(f(x)) < f(x) < x$, which again is impossible. Therefore, $f(x) = x$. Since x was arbitrary, this shows that the unique solution to the functional equation is the identity function $f(x) = x$. □

551. Do there exist continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = x^2$ and $g(f(x)) = x^3$ for all $x \in \mathbb{R}$?

552. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ with the property that

$$f(f(x)) - 2f(x) + x = 0, \quad \text{for all } x \in \mathbb{R}.$$

3.4.2 Ordinary Differential Equations of the First Order

Of far greater importance than functional equations are the differential equations, because practically every evolutionary phenomenon of the real world can be modeled by a differential equation. This section is about first-order ordinary differential equations, namely equations expressed in terms of an unknown one-variable function, its derivative, and the variable. In their most general form, they are written as $F(x, y, y') = 0$, but we will be concerned with only two classes of such equations: separable and exact.

An equation is called separable if it is of the form $\frac{dy}{dx} = f(x)g(y)$. In this case we formally separate the variables and write

$$\int \frac{dy}{g(y)} = \int f(x)dx.$$

After integration, we obtain the solution in implicit form, as an algebraic relation between x and y. Here is a problem of I.V. Maftei from the 1971 Romanian Mathematical Olympiad that applies this method.