PROBLEM: Show that the only integer values in the set

$$\left\{ \frac{a^2+b^2}{ab+1} \mid a,b \in \mathbb{Z}, a,b \geq 0 \right\}$$

are the perfect squares.

SOLUTION:

We can assume without loss of generality that $0 \leq b \leq a$.

To simplify notation, let us introduce

$$x(a,b) = \frac{a^2+b^2}{ab+1}.$$

We do long computations, based on the idea that we should write $a = bk + r$ where $k, r \in \mathbb{Z}$, $0 \leq r < b$. At the end, we observe that if (a,b) is a pair with $x(a,b) \in \mathbb{Z}$ then $x(a,b) = x(b, b-r)$, that is, $(b, b-r)$ is another solution. Computing r explicitely gives

$$b - r = b - \left[a - b \left(\frac{a^2+b^2}{ab+1} - 1 \right) \right] = \frac{b^3 - a}{ab+1}$$

Thus, we can forget about the computation and use only the following fact:

Given a pair of real numbers (a,b) with $0 \leq b \leq a$, consider the pair (b,c), where

$$c = \frac{b^3 - a}{ab+1}.$$

Then:

(1) $x(a,b) = x(b,c)$

(2) $c \leq b$, with equality only if $a = b = 0$.

(3) if $a, b, x(a,b) \in \mathbb{Z}$ and $b > 0$ then $c \geq 0$.

Once we know this (to be checked later), the proof is short.

Assume that $0 \leq b \leq a$ are integers for which $x(a,b) \in \mathbb{Z}$. Then, provided $b \neq 0$, we can construct another integer solution, $(a', b') = (b, c)$, with $0 \leq b' < a' = b \leq a$ (use (2) and (3)). That is, we obtain a **smaller** solution, without changing $x(a,b)$.

This process must end in finitely many steps. We cannot continue it when $b = 0 \leq a$, which gives $x(a,b) = a^2$. Therefore, for the whole sequence of solutions that led to $(a, 0)$, the value of the quotient x is equal to a^2, a perfect square, as desired.

The reverse operation is

$$(b,c) \mapsto (a,b), \qquad a = \frac{b^3 - c}{bc+1},$$

and we can reconstruct the (increasing) sequence of solutions:

$$(a, 0) \to (a^3, a) \to (a^5 - a, a^3) \to \dots$$

It remains to check conditions (1)-(3).

The first one is just a direct computation.

The inequality $c \leq b$ is equivalent to $b(b^2 - 1) \leq a(b^2 + 1)$, which explains (2).

Finally, notice that if $ab + 1$ divides $a^2 + b^2$, then it also divides $b^4 + 1 = (a^2 + b^2)b^2 - (ab + 1)(ab - 1)$. Hence, $ab + 1 \leq b^4 + 1$. If $b > 0$, this implies that $a \leq b^3$, hence that $c \geq 0$.

OPEN QUESTION: What is the theory behind this?

Note that the result is not true if a and b can have opposite signs: $x(2, -1) = -5$.