Some partial answers to natural questions in II_1–factors.

The following are natural questions in a II_1–factor $\mathcal{M}$, because their direct analogues in $M_n(\mathbf{C})$ have positive answers.

(I) Is every element in $\mathcal{M}$ whose trace value is zero equal to a single commutator of elements of $\mathcal{M}$?

(II) If A is a maximal abelian self–adjoint subalgebra (called a *masa*) in $\mathcal{M}$ and if $x \in A$ satisfies $0 \leq x \leq 1$, is there a projection $p \in \mathcal{M}$ such that $E_A(p) = x$, where E_A is the trace–preserving conditional expectation onto A?

Both questions are still open, and in this talk we'll discuss some partial answers and a related question.

We'll discuss partial answers to (I), from joint work with Anna Skripka [DS].

Question (II) is known as the Carpenter problem, thus named by R.V. Kadison [Ka02a], [Ka02b]. It is a particular case of the Schur–Horn problem in II_1–factors, which was formulated by Arveson and Kadison in [AKa06]:

(II′) For a masa $A \subseteq \mathcal{M}$, if $x = x^* \in \mathcal{M}$ and if $b \in A$ is such that their spectral distributions satisfy $\mu_b \preceq \mu_x$, (this is a suitable notion of spectral dominance, requiring also $\tau(b) = \tau(x)$), does it follow that we have $b = E_A(y)$ for an element y of the norm closure of the unitary orbit of x?

The reverse direction, namely, that we have $\mu_{E(y)} \preceq \mu_y$ for all self–adjoint $y \in \mathcal{M}$, was established in [AKa06]. The corresponding statement (going in both directions) in a matrix algebra $M_n(\mathbf{C})$ is known as the Schur–Horn Theorem.

We'll discuss answers to (II′) in some cases, from joint work with Junsheng Fang, Don Hadwin and Roger Smith [DFHS].

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