Question 1

 Tic-Tac-Toe is commonly played on a 3 × 3 grid, and it is known that if both players make

optimal moves then it will result in a tie; i.e., one player cannot force a win.

(i) Can either player force a win in 4 × 4 tic-tac-toe (where a win is four in a row, four in a

 column, or four along one of the two main diagonals)?

(ii) (BONUS) Can either player force a win in n × n tic-tac-toe, where n > 4?

Answer (for both): No, it is not possible to force a win for any square (n x n) tic-tac-toe game where $n\geq 3$.

Definitions:

* Win paths- lines (n long) that result in victory, so for a 3x3 this would be any line 3 units long on the board, which result in a win.
* Optimal block number- the number of turns it takes for the opponent, playing optimally, to block every possible win path.
* Playing optimally- Player opening the maximum win paths per turn, and the opponent closing the maximum win paths per turn.

For each number n, the number of possible “wins” is given by n(n+1), and the optimal block number is given by n+1. The actual optimal block number is n, but if both are playing optimally, this number becomes n+1, as the table below demonstrates, determined by experimentation.

|  |  |  |
| --- | --- | --- |
| n | Optimal block number | Number of possible wins |
| 3 | 4 | 8 |
| 4 | 5 | 10 |
| 5 | 6 | 12 |
| 6 | 7 | 14 |
| n | n+1 | n(n+1) |

In order to prove that there is no way to force a win, we must prove that, for the last two steps before all possible win paths are closed, the number of moves your opponent has left is greater than or equal to the number of existing win paths. The game is always won in the last two moves, so if we can prove that the opponent has more (or equal) moves than there are paths left, the player cannot force a win.

To do this we examine the number of paths blocked by each consecutive move by the opponent.

We already know that this is true for the 3x3, but here is the proof.

3x3

|  |  |  |
| --- | --- | --- |
| Opponent move number | Number of paths blocked | Paths remaining |
| 1 | 3 | 5 |
| 2 | 2 | 3 |
| 3 | 2 | 1 |
| 4 | 1 | 0 |

Note that in moves 3 and 4, there are always more moves by the opponent than win paths remaining, so the player cannot force a win.

Proof for 4x4:

4x4

|  |  |  |
| --- | --- | --- |
| Opponent move number | Number of paths blocked | Paths remaining |
| 1 | 3 | 7 |
| 2 | 3 | 4 |
| 3 | 2 | 2 |
| 4 | 1 | 1 |
| 5 | 1 | 0 |

With the player and opponent opening or closing the maximum number of win paths, respectively, it is impossible for the player to win. Note that in moves 4 and 5, there are always more moves left (for the opponent) than paths remaining, which is what we set out to prove. The opponent only needs 5 moves to block every possible win path, the player will never be able to get more than 3 in a row, less than the 4 required, thus, it is impossible for a player to force a win in a 4x4 game of tic-tac-toe.

Proof for 5x5

5x5

|  |  |  |
| --- | --- | --- |
| Opponent move number | Paths Blocked | Possible paths remaining  |
| 1 | 3 | 9 |
| 2 | 3 | 6 |
| 3 | 2 | 4 |
| 4 | 2 | 2 |
| 5 | 1 | 1 |
| 6 | 1 | 0 |

Note that in moves 5 and 6, the last two moves, there are always more moves left (for the opponent) than paths remaining, which is what we set out to prove. Thus, we have proved for both odd and even tables that there can be no forced win, so it must be true for all n>=3