PROBLEM OF THE WEEK - FALL 2014 - WEEK OF 11.3.14

Question 1

[*Proposed by Sergey Sarkisov*] You had a collection of critters weighing 3000 lbs, and 99% of their mass is water-weight. Then, they exercise, thus losing water-weight until their total weight is only 98% water. How much do these critters now weigh?

Question 2

[*Proposed by Dr. Glowinski*]

Let L be a positive number. With $f \in L^2(0, L)$ we associate the function φ defined by

$$\varphi(x) = \frac{1}{x} \int_0^x f(\xi) d\xi, \forall x \in (0, L). \tag{1}$$

It is well-known that $\varphi \in L^2(0, L)$, $\forall f \in L^2(0, L)$, and (***Hardy's inequality***) that

$$\|\varphi\|_{L^2(0,L)} \leq 2\|f\|_{L^2(0,L)}, \forall f \in L^2(0, L). \tag{2}$$

Questions:

i) Show that the inequality (2) is *sharp*, i.e., 2 is the best constant.

ii) We denote by T the linear operator from $T : L^2(0, L) \to L^2(0, L)$ which with f associates φ via relation (1). Show that T is ***not compact*** from $L^2(0, L)$ to $L^2(0, L)$.

iii) Show that T is ***compact*** from $L^2(0, L)$ to $L^s(0, L)$, for $1 \leq s < 2$.

Remark. Of course the integral in (1) is ***Lebesgue's***.