On the Curvature of Rational Surfaces

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Outline

- The Types of Curvatures on a Complex Manifold
- A Relationship Between Curvature and Rationality
- Special Example: The Hirzebruch Surface
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► A Relationship Between Curvature and Rationality
► Special Example: The Hirzebruch Surface
Definition
Let $M$ be an $n$-dimensional complex manifold. Let $p \in M$. A hermitian metric on $M$ is a positive definite hermitian inner product
\[ g_p : T'_p M \otimes \overline{T'_p M} \to \mathbb{C} \]
which varies smoothly for each $p \in M$. 
By “varying smoothly”, we mean that if $z = (z_1, ..., z^n)$ are local coordinates around $p$ and $\{\frac{\partial}{\partial z^1}, ..., \frac{\partial}{\partial z^n}\}$ is the standard basis for $T'_p M$, then the functions

$$g_{ij} : M \to \mathbb{C}, \quad p \mapsto g_p \left( \frac{\partial}{\partial z^i}(p), \frac{\partial}{\partial z^j}(p) \right)$$

are smooth.
Let \( \{dz^1, \ldots, dz^n\} \) be the dual basis of \( \{\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n}\} \). Then locally, the hermitian metric can be written as

\[
g = \sum_{i,j=1}^{n} g_{ij} dz^i \otimes d\bar{z}^j
\]

where \([g_{ij}]\) is an \( n \times n \) positive definite Hermitian matrix of smooth functions.

The metric \( g \) can be decomposed into two parts: (1) The Real Part, (2) The Imaginary Part.
The real part, $Re(g)$, gives an ordinary inner product called the *induced Riemannian metric* of $g$.

Because $T_{\mathbb{R},p}M \cong_{\mathbb{R}} T'_pM$, we write

$$Re(g) : T_{\mathbb{R},p}M \otimes T_{\mathbb{R},p}M \to \mathbb{R}$$

The imaginary part,

$$Im(g) : T_{\mathbb{R},p}M \otimes T_{\mathbb{R},p}M \to \mathbb{R}$$

is alternating and thus represents an $\mathbb{R}$-differential 2-form.

Write $g = Re(g) + \sqrt{-1}Im(g)$ and let $\omega = \frac{-1}{2}Im(g)$. 
Definition
The \((1, 1)\)-form \(\omega := -\frac{1}{2} \text{Im}(g)\) is called the associated \((1, 1)\)-form of \(g\).

Definition
The hermitian metric \(g\) becomes a \textit{Kähler metric} if \(\omega\) is closed.
Definition
A connection on a complex manifold $M$ is a $\mathbb{C}$-linear map

$$D : \Gamma(TM) \to \Gamma(T^*M \otimes TM)$$

which satisfies the Leibniz rule:

$$D(fX) = df \otimes X + fDX$$

$\forall X \in \Gamma(TM)$ and $\forall f \in C^\infty(M)$. 
Choose $D$ to be the canonical metric connection (i.e. a connection that is compatible with the metric $g$ and the complex structure of $M$).

Locally, if $z = (z^1, ..., z^n)$ is a local chart on $M$ around $p$ and $\left\{ \frac{\partial}{\partial z^i} \right\}$ is the standard basis for $T'_p M$, then $\left\{ dz^i \otimes \frac{\partial}{\partial z^j} \right\}_{i,j=1}^n$ forms a basis for the vector space $\left\{ T^*_p M \otimes T_p M \right\}$. 
Let $\frac{\partial}{\partial z^i} \in \Gamma(TM)$. Then by definition,

$$D\frac{\partial}{\partial z^i} = \sum_{j,k=1}^{n} \Gamma^k_{ij} dz^j \otimes \frac{\partial}{\partial z^k}$$  \hspace{1cm} (1)$$

where $\Gamma^k_{ij}$ are smooth functions called *Christoffel symbols*. 
Let $\Theta^k_i := \sum_{j=1}^n \Gamma_{ij}^k dz^j$.

Because $D$ was chosen to be the canonical metric connection, we have

$$\Theta^k_i := \sum_{j=1}^n \Gamma_{ij}^k dz^j = \sum_{j=1}^n g^{jk} \partial g_{ij}$$

So equation (1) becomes

$$D \frac{\partial}{\partial z^i} = \sum_{k=1}^n \Theta^k_i \otimes \frac{\partial}{\partial z^k}$$

Let

$$\Theta := [\Theta^j_i]_{i,j=1}^n$$
**Definition**
The \( n \times n \) matrix \( \Theta \) is called the *connection matrix* of \( D \) w.r.t \( \left\{ \frac{\partial}{\partial z^i} \right\} \).

**Definition**
The matrix of differential 2-forms \( \Omega := d\Theta - \Theta \wedge \Theta \) is called the *curvature matrix* w.r.t \( \left\{ \frac{\partial}{\partial z^i} \right\} \).
Computing the elements of $\Omega = [\Omega_{ij}]_{i,j=1}^n$ using the structure equation yields

$$\Omega_{ij} = \sum_{i,j=1}^n R^j_{ikl} dz^k \wedge d\bar{z}^l$$

where

$$R^j_{ikl} = \sum_{p=1}^n \frac{\partial g_{ip}}{\partial k} \frac{\partial g^{pj}}{\partial \bar{z}^l} - g^{pj} \frac{\partial^2 g_{ip}}{\partial z^k \partial \bar{z}^l} \tag{2}$$
Introducing the covariant tensor

\[ R_{ijkl} = \sum_{p=1}^{n} g_{pj} R_{ikl}^{p} \]  

(3)

yields

\[ \Omega_{ij} = R_{ijkl} dz^{k} \wedge d\bar{z}^{l} \]

where, by equations (2) and (3),

\[ R_{ijkl} = \frac{-\partial^{2}g_{ij}}{\partial z^{k}\partial \bar{z}^{l}} + \sum_{p,q=1}^{n} g^{pq} \frac{\partial g_{ip}}{\partial z^{k}} \frac{\partial g_{qj}}{\partial \bar{z}^{l}} \]  

(4)
The $R_{ijkl}$ are the components of the *Riemann Curvature Tensor*, and are used to define the following curvatures.
Definition
Let \( \xi = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial z_i} \) be a complex unit tangent vector.

The **holomorphic sectional curvature**, \( K \), in the direction of \( \xi \) is

\[
K(\xi) = \sum_{i,j,k,l=1}^{n} 2R_{ijkl} \xi_i \bar{\xi}_j \xi_k \bar{\xi}_l
\]

The **Ricci curvature**, \( \text{Ric} \), in the direction of \( \xi \) is

\[
\text{Ric}(\xi) = \sum_{i,k,l=1}^{n} R_{iikl} \xi_k \bar{\xi}_l
\]

The **scalar curvature**, \( R \), is just the trace of \( \text{Ric} \),

\[
R = \sum_{i,j=1}^{n} R_{iijj}
\]
Some Remarks:

- $R_{ijkl} = \bar{R}_{jilk}$
- $R_{ijkl} = R_{kjil} = R_{ilkj}$
- Due to these symmetry conditions, all curvatures are real-valued.
- $Ric > 0 \implies R > 0$
- $K > 0 \implies R > 0$ (M. Berger, 1965)
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Recall:

**Definition**
An algebraic surface $M$ is *rational* if there exist a birational map $M \to \mathbb{P}^2$.

**Definition**
Let $M$ be a complex surface. Then $M$ is *regular* if $b_1(X) = \dim H_1(M, \mathbb{C}) = 0$. 
Proposition

Let $M$ be a regular Kähler surface. If $M$ has positive scalar curvature, then $M$ is rational.
Proof.
By the Vanishing Theorem of Kobayashi and Wu, if a surface has positive scalar curvature, then the plurigenera $P_m = \dim H^0(M, \mathcal{O}(K^m_M)) = 0, \forall \ m > 0$.

By Kodaira, we know that a complex surface with an even first betti number and $P_1 = 0$ is algebraic. Because $b_1 = 0$, and $P_m = 0, \forall \ m > 0$, we have that $M$ is algebraic.

The Theorem of Castelnuovo-Enriques states that if an algebraic surface has irregularity, $q = 0$ and $P_2 = 0$, then the surface is rational. We already have that $P_m = 0$. Also, because $M$ is Kähler and regular, $q = \frac{0}{2} = 0$. Thus, $M$ is rational.
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Definition

The $n^{th}$ Hirzebruch surface, $S_n$, is the ruled surface

$$S_n = \mathbb{P}(H^n \oplus 1)$$

Using sections of the line bundles, $H^n$ and 1, and projectivizing, we have special curves on $S_n$: 

![Diagram showing special curves on $S_n$]
Remarks:

- $E_\infty$ is the only irreducible curve on $S_n$ with negative self-intersection.
- For $n \neq 0$, $S_n$ is the only $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ which has an irreducible curve with negative self-intersection, $-n$. (⋆)
- For each $n$, the spaces $\{S_n\}_{n \geq 0}$ are all distinct as complex manifolds, and are indeed rational.
Proposition

All surfaces \( \{ S_n \}_{n \geq 0} \) are rational.

Proof: We will do this inductively. For \( n = 0 \) (the simplest Hirzebruch surface), we have

\[
S_0 = \mathbb{P}(H^0 \oplus 1) = \mathbb{P}^1 \times \mathbb{P}^1
\]

We will construct a birational map between \( S_0 \) and \( \mathbb{P}^2 \) as follows: Let \( p, q \in \mathbb{P}^2 \) and let \( \ell = \overline{pq} \) be the line segment joining \( p \) and \( q \). Take the blow-up, \( \pi : \tilde{\mathbb{P}}^2 \to \mathbb{P}^2 \). Let \( \tilde{\ell} \) be the strict transform of \( \ell \), \( E_p = \pi^{-1}(p) \), and \( E_q = \pi^{-1}(q) \). Because \( \ell \) was a 1–curve, blowing up two points on it makes \( \tilde{\ell} \) a \((-1)\)-curve.
By the Castelnuovo-Enriques Criterion, \( \tilde{\ell} \) is an exceptional divisor of some point in \( \mathbb{P}^1 \times \mathbb{P}^1 \). So \( \tilde{\ell} \) can be blown down to a point, say \( z \in \mathbb{P}^1 \times \mathbb{P}^1 \), and the exceptional divisors become 0 – curves. Because the blow-up (and blow-down) is a birational map, and the composition of birational maps is birational, we have that \( S_0 = \mathbb{P}^1 \times \mathbb{P}^1 \) is birational to \( \mathbb{P}^2 \) and thus \( S_0 \) is rational.
Now assume that $S_{n-1}$ is rational. We will show that $S_n$ and $S_{n-1}$ are birational to each other, $\forall \ n \geq 1$.

Let $C_\lambda$ be an irreducible curve on $S_n$ and let $x \in S_n$ be any point on $C_\lambda$, not on $E_\infty$. Blow up $C_\lambda$ at $x$, $\pi_1 : \tilde{S}_n \to S_n$. Because $C_\lambda$ is 0-curve, the strict transform $\tilde{C}_\lambda$ is a $(-1)$-curve in $\tilde{S}_n$. Thus, $\tilde{C}_\lambda$ is the exceptional divisor of some point $y \in S$, where $S$ is some smooth algebraic surface.

Let $\pi_2 : \tilde{S}_n \to S$ be the blow-up of $S$ at $y$. If $\pi_1^*E_\infty$ is the total transform of $E_\infty$, then $y \in \pi_2(\pi_1^*E_\infty)$. Because $\{\pi_2(\pi_1^*C_\lambda)\}_{\lambda \in \mathbb{P}^1}$ forms a pencil of irreducible rational curves on $S$ with self-intersection 0, $S$ is a rational ruled surface.
Note that
\[ \pi_2(\pi_1^*E_\infty).\pi_2(\pi_1^*E_\infty) = E_\infty.E_\infty + 1 = -n + 1 = -(n - 1) \]

So by (⋆), S is biholomorphic to \( S_{n-1} \), and thus is birational to \( S_{n-1} \). Therefore, we have

\[ \mathbb{P}^2 \xrightarrow{\text{birational}} S_0 \xrightarrow{\text{birational}} S_1 \xrightarrow{\text{birational}} \cdots S_{n-1} \xrightarrow{\text{birational}} S_n \]

Thus, \( \forall \ n \geq 0, \ S_n \) is birational to \( \mathbb{P}^2 \) and thus are all rational.
Now to consider curvature on $S_n$...
Proposition

\( \text{Ric}(S_n) \) is not positive for all \( n \geq 2 \).

Proof: It suffices to show that \( -K_{S_n} \) is not ample for \( n \geq 2 \). By Nakai’s Criterion, a line bundle is ample if its self-intersection number is positive and its intersection with any irreducible curve is positive. Consider \( C = E_\infty \) on \( S_n \). Using the adjunction formula, we have

\[
K_{E_\infty} = K_{S_n}|_{E_\infty} \otimes [E_\infty]|_{E_\infty}
\]
Then

\[
\deg K_{E_\infty} = \deg K_{S_n}|_{E_\infty} + \deg [E_\infty]|_{E_\infty} \\
= \deg K_{S_n}|_{E_\infty} + E_\infty \cdot E_\infty \quad \text{(def. of intersection number)} \\
= \deg K_{S_n}|_{E_\infty} + (-n) \\
= K_{S_n} \cdot E_\infty - n
\]

With \( \deg K_{E_\infty} = 2g - 2 \) and \( g = 0 \), \( \deg K_{E_\infty} = -2 \). Thus

\[-2 = K_{S_n} \cdot E_\infty - n \implies -K_{S_n} \cdot E_\infty = 2 - n \leq 0\]

Thus, \(-K_{S_n}\) is not ample and \(\text{Ric}(S_n)\) cannot be positive, for \( n \geq 2 \).
Proposition

For all $n \geq 0$, $S_n$ admits [Hodge] metrics of positive holomorphic sectional curvature.
Before proving positivity, let us first look at the general case for any Hermitian vector bundle over a Kähler manifold, $\pi : E \to X$, and use this as a model to define a Kähler metric on $S_n$.

Consider $\pi : E\{0\} \to X$ be the projection without the 0 section. Let $\omega$ be the associated $(1, 1)$-form of $X$ (which we know is closed). On $E\{0\}$, define a form $\bar{\phi}$

$$\bar{\phi} = \pi^* \omega + \sqrt{-1} s \partial \bar{\partial} \log ||w||^2$$

where $w \in E\{0\}$ and $s \in \mathbb{R}^+$.  

Projectivizing $E$, $\mathbb{P}(E)$ means we projective $\bar{\phi}$, which yields a closed form $\varphi$ which is positive definite for $s$ small enough.
We use this construction on $S_n$, where $M = \mathbb{P}^1$:

Take the standard Kähler metric on $\mathbb{P}^1$, the Fubini-Study metric

\[
\frac{dz_1 \wedge d\bar{z}_1}{(1 + z_1 \bar{z}_1)^2}
\]

Since $H^{-2} = K_{\mathbb{P}^1}$, $H^2 = T\mathbb{P}^1$.

Thus we have a natural hermitian metric on $H^n \oplus 1$. 
Let $z_1$ be an inhomogeneous coordinate on $\mathbb{P}^1$.

Then $dz_1 \in K_{\mathbb{P}^1} \implies dz^{-1} \in T\mathbb{P}^1$.

Write $w \in H^n \oplus 1$ as

$$w = (z_1, w_1(dz_1)^{-\frac{n}{2}}, w_2)$$

where $w_1, w_2 \in \mathbb{C}$ and $(dz_1)^{-\frac{n}{2}} \in H^n$. 
Using Fubini-Study, the metric on $H^n \oplus 1$ is given by

$$||w||^2 = w_1 \bar{w}_2 (1 + z_1 \bar{z}_2)^n + w_2 \bar{w}_2$$

Taking local inhomogeneous coordinates $z_2 = w_2/w_1$, yields the form on $H^n \oplus 1$

$$\bar{\phi} = \pi^* \omega + \sqrt{-1} s \partial \bar{\partial} \log ||w||^2$$
Projectivizing yields a closed form $\varphi$.

Using the fact that the Kähler form of Fubini-Study is
\[ \omega = i \partial \bar{\partial} \log |z|^2 = \sqrt{-1} \partial \bar{\partial} \log(1 + z_1 \bar{z}_1) \]
and that $z_2 = w_2/w_1$ yields

\[ \varphi = \sqrt{-1} \partial \bar{\partial} [\log(1 + z_1 \bar{z}_1) + s \log((1 + z_1 \bar{z}_1)^n + z_2 \bar{z}_2)] \]

which is the metric we must calculate the curvature.
Remark:

Because $SU(2)$ acts on $\mathbb{P}^1$ as an isometry of Fubini-Study, preserves the fiber metric when lifted to the bundle, and acts transitively on $\mathbb{P}^1$, WLOG, we can simplify our computations by setting $z_1 = 0$. 
Let $G := \log(1 + z_1 \bar{z}_1) + s \log((1 + z_1 \bar{z}_1)^n + z_2 \bar{z}_2)$.

Thus $\varphi = \sqrt{-1} \partial \bar{\partial} G = \sqrt{-1} g_{ij} dz_i \wedge d\bar{z}_j$.

Direct computation yields

$$
g_{ij} = \begin{bmatrix}
\frac{1+z_2 \bar{z}_2+sn}{1+z_2 \bar{z}_2} & 0 \\
0 & \frac{s}{(1+z_2 \bar{z}_2)^2}
\end{bmatrix}, \quad g^{ij} = \begin{bmatrix}
\frac{1+z_2 \bar{z}_2}{1+z_2 \bar{z}_2+sn} & 0 \\
0 & \frac{(1+z_2 \bar{z}_2)^2}{s}
\end{bmatrix}
$$
Equation (4) yields

\[ R_{1111} = \frac{2(-nsz_2\bar{z}_2 + (1 + z_2\bar{z}_2)^2 + n(s + sz_2\bar{z}_2))}{(1 + z_2\bar{z}_2)^2} \]

\[ R_{1122} = \frac{ns(1 + ns - z_2^2\bar{z}_2^2)}{(1 + z_2\bar{z}_2)^3(1 + ns + z_2\bar{z}_2)} \]

\[ R_{2222} = \frac{2s}{(1 + z_2\bar{z}_2)^4} \]

while the other terms (except those obtained from symmetry) are zero.
Plugging in these values into the formula:

\[ K(\xi) = \sum_{i,j,k,l=1}^{n} 2R_{ijkl} \xi_i \bar{\xi}_j \xi_k \bar{\xi}_l \]  

gives us

\[ K(\xi) = K(\xi_1, \xi_2) = \frac{4}{(1 + z_2 \bar{z}_2)^4}(\xi_1 \bar{\xi}_1)^2(1 + z_2 \bar{z}_2) \]

\[ + \frac{4s}{(1 + z_2 \bar{z}_2)^4} \left( (\xi_1 \bar{\xi}_1)^2(1 + z_2 \bar{z}_2)^3 + (\xi_2 \bar{\xi}_2)^2 \right) \]

\[ + \frac{4s}{(1 + z_2 \bar{z}_2)^4} \left( -\frac{(\xi_1 \bar{\xi}_1)^2 n^2 z_2 \bar{z}_2 (1 + z_2 \bar{z}_2)^2 + 2n(\xi_1 \bar{\xi}_1)(\xi_2 \bar{\xi}_2)(1 + z_2 \bar{z}_2)(1 + ns - z_2^2 \bar{z}_2^2)}{1 + ns + z_2 \bar{z}_2} \right) \]

\[ > 0 \]

due to the first term being positive and letting \( s \) being very small. Thus \( S_n \) has a metric which admits positive holomorphic sectional curvature \( \forall \ n \geq 0 \).
Thank you.